Analysis and Design Principles of Microwave Antennas
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Lecture – 31
Solution of Wave Equation for electric and Magnetic Current Densities

Welcome to this NPTEL lecture. Today we will see that if we have a Magnetic Current Density M, then how to find the vector potential from it and then, we will find what is the how to solve the wave equation with the help of that vector potential. And then, we will go to actually solution of the general case. So, today since in the last lecture we have introduced the concept of magnetic current. So, let me rewrite the Maxwell's equation, when both J and M; that means, electric current and magnetic current are present I can write them Maxwell's equation.

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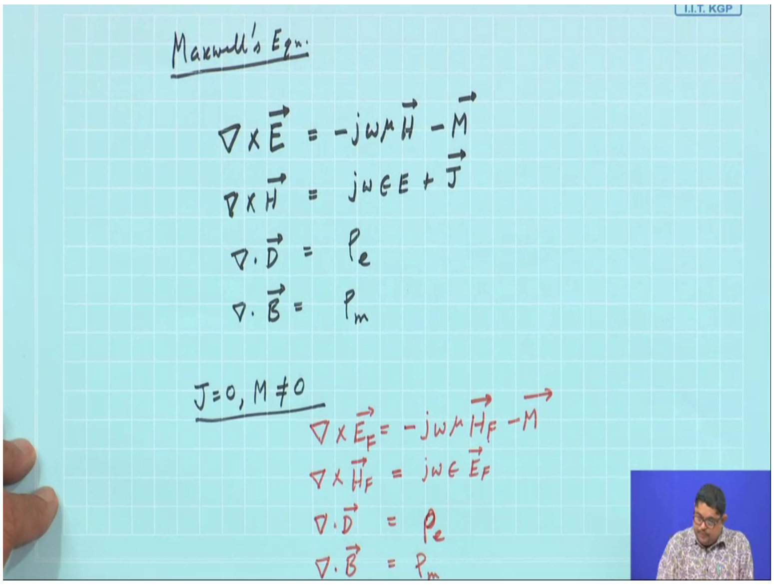

So, I can write del cross E is equal to minus j omega mu H minus M, then del cross H is equal to j omega epsilon E plus J. Del dot D is equal to rho e and del dot B is equal to rho m. So, this is the generalized Maxwell's equation. Time harmonic case; that means, both J and M are time harmonic currents. Now since, we already have seen the case when M is equal to 0; that means, only J present; we will not do it again.

But for the case when we have that J is 0, but M is not 0; then, I can rewrite this equation which I will now solve. So, del cross E is equal to minus j omega mu, but here since actually this solution will be in terms of the vector potential F actually. As in case of electric current, we introduced a the magnetic vector potential in case of M, will introduce the electric vector potential F; that is why I am saying. But that this I am considering this case, where M is present J is absent; so, I am putting a subscript F in the field quantities.

So, E F, this F subscript means that I will actually introduce the potential function F electric vector potential, I will define that. So, that is why I am writing this M and I can write del cross H F that will be j omega epsilon E F plus nothing because J is 0 and del dot D that will become 0, the or let me write first rho e, I will just now prove and del dot B is definitely rho m.

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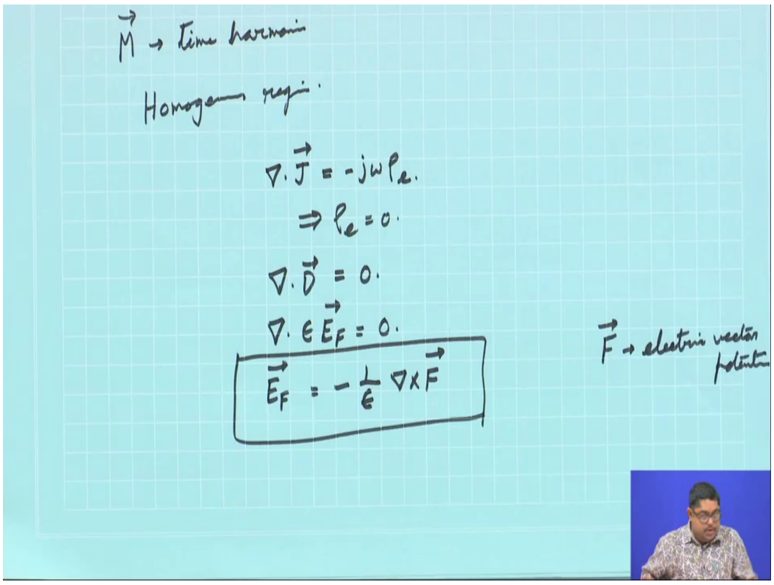

Now, I give the qualifications of M that M is a time harmonic magnetic current, time harmonic magnetic current also the whole region is a Homogeneous region. So, now, continuity equation holds. So, I just first invoke the continuity equation. So, I write what is del cross J is minus j omega rho e. Now, this rho e is 0 sorry; now J is 0. So, del dot J is 0 that gives this implies that rho e is 0.

So, if rho e is 0, then actually I can correct this equation that this thing becomes 0 and del dot D that now becomes 0. So, I can say del dot epsilon E F that also becomes 0. So,. So, electric field you see that the divergence of the electric field due to the magnetic current that is 0. So, definitely now I can express E F as minus 1 by epsilon del cross F.

So, I introduce the vector potential. So, this is the electric vector potential. F is electric vector potential because electric field is getting defined by this vector function. So, now, I can substitute this E F in the Maxwell's Second Curl equation. I now have an expression for E F in terms of F. So, this is my Second Curl equation.

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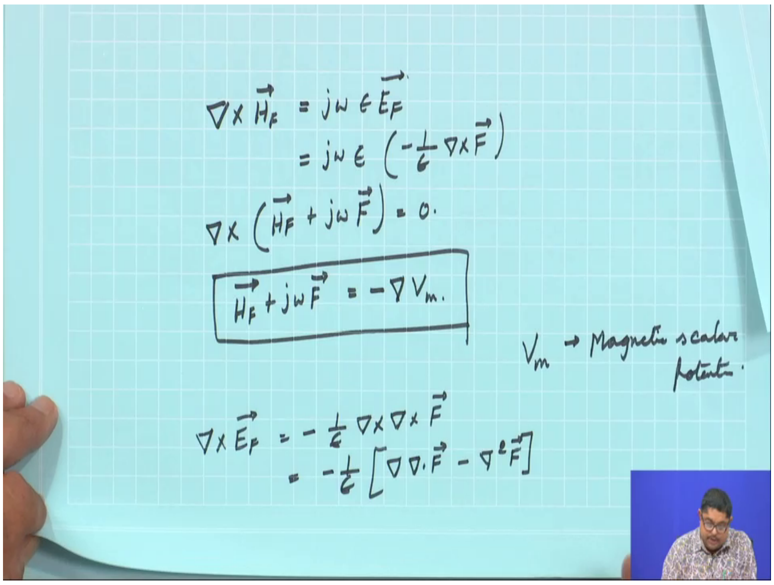

So, here I can put and that gives me del cross H F is j omega epsilon E F.

Now, so this E F value I can write that j omega epsilon, then minus 1 by epsilon del cross F and or I can write del cross H F plus j omega F that becomes 0. So, I can write H F plus j omega F that since curl of this is 0, I can say that gradient of these is a scalar function Vm. Now, this minus as you see here I am choosing deliberately because actually in my curl equation you will see that M is negative. So, to adjust with that we are taking both these electric vector potential and this magnetic scalar potential, we are choosing by definition negative.

So, this is another defining equation that another scalar function potential function I am saying, this is magnetic scalar potential. Now with this, if I take curl of the this equation E F is equal to this equation if I take the curl, this defining equation. So, then I get del cross E F is equal to minus 1 by epsilon del cross del cross F and that gives me minus 1 by epsilon del vector identity. Also from Maxwell's first curl equation; I have the value for this del cross E F, curl of E F. From here I have another.

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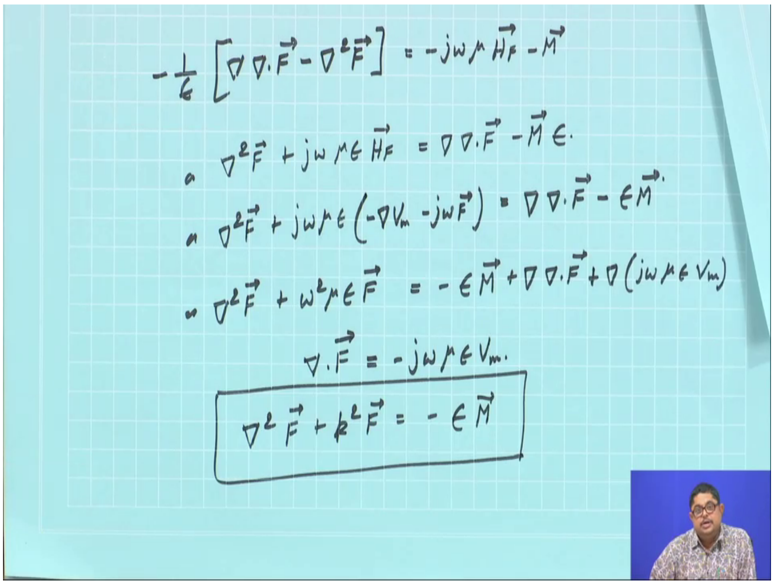

So, I can equate these two and that gives me minus 1 by epsilon del del cross F minus del square F is equal to minus j omega mu H F minus M or from here I can write, del square F plus j omega mu epsilon H F is equal to del del dot F minus M into epsilon. So, putting now the defining equation of the magnetic scalar potential Vm in place of H F, I can write del square F plus j omega mu epsilon.

Here, I am putting that definition of magnetic potential. So, from here I can write what is the value of H F will be this minus this. So, minus or now, we have not defined the divergence of F. We have taken defined already the divergence of curl. So, we have 1 a left. So, that is the gauss condition. Now, we put that del dot F is equal to minus j omega mu epsilon Vm, if we do that; then, these term cancels and we get del square F plus k square F is equal to minus this. This is the in homogeneous vector Helmholtz equation in F.

So, this needs to be solved, but you know this, already we have solved this equation earlier. So, we can similarly find this. So, solution is obtained here. So, once I solve this equation; that means, F is now known.

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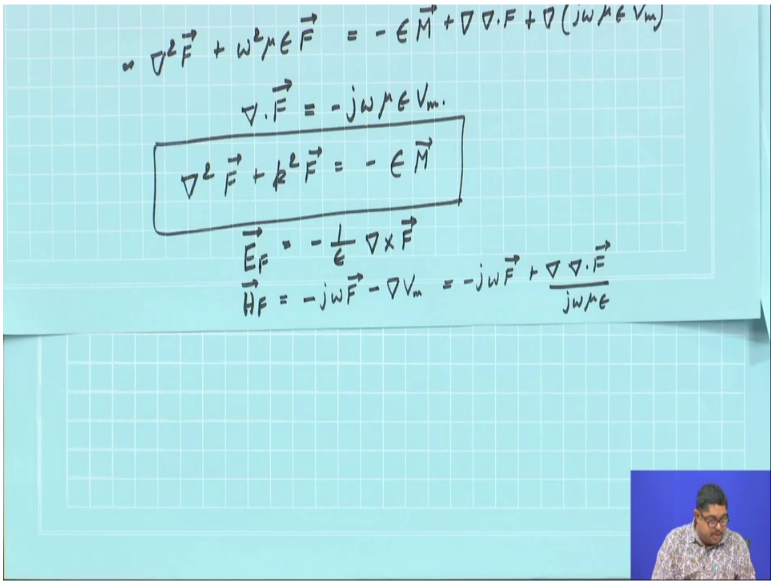

So, I can easily write that E F will be minus del cross F because F is known. So, E F will be known and H F, I can find out is equal to minus j omega F minus del Vm is equal to minus j omega F plus del del cross F by j omega mu epsilon.

Usually I do this; that means, H F in terms of j, but once E F is known with the help of Maxwell's equation also you can take the curl of this and that will give you H F, either of that people do depending on complexity of the solution F at hand. So, this says that potential for F is known. Now, what we will do that we recall that, we have solved this equation for A. But that. Now, for electric sources J, we have seen that the in homogeneous vector homogeneous you know in homogeneous vector Helmholtz equation in terms of magnetic vector potential, we have seen now.

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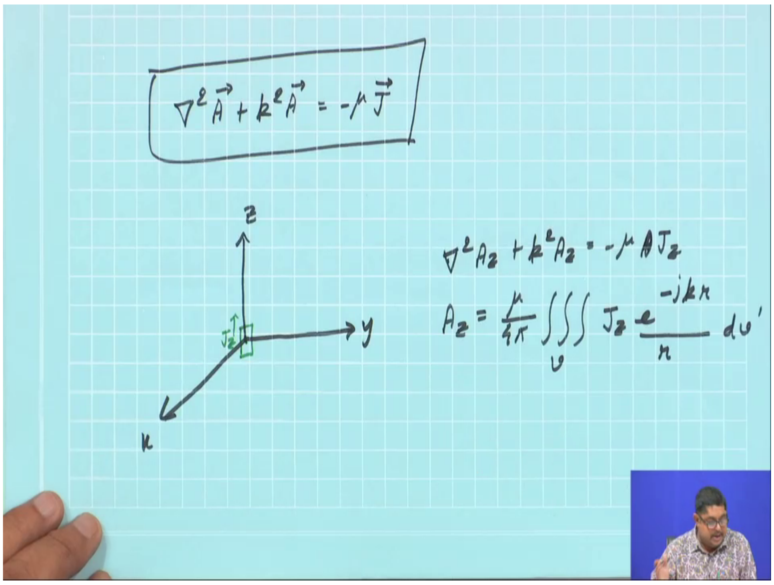

And this equation already we have seen this equation earlier in terms of magnetic vector potential that del square A plus k square A is equal to minus mu J. This was the Helmholtz equation in homogeneous vector Helmholtz equation earlier.

This equation we have solved, twice. I tell you once for current element when we saw we have solved it, the in homogenous part; that means, the singularity also we extracted; that means, with that source we did. Another time for a long dipole also we have solved it, but both those solutions on both the things we simplified the vector Helmholtz equation to a scalar Helmholtz equation because in both those cases the current we was only Z directed. So, it was a one dimensional source that is why we got away without solving the vector Helmholtz equation, we actually solved the scalar Helmholtz equation.

But now, if current is three dimensional and arbitrarily; that means, Z is 3 dimensional, how to solve that that we will see now, that is a small part. So, to do that again, we just recall the diagram that this was our diagram for solving the things y x z in Cartesian coordinate because source generally we solve in Cartesian coordinates and let us say that I have a current density Jz in this direction at the origin of the coordinate system. So, since Jz has only z component. So, we can easily say that A will also be z directed and vector Helmholtz equation turns to be that del square Az plus k square Az is equal to minus mu sorry, minus mu Jz.

So, this equation is a scalar equation and we saw that what is it solution. You see, differential equation of same type always gives same solution. So, we saw that Az gives mu by 4 pi Jz e to the power minus jkr by r dv dashed ok. The source is existing over this. So, this we saw as the solution. Now we say that instead of z component suppose J also has a x component; that means, it has a Jx and also a Jy. So, then, one by one we will consider that.

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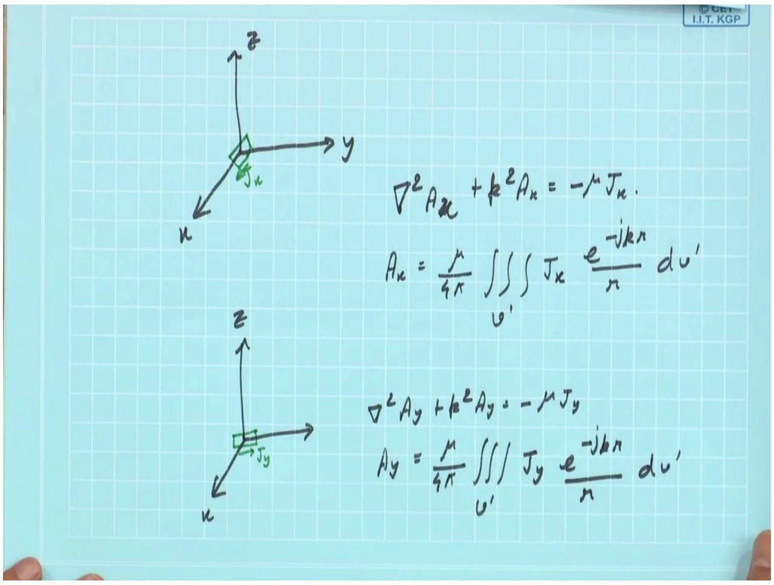

So, suppose I have a Jx; that means, and I have a Jx here. So, then, we can write that the scalar equation will be del square A x plus k square A x is equal to minus mu J x and its solution Ax will be mu by 4 pi and also if we have A y directed current, current density; then, our equation will be del square Ay plus k square Ay is equal to minus mu J y and Ay will be mu by 4 pi the v dash J y e to the power minus jkr by r, then dv dash ok.

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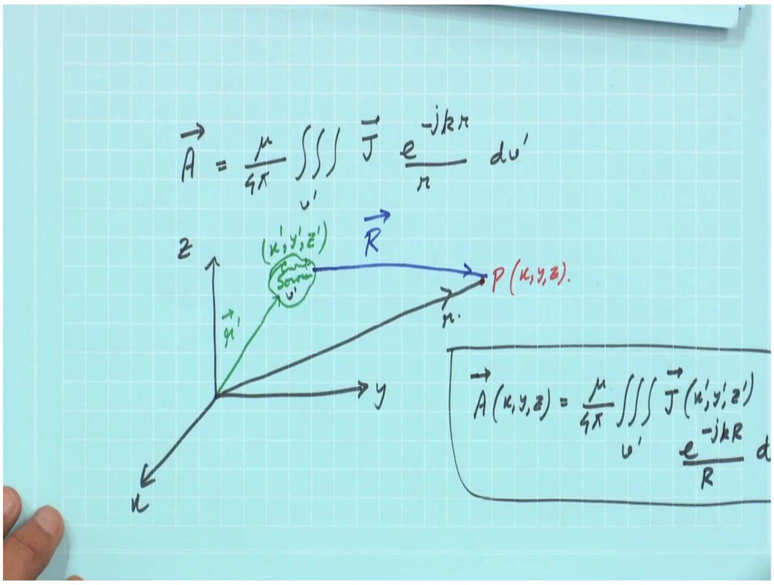

So, if all these components are present; I can easily write that A will be mu by 4 pi J e to the power minus jkr by r dv dash. So, this is a solution of the vector Helmholtz equation. Now, if the source is removed from the origin because this is not the general thing. Source will be in general in a different points. So, if the source is removed from the origin and placed at a position represented by prime coordinates that means, if I has this diagram that this is my coordinate system xyz.

But my source is somewhere here and this coordinate is x dashed y dashed z dashed. This is my source, current source. I will should say current source. Then and let us say that I am observing at a point P, whose coordinate is x y z. So, I can write that this is my r vector. This will be my r dashed vector. So, my source to the observation point vector, let me call capital R. So then, I can say that a which is a function of xyz that will be mu by 4 pi. This is mu v dash J x dashed y dashed z dashed e to the power minus jkr by R db dashed ok.

In a similar fashion, so let me say this is A.

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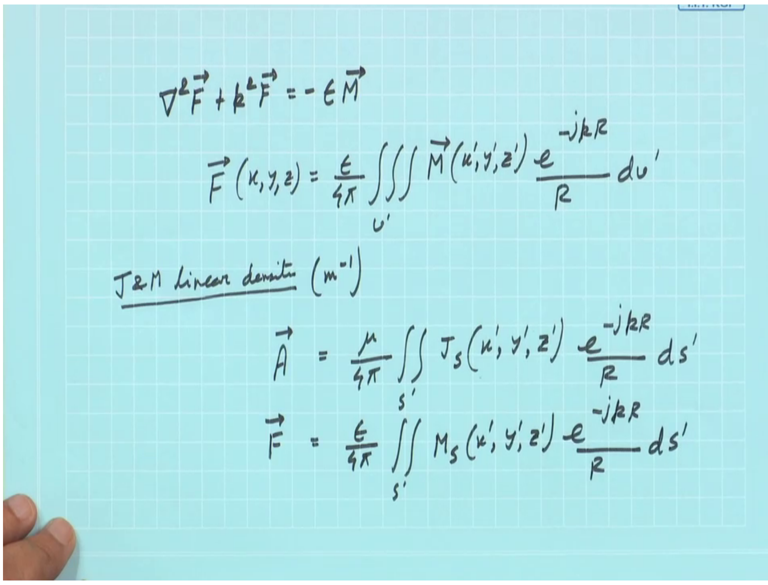

In a similar fashion, we can show that the solution of this vector Helmholtz equation in terms of electric vector potential F. So, that will be F xyz is equal to epsilon by 4 pi M x dashed y dashed z dashed e to the power minus jkr by R dv dashed. Now, if J and M represent linear densities because these J and M we assume current density. They are ampere per meter square; but if J and M represent. So, let me say will J and M linear densities per meter quantities.

Then, this volume integrals will reduce to surface integrals. So, that time my A will be mu by 4 pi J s x dashed y dashed z dashed e to the power minus jkr by R ds dash and F will be epsilon by 4 pi Ms x dashed y dashed z dashed e to the power minus jkr by R ds dashed.

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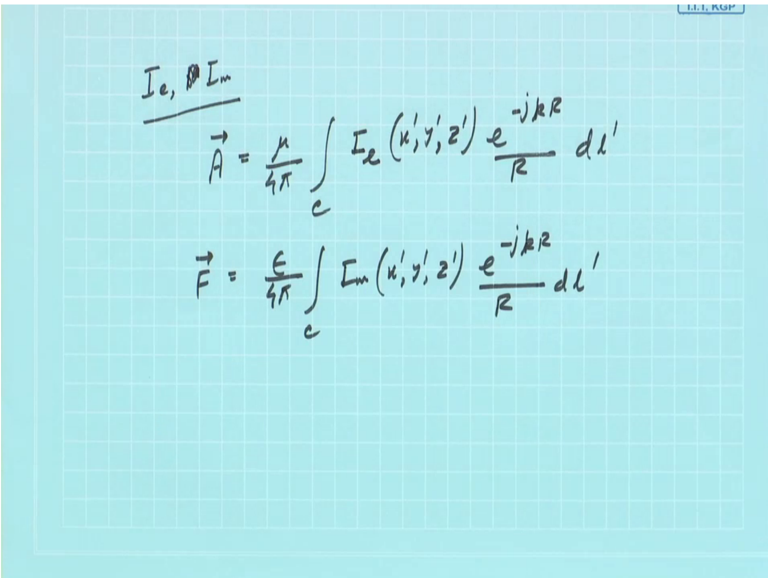

Now, if we have electric and magnetic currents I e and I m, then the integrals will become line integrals. So, A is equal to mu by 4 pi I e x dashed y dashed z dashed e to the power minus jkr by R dl dashed and F is equal to epsilon by 4 pi ok.

So, we have seen these potentials their solutions. So now, we will see that what is the solution of electromagnetic fields radiated, when both J and M are present because we have seen separately the J is present or M is present.

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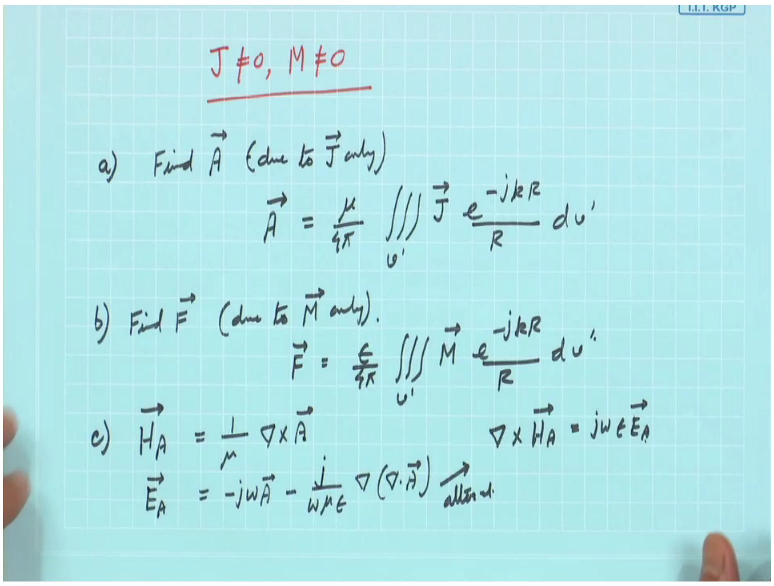

So, when both J is present and M is present. So, the procedure I am not think, so, first procedure is Find A, the magnetic vector potential due to J only that time assume that M is 0. So, you will get either of those 3 expressions.

So, let me take that I have a current density volume say. So, mu by 4 pi that J e to the power minus jkr by R dv dash ok. Then, find F due to M only. So, F will be epsilon by 4 pi v dashed M e to the power minus jkr by R dv dash. Then, from a you find H A. Now you see I am writing H A, it is denoting that for electric current present only the field. So, H A will be 1 by mu del cross A. So, you know a you can find H A, then you find E A you can find A.

So, since you already know a in terms of A, it will be minus j omega A minus 1 minus j by omega mu epsilon del del cross A. Alternately, if this expression is a bit more complex, you can find once you find H A, you can find directly from Maxwell's equation you can put that. This is one alternate thing, alternately. So, already I assume that H A is known from here. So, you can find del cross H A is equal to j omega epsilon E A.

You remember that in this case J is 0 because this I am doing at the observation point there is no J. So, put J in the Maxwell's equation you get this. So, from here you can get E A. So, E A will be del cross H A by j omega epsilon ok. So, that means, this E A, H A, you have found out.

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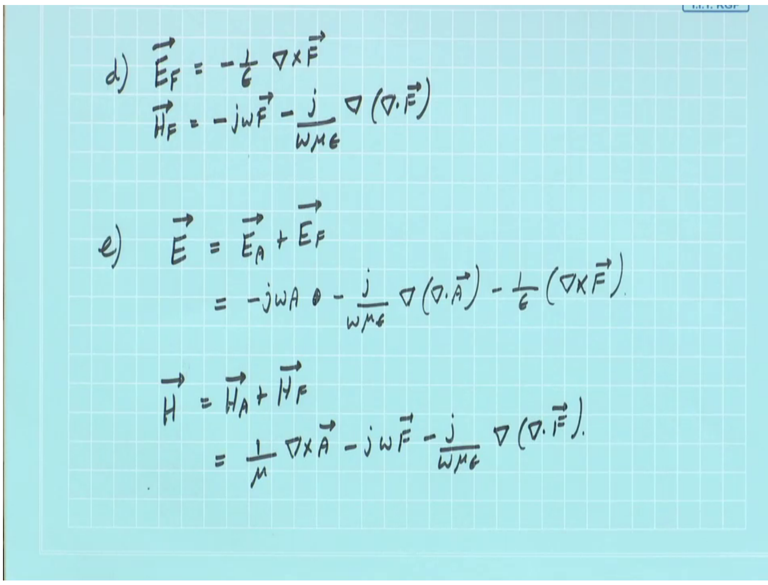

Then, from F find similarly E F. So, E F is minus 1 by epsilon del cross F and H F is minus j omega F minus j by omega mu epsilon del del cross F or you can say alternately, you can find from the Maxwell's equation you take the curl of E F remembering that M is equal to 0 here and you can find.

So, then, you apply Superposition principle. So, total electric field radiated will be E A plus E F. So, that will be minus j omega A minus j by omega mu epsilon del del cross A minus 1 by epsilon del cross F. You can have alternate expression, but I am keeping these because this will be easier for us when we will go to the next step that far field approximation, this gives a better a better I means reduction of complexity. So, this.

So, that is why I am not writing the alternative and also you find H. So, H will be H A plus H F and that you can write as 1 by mu del cross A minus j omega F minus j by omega mu epsilon del del dot F ok. So, you see that now we got that the general solution of the radiated fields we got.

From here actually, we will next in the next lecture, we will make the far field radiation because we are not so interested to know the near field, but if you want to know near field, this is the exact solution from here you can find and if because if you want to compute suppose the near field, what is the reactive impedance etcetera. Then, this equation will help you and, but actually we see that we want to find the far field.

So, in the next class, we will do the Far field approximation and we will simplify this.

Thank you.