Networks, Signals and Systems
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Lecture - 09
Tutorial on Laplace Transform – Application to Circuit Problems

Okay, Good afternoon friends. We will continue with Laplace transform. Today we will have some tutorial session; we will be solving some problems employing Laplace transform.

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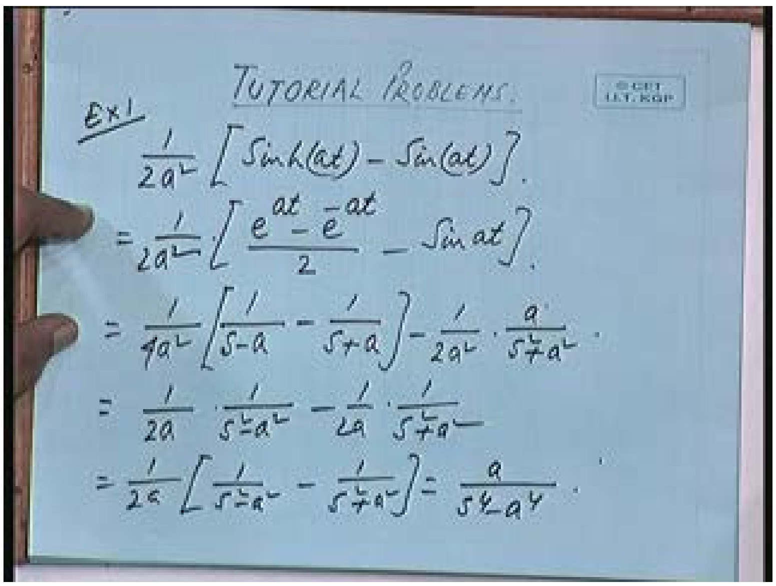

Let us take simple functions what be the Laplace transform of 1 by 2 a squared sin h a t minus sin a t, what be the Laplace transform of this? So this one can be written as 1 by 2 a square what is sin hyperbolic a t minus e to the power minus a t divided by 2 minus sin a t so it is pretty simple 1 by 4 a squared e to the power at will be 1 by s minus a minus 1 by s plus a is that all right? minus 1 by 2 a squared sin a t will give you a divided by s square plus a square is that okay. So if I add these 2 it will be s plus a minus s plus a so twice a so that will be 1 by 2 a in to s squared minus a square minus here 1 a goes so 1 by 2 a, 1 by s square plus a square, so 1 by 2 a I can take out it will become 1 by s square minus a square minus 1 by s square plus a square and that gives me s squared plus s squared minus s square, so twice s squared twice a will go.

So a by s to the power 4 minus a to the power 4 okay, it is very simple. What would be the Laplace transform of t squared e to the power minus a t cos omega t? Okay t squared e to the power minus at cos omega t, now Laplace transform of e to the power minus a t in to e to the power minus a t in to f (t) is how much f (t) is multiplied by e to the power minus s t for integration along with that you are having e to the power minus a t, so it will be s plus a, so f (s) plus a. So if you take only this part what will be the Laplace transform of e to the power minus a t cosine omega t, s plus a whole squared so s plus a by very good s plus a whole square plus omega square okay.

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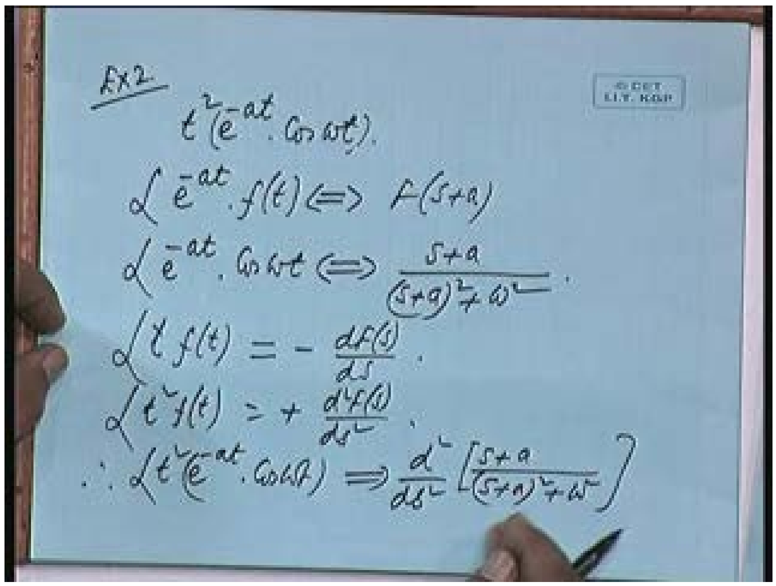

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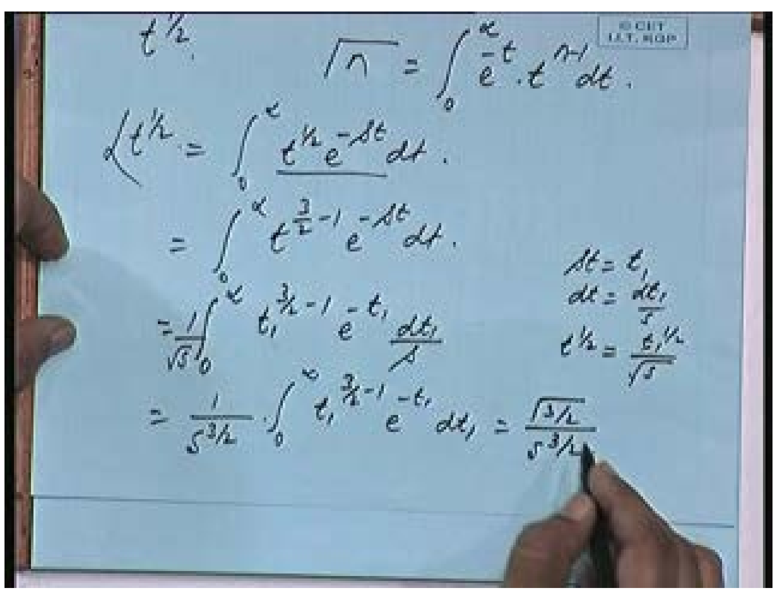

Now in the morning, we discussed t in to f (t) if I take the Laplace transform what was it? what was t in to f (t) minus f (s) by ds is it not? therefore t square f (t), so it will be plus plus d square f

(s) by ds squared, okay. So t square in to this will be second derivative of this therefore Laplace transform of t square e to the power minus a t cosine omega t will be d square of this function s plus a by s plus a whole squared plus omega square, you can do the derivation okay second derivative of this you can extract, so that will be the Laplace transform. Next t to the power half this is very different kind of function you have if you have already been expose to gamma function, gamma n is 0 to infinity e to the power minus t, t to the power n minus 1 dt okay. So t to the power half t to the power half Laplace transform of this will be 0 to infinity t to the power half, t to the power minus s t dt and t to the power half can be written as t to the power 3 by 2 minus 1 I will write in this form t to the power n minus 1 e to the power minus s t dt okay so I can take s t as a variable s t as some t 1 so dt will be t_1 by s okay so t to the power half will be t 1 to the power half by root s so this can be written as 0 to infinity t 1 to the power 3 by 2 minus 1 1 by root s, okay t to the power half can be written as 1 by root s in to t t_1 to the power half.

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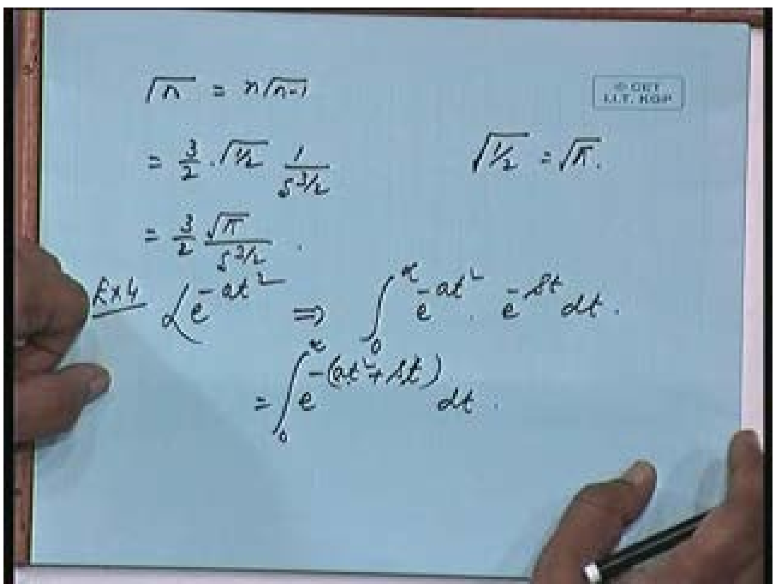

So dt will be t_1 dt 1 by s so e to the power minus t 1 dt 1 by s, so I get s in to root s s to the power 3 by 2 in to 0 to infinity, what is this gamma 3 by 2 and gamma 3 by 2 is gamma n will be n in to gamma n minus 1 okay. Therefore this one will be 3 by 2 in to gamma half divided by s to the power 3 by 2 okay. So 3 by 2 gamma half is root phi this is a standard result we are not going to the proof of it gamma half is root phi, so 3 by 2 phi by s to the power 3 by 2 okay.

Next what would be the Laplace transform of e to the power minus a t squared you may go in to the okay start from the beginning from the fundamentals 0 to infinity, e to the power minus a t squared this is one approach then this gives me e to the power minus a t square plus s s t in to dt you may adjust some terms to get some t plus alpha whole squared okay, in that form and then take out something in terms of s and then integrate okay. This is one approach there are standard results like e to the power minus x square dx what will be the integration of this is a standard one.

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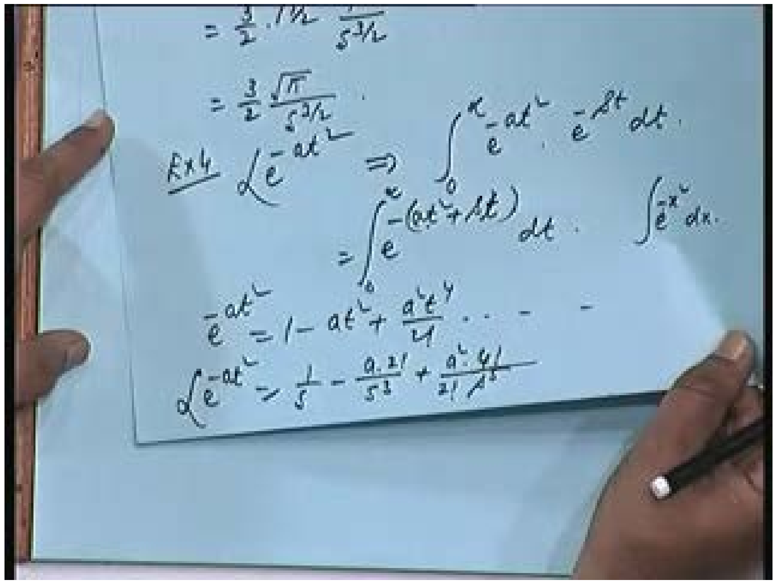

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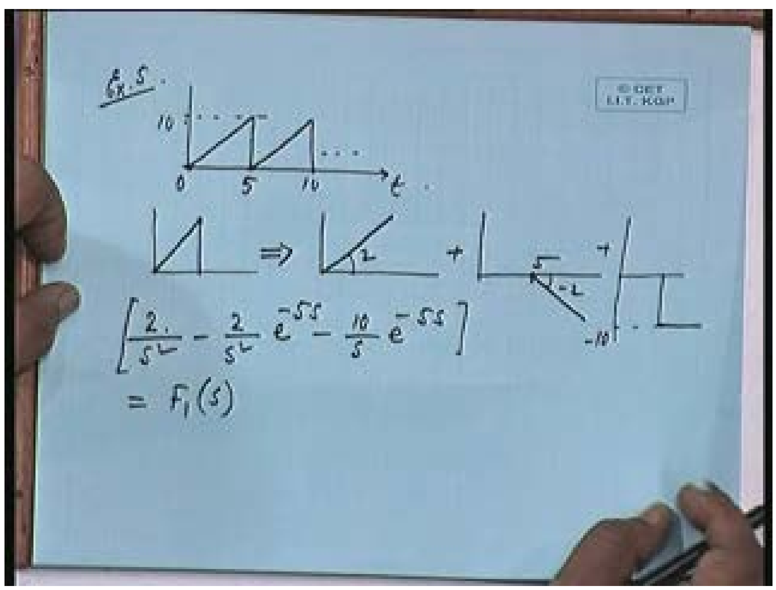

So if you can bring it in this form you can evaluate these the other alternative is a very simple straight forward approach. Take e to the power minus a t square as a series 1 minus a t square plus a square t to the power 4 by factorial 2 and so on and then take the Laplace transform of individual terms that is one will give me 1 by s minus a t square will give me a by s cube s cubed that is all factorial 2. So I will write 2 plus a square t_4 by factorial 2 t_4 what will be this factorial factorial 4 by s to the power 5 and so on. Okay you can write the series.

Now you are given some functions like this a periodic function say 0, 5, 10 and so on say this is 10 it is continuing, what will be the Laplace transform of this? So for this what you do you take only the first element first period and determine it is Laplace transform, what is the Laplace transform of this? you bring it up in to standard functions this is a ramp function plus what do I do here? if I apply a negative ramp will that be all right should I add anything more, if I add a negative ramp of the same slope here the slope is 10 by 5, 2, is it not? slope is 2 here it should be minus 2 if I start subtracting then it will remain there. So I will have to also subtract another stay function of what magnitude 10 is that all right minus 10.

So what would be the Laplace transform corresponding to the first element 2 by s square, second one 2 by s square shifted by 5 seconds, so e to the power minus 5 s. Next minus 10 by s, e to the power minus 5 s, so this is the Laplace transform of the first element $F_1(s)$ if it is coming repeatedly then f (s) will be $F_1(s)$ in to 1 plus e to the power minus 5 s plus e to the power minus 10 s and so on, is it not? so that will be a GP series $F_1(s)$divided by 1 minus e to the power minus 5 s okay 1 minus the common ratio and $F_1(s)$is this much, substitute that okay. Let us have another example, what would be the Laplace transform of this function? This is a sinusoidal function but it is truncated at 90 degree and it continues 0, 4, 8, 12 and so on this is a maximum value this is 10.

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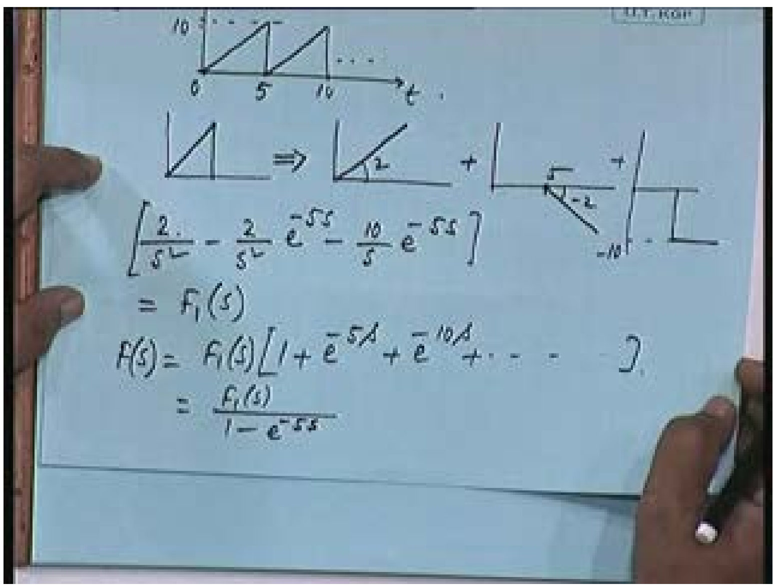

So this is the first part if I write it this is a sine wave given this is a sine wave and this is a maximum value 10 sine omega t. So you are stopping at a maximum value that means corresponding to 90 degrees. So what would be the breakup of this what are the components that will be adding up to this, 10, it will be 10 sin omega t and what is omega 2 phi by t, what is a period ?16, 16 okay 10 sin 2 phi by 16 t is this one plus at this point you are adding a negative function and what is this function cos, cos so plus this will be 10 will it be plus or minus, minus because it has to be subtracted minus 10 cosine 2 phi by 16, t minus 4 in to u t minus 4 I am not writing that it is implied okay.

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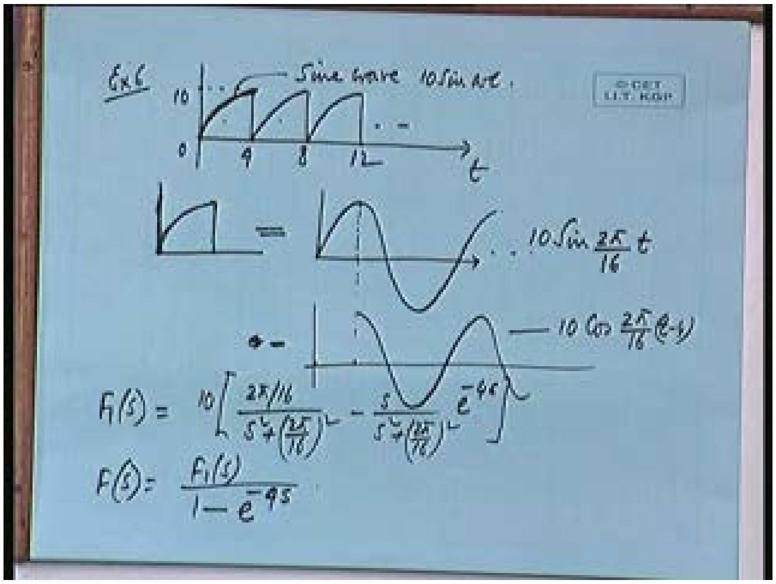

So this plus this will give me this one. So if I write F_1(s)as the Laplace transform of the first element that will be equal to 10 sin 2 phi by 16 t so how much is it 2 phi by 16 divided by s square plus 2 phi by 16 whole square minus cosine of this will be giving me s by s square plus 2 phi by 16 whole square in to e to the power minus 4 s is that all right because it is a shifted function so F_1(s) is being repeated a every 4 seconds. So what will be F (s) in terms of F_1(s)straight away F_1(s) divided by 1 minus 1 minus because it is all plus, so in the GP series sum it would be 1 by 1 minus e to the power minus 4 s in to F_1(s) F_1(s)is this is that all right. So whenever you come across such terms 1 by 1 minus or 1 plus some e to the power minus tau s, it is a periodic function. So if you drop out that term rest of it will give you the Laplace transform of the first period okay. Next a periodic triangular wave so this is 5, 10, 15, 20 and so on this miniature is given twenty so once again. Let us take the first period you need not take 1 period you can take just 1 half, okay that is coming alternative plus and minus all right.

So let me calculate Laplace transform of this function okay this one how do you break up a ramp of slope 20 by 5, 4 is it not? So this is 4 t slope is 4 okay plus what do I do at 5 second at this this time at the fifth second what do I do I add a ramp in the negative direction and what is the slope of that double the slope then only a line that was going up will be brought down to minus 4 t. So this slope is 8 minus 8 okay and shifted by 5 seconds will that surprise this plus this will that give me this after 10 seconds after 10 seconds what do I add another ramp of what slope, if I add plus 8 what do I get see this plus this net result is 4.

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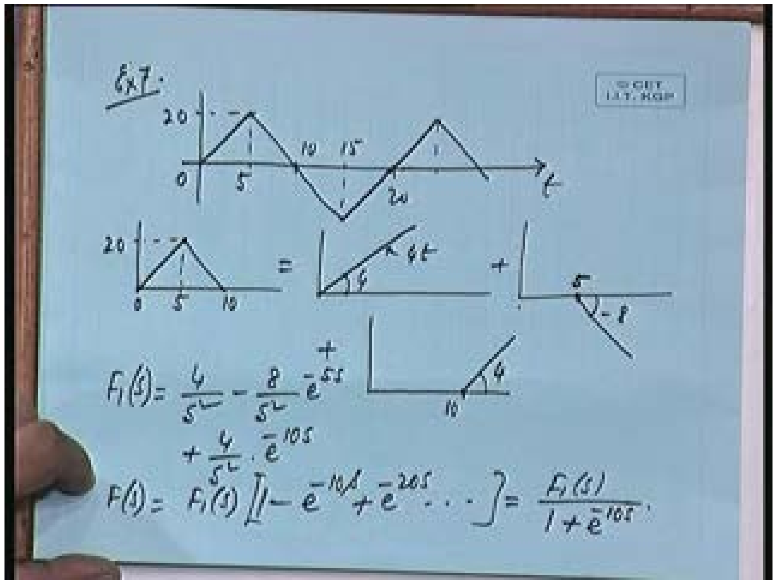

So it would have continued with minus 4, so now to bring it to the horizontal position I have to add only plus 4 So slope is plus 4 all right so what will be the Laplace transform of these 3 comp1nts say F_1 (s) will be 4 by s squared it is a ramp function minus 8 by s square in to e to the power minus 5 s plus 4 by s square in to e to the power minus 10 s, is that all right? Therefore f (s) is F_1 (s) in to this is 1 next it is negative e to the power minus 10 s the whole thing is shifted by 10 s, 10 seconds and given a negative sign so minus 10 s plus e to the power minus 20s and so on so it is a GP series again. so it will be F_1(s) divided by 1 plus plus e to the power minus 10 s is that all right common ratio is now minus e to the power minus 10 s.

So it is this there are a few more problems, okay there is an interesting problem, let us see. You have an exponentially decaying function has the envelop of a pulse train it continues this envelope is e to the power minus t by 2, this is the envelope that is you are having rectangular pulses, periodic pulses like this 0. So this is 0, 1, 2, 3, 4, 5 and so on this has been multiplied by e to the power minus t by 2 an exponential function to give you this product okay. This and this multiplied together has given me this exponentially decaying pulse what will be the Laplace transform of this function f (t), f (t) is basically f_1 (t) in to f_2 (t) what will be the Laplace transform of this F_1(s)? what is F_1(s) it is 1 pulse 1 by s in to if I consider only the first pulse will be 1 by s in to 1 minus e to the power minus s okay? Then that gets repeated every 2 seconds, so it will be the geometric s a series the sum will be e to the power minus 2 s 2 s.

So that gives me 1 by s in to 1 by 1 plus e to the power point minus s this can be written as e to the power minus 1 minus e to the power minus s in to 1 plus e to the power minus s. So that will be simplified in this, so what would be f (s) if a function is multiplied by an exponential function e to the power minus alpha t, where alpha is .5 now e to the power minus alpha t alpha is half, .5 then s will be replaced by s plus half. So it will be 1 by s plus point 5 in to 1 plus e to the power minus s plus .5, is that okay? is this all right? Another example, can you tell me what would be the Laplace transform of this function? 2, 2, this is t, what would be the Laplace transform of this? How do I break it up first of all, I can since it is starting from 2 I can write like this plus ramp like this that means this is 2 by 2, the slope is 1 all right.

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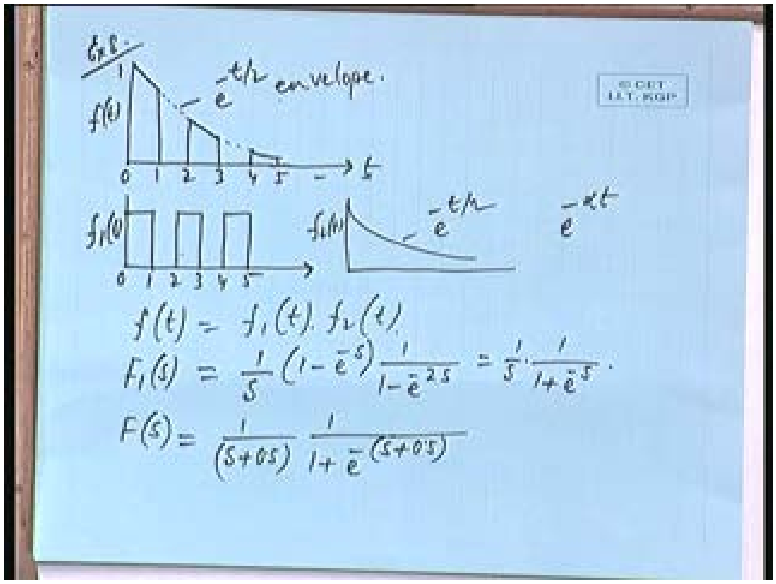

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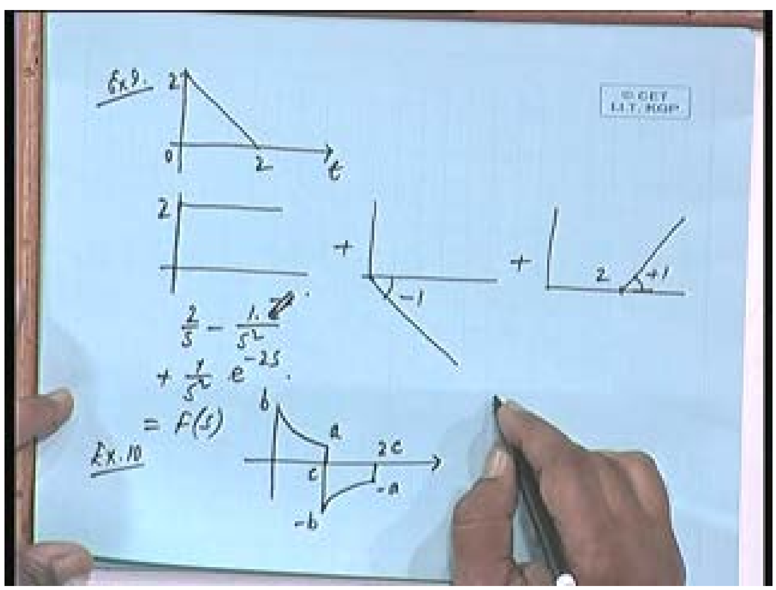

So this is minus 1, so from a step function if I keep on subtracting a ramp of this slope I will start getting this it would have continued so at this point I should add at 2 second this 1 step at t equal to 2 second this will start okay with the same slope plus 1 so what would be the Laplace transform of this be 2 by s minus 1 by s square e to the power minus 2 s, is it nothing? it is starting from 0 what about this 1 plus 1 by s square e to the power minus 2 s. So that will be the Laplace transform of the given function okay. Another function we take exponentially decaying function then again this is b, this is a, this is c, this is 2 c, it is going to minus b again then minus a and so on it is keeps on repeating. Let us take this itself, if it is repeated there is no problem it is just a GP series we get, let us see what will be the Laplace transform of this.

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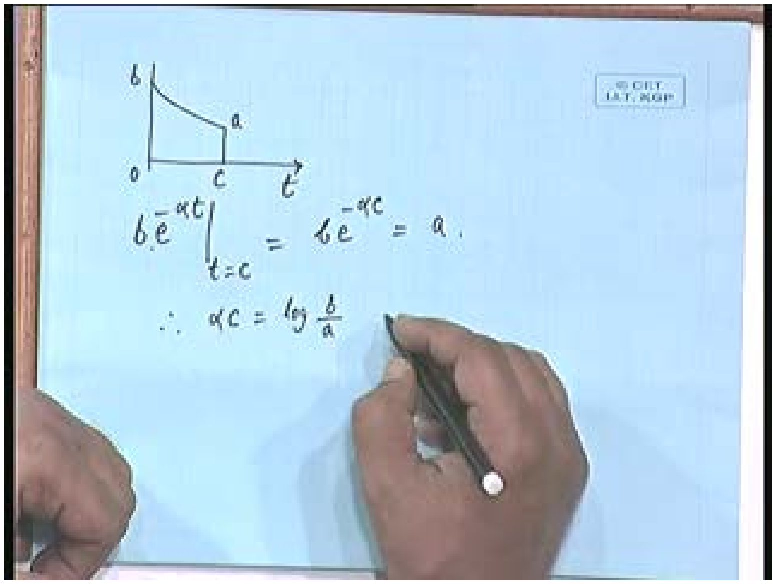

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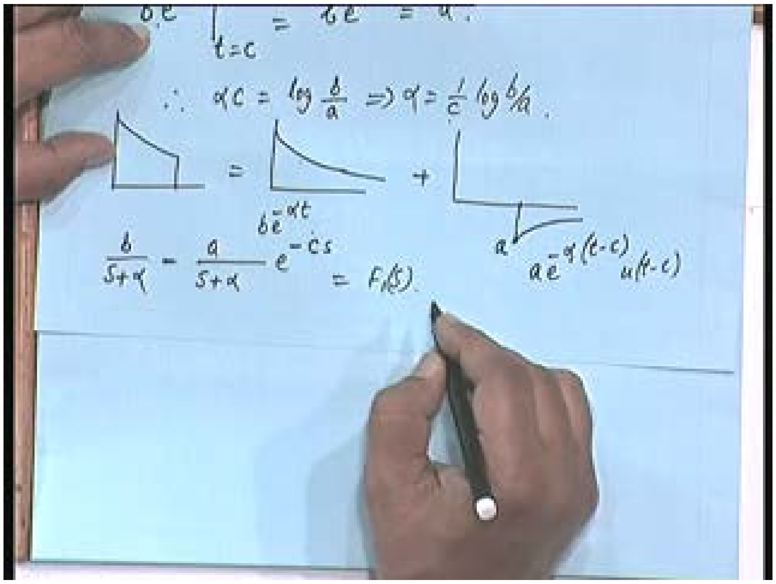

So let us find out for the first block, what is the Laplace transform of this? it is starting from b coming up to a in time c okay. So how do I write like this, write the equation for this e to the power from minus alpha t is this at t equal to 0 whatever be this at t equal to 0 it is always 1 so it must be b times e to the power minus alpha t if it starts from b it has to be b to the power b in to e to the power minus alpha t. Okay and at t equal to c what is the value its value is a that is b in to e to the power minus alpha c is equal to a, so how much is alpha. Therefore, alpha c is log negative so it will be b by a log of b by a will be it give the alpha c, so alpha c is this.

So alpha equal to 1 by c log b by a, so if b, a, c are given you can calculate alpha. So the function is b in to e to the power minus alpha t sorry, alpha c is log b by a. So alpha is just bring the c down 1 by c, so what would be this function that is we have calculated alpha there is the expression for this we have not yet broken up in to different components so this function can be written as this going up to infinity plus what plus what a function which starts from a starts with the value a but the same exponentiation that is alpha t all right.

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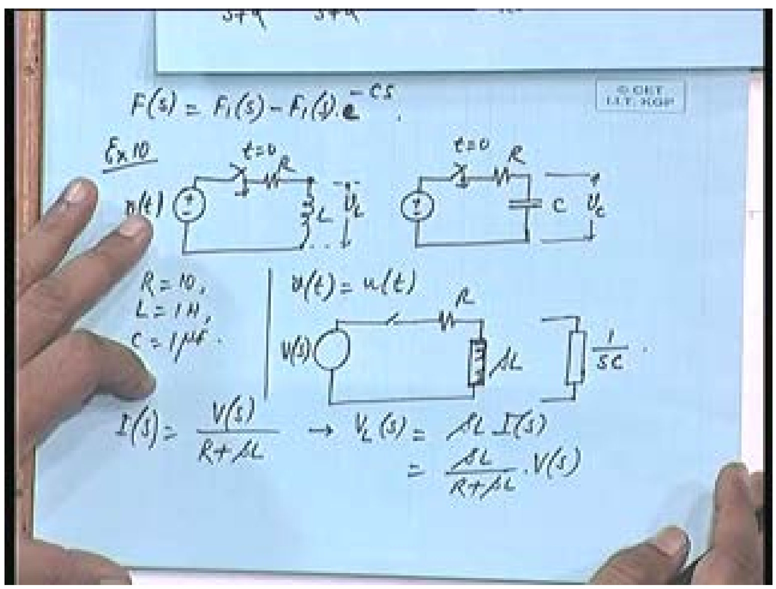

So this one is b, e to the power minus alpha t and this one is a, e to the power what should I write alpha t minus c, u t minus c okay because it is starting from a is the same rate of all. So alpha remains same t is replaced by t minus c okay. So what will be the Laplace transform of this b by s plus alpha and minus a by s plus alpha again but because it is shifted e to the power minus c s, is that all right? This will be the Laplace transform of this I will call it F_1 (s), so f (s) is if it is if the given function is only for these 2 parts and after that it remains 0 then what will be f (s) it will be F_1(s)minus this is evaluated as F_1(s).

So this 1 is minus F_1(s) F_1(s) e to the power minus c s, F_1(s) e to the power minus c s okay, what about 2 c after that it does not come nothing is there beyond 2 c. So do you feel 2 c s some e to the power minus 2 c s term should appear somewhere in this, is it not missing? it will come out of F_1 (s) F_1 (s) has embedded in it sorry where is that F_1(s) is here there is 1 e to the power minus c s. So when it gets multiplied by another e to the power minus c s so it will be e to the power minus 2 c s. So that means after 2 seconds there is some termination so you can substitute the value of F_1(s)from here and then see the result okay.

Now there is an interesting circuit problem you are ask to calculate the current in these 2 circuits calculate the current in this circuit as well as current in this circuit before the switching operation both these elements were neutral they were there is no energy stored. okay this is R, this is C, this is R, this is L. Now R is given as 10 ohms L is 1 Henry, C is 1 micro farad okay what would be the current also you are ask to calculate the voltage across the inductor and the capacitor. Now you know you can replace R L and C by you are applying here step voltage. First you are applying a step voltage and then you are applying a different types of voltages that you have discussed earlier okay that is some of the voltages that will be applying are say the triangular wave, rectangular pulse and so on. So let us start with a simple voltage step voltage v (t) is u (t).

Now in that transform domain I can write this as v (s) this will be R and this will be sL and in the other case I need not show the inductor any more the moment I put s, I might has a put just a block okay and in the other case it is we discussed the other day 1 by s c, is it not? For simple circuits so what will be the current in the first case it is v (s) divided by R plus sL correspondingly what will be VL, Laplace transform of this VL small here it will be current multiplied by this impedance sL. So sL in to I (s) that means sL by R plus sL in to v (s). Okay in the second case it will be V_c (s). So similarly, V_c (s) will be when the capacitance is there so that current multiplied by the impedance, so v (s) by R plus 1 by c s is the current multiplied by the impedance 1 by s c okay. So that will be v (s) if I multiply with this r c s plus 1 okay. So let us take any of the functions say I may take this one suppose, this is the applied voltage what would be the current applied voltage is this one okay.

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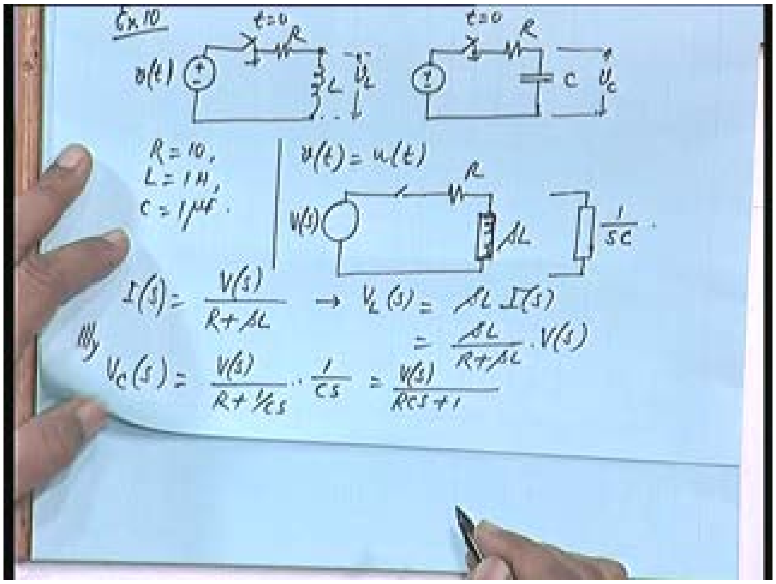

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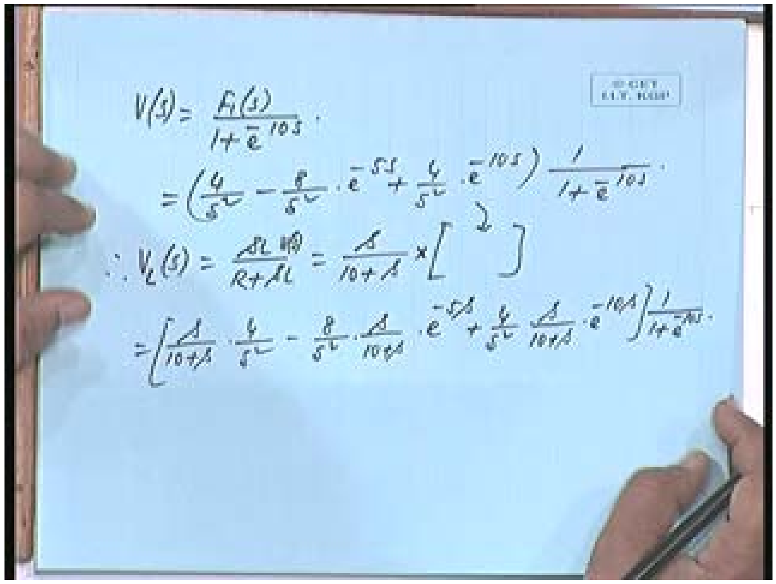

So v (s) is if you remember f (s) was derived as F_1 (s) by this so F_1(s) is known where F_1(s) is 4 by s square minus 8 by s square in to e to the power minus 5 s plus 4 by s square in to e to the power minus 10 s divided by 1 plus e to the power minus 10 s, so this is v (s). So let us take the simple value R and L 10 Henry 10 ohms and 1 Henry so this 1 is s l by R plus s l this is the voltage across the inductance in to v (s) okay, so sL therefore V across inductor will be sL by R plus s l which means L was 1 Henry in to v (s) and v (s) is this much in to this whole quantity term by term you write s by 10 plus s in to 4 by s square minus 8 by s square in to s by 10 plus s in to e to the power minus 5 s plus 4 by s square s by 10 plus s in to e to the power minus 10 s okay multiplied by this some of GP series okay.

So whatever is this final result in this bracketed quantity, whatever expression you have got, whatever is the time domain function corresponding to this that gets repeated with alternate plus and minus signs, so that can be the that will come out of this. So that can be kept aside for the time being will take the Laplace inverse of this and then we will repeat that in that same fashion. So it will be what will be the Laplace inverse of the first term s will go. Okay let me write in the simplified form 4 by s in to s plus 10 okay minus 8 by s in to s plus 10 e to the power minus 5 s plus 4 by s in to s plus 10 e to the power minus 10 s this can be broken up in a partial fraction form how much is this 1 by s minus 1 by s plus 10 gives me 10 in the denominator all right.

So I can always write this as 4 by 10 in to this minus 8 by 10 in to 1 by s minus 1 by s plus 10 e to the power minus 5 s once again 4 by 10 in to 1 by s minus 1 by s plus 10 in to e to the power minus 10 s. This is corresponding to this bracketed quantity I call it say p (s) and then that will be multiplied by this okay. So what will be corresponding current a corresponding voltage corresponding to v p or this P part. So if I call it V_p (t) which is only arising out of this function it is basically corresponding to 1 period if you remember 1 period gives me this so what will be V_p (t) 4 by 10 in to 1 by s gives me u t minus 1 by s plus 10 gives me e to the power minus 10 t whole thing in to 4 by 10 minus 8 by 10, I am not writing it it is multiplied by e to the power minus 5 s.

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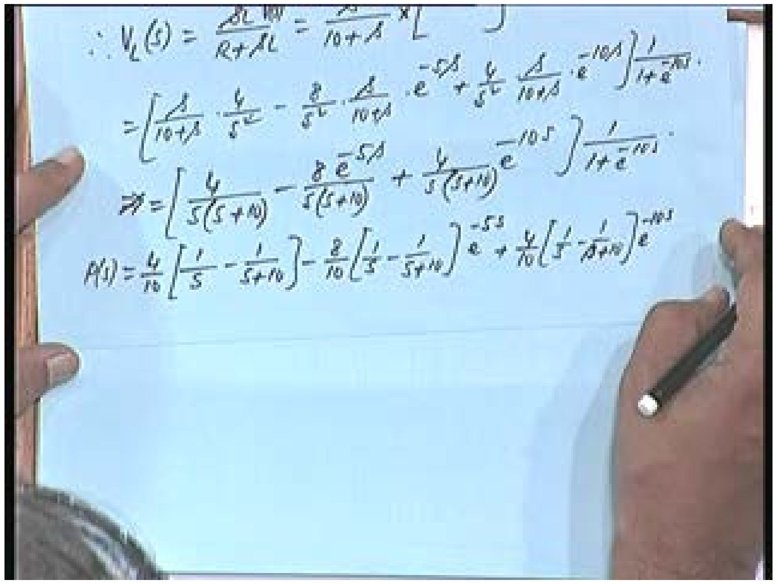

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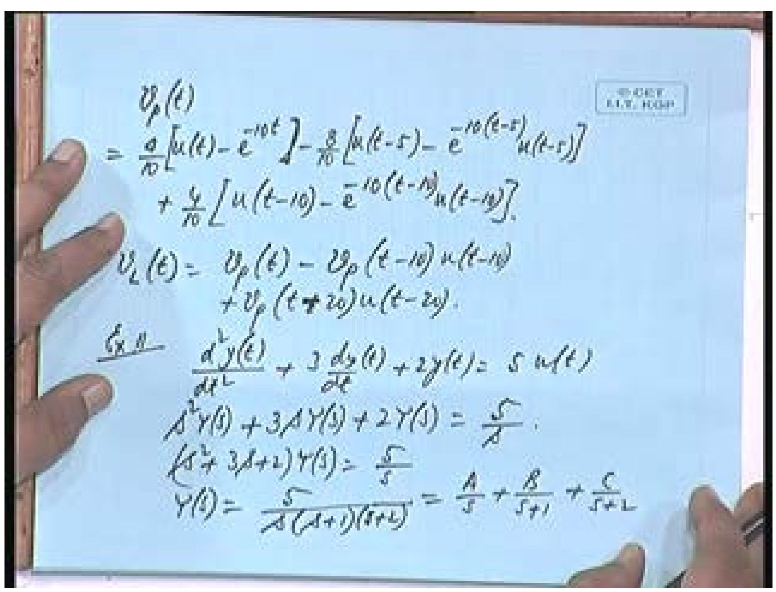

So whatever I get I will be delaying it by 5, so it will be u t minus 5 minus e to the power minus 10 t minus 5 u t minus 5 okay. The second part similarly, the third part will be 4 by 10 u t minus 10 minus e to the power minus 10 t minus 10 u t minus 10 okay. So this is whatever it is this is the first period and 1 by 1 plus e to the power minus 10 s means it is common ratio is minus so alternatively it will be plus minus plus minus okay. So it will be if V_p (t) means this therefore actual v_l (t) will be V_p (t) minus V_p (t) minus 10 u t minus 10 plus V_p (t) plus 20 sorry t minus 20 minus 20 u t minus 20 and so on alternatively plus and minus is that okay. I hope you can solve it for the capacitive element also with different values. We shall be taking up some differential equations simple differential equation how to write in the Laplace domain d square y by dt squared plus 3 times dy by dt plus 2 times y (t) is equal to 5 times u (t) is the forcing function and this is the differential equation you are ask to calculate you are ask to calculate y (t) by using Laplace transform okay. It is a second order differential equation, so second order term the second derivative you can write in terms of f (s) so y (s) so this will give you s squared y (s) if you initially assume all the initial conditions to be 0 will take up initial conditions later on for the time being, let us take this to be 0 then this will be 3 times s in to y (s) plus 2 times y (s) is equal to 5 by s.

So it is s squared plus 3 s plus 2 in to y (s) which is 5 by s. So how much is y (s) 5 divided by s in to s square plus 3 s plus 2 can be written as s plus 1 in to s plus 2 okay. I can write it as A by s plus B by s plus 1 plus C by s plus 2 and taking the inverse of these 3 I will get the final y (t) as A in to u (t) plus B in to u (t) the power minus t plus C in to u (t) the power minus 2 t where A, B, C residues can be calculated from here by standard partial fraction technique okay. So will stop here for today will take it up in the next class with further examples using Laplace transform for networks problems. Thank you very much.

Preview of next Lecture

Lecture - 10

Frequency Response-Bode Plot

Good morning friends, before we start our next topic I would like to correct a small tutorial problem that we did last time if you remember there was a function t to the power half for which we are computing the Laplace transform. So gamma n is this, now gamma n is basically n minus 1 gamma n minus 1. So here it should be gamma 3 by 2 which is half there was a slip here it should be n minus 1 gamma n minus 1. So it should be half, gamma half.

So please make this correction I am extremely sorry it will be half root phi s to the power 3 by 2 okay. Today we shall be starting frequency response of systems, frequency response of various types of networks, what you mean by frequency response? When you excite a system by say we have a sinusoidal input v_m sin omega t applied to a network whose impedance function is z (s) all right in the Laplace domain. In the time domain we are applying a function v_m sin omega t so in the Laplace domain this will appear as v_m in to omega by s squared plus omega square. So this is our input function z s is the impedance function what will be the output now in this we may treat the current as the output okay. So I (s) will get as v (s) by z (s), you can write y (s) in to z (s), so for any system we define y (s) in to v (s), thank you y (s) in to v (s).

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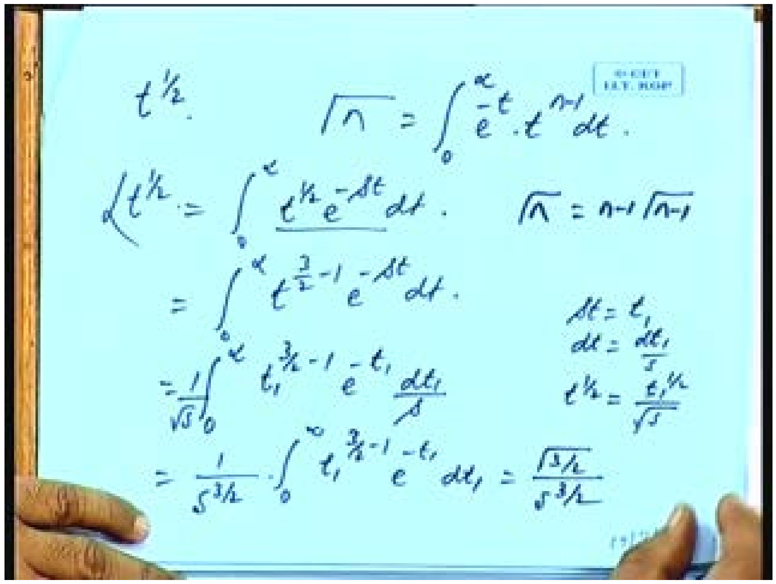

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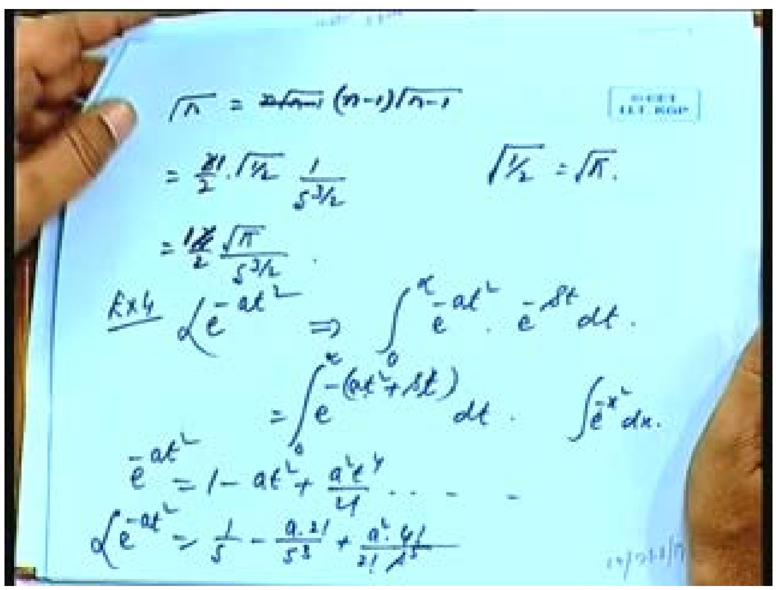

So if v (s) is the input and I (s) is the output 1 by z (s) is the multiplier, we call it a transfer function okay. So input multiplied by a transfer function is the output in this particular case it happens to be 1 by z (s) the transfer function depends on how you define the input and output for a 2 port network we may define output as V_2 and input as V_1, so V_2 (s) by V_1 (s) will be the transfer function somebody may be interested in the current that is flowing under open circuit condition or may be under short circuit condition. So it is not necessary that it should be always V_2 by V_1 the transfer function depends on how you define the output okay.

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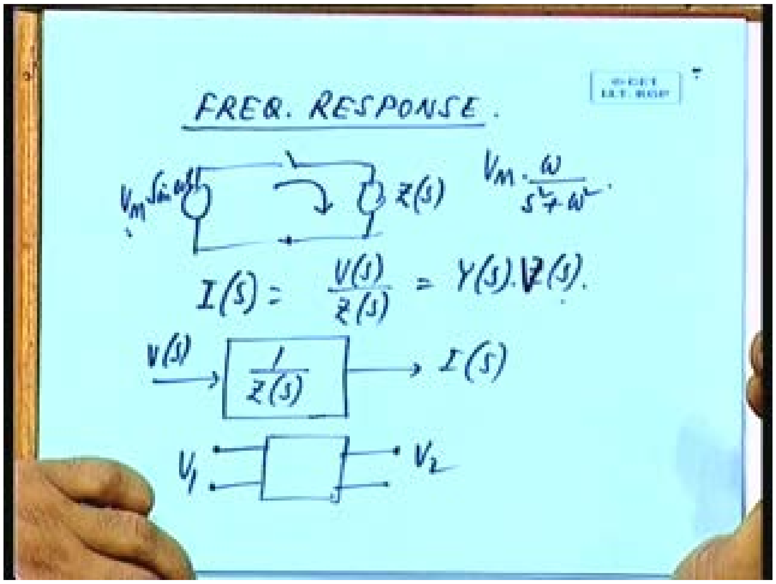

Now will evaluate right from the start what would be the output in the time domain and at steady state, at steady state, what will be the response like? When you see an image or when you here a sound if the energy of the sound or the light is just doubled, if you just doubled the intensity the perceived intensity will be log of 2, in some logarithmic scale that means if I change the brightness if I give the input energy to the light double the previous value then I will not be receiving I will not be feeling that the light has been doubled, the intensity has been doubled, it will be in a logarithmic scale.

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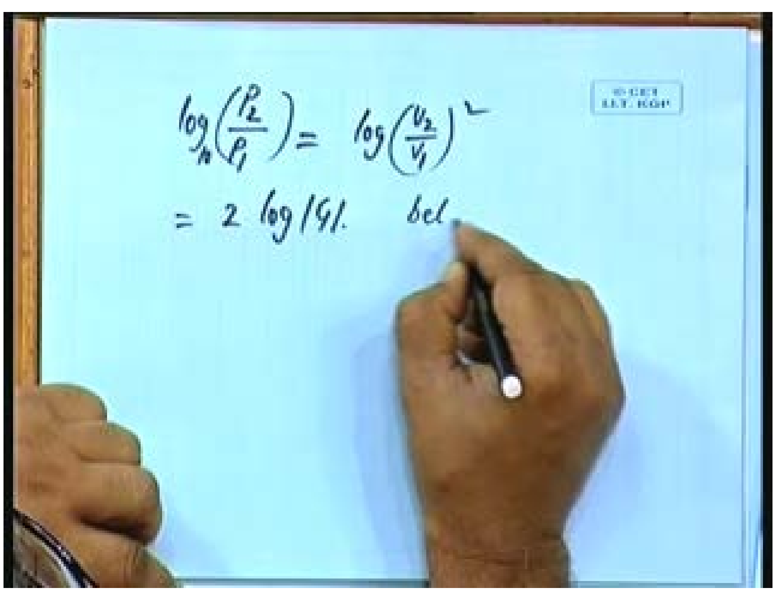

Similarly, when you here a sound if the energy of the sound has been doubled then I will be receiving it as log of 2, the perceived intensity will be varying logarithmically okay. So it is better to define new game function which will be log of the original game function P_2 by P_1 we choose a base 10, base is not so much important we could have selected the natural log also that will be giving just a constant how much is that constant, log of 10 to the base e 2.303 okay. So log of say V_2 by V_1 squared that means basically twice log of original G okay we call this unit as bel the integers when zeta is more than more than say how much okay when zeta is equal to 1 when zeta is equal to 1 then it will be 1 by s squared plus twice omega n s plus omega n squared that means 1 by s plus omega n whole squared there are 2 real roots, 2 equal real roots. So at omega n straight away there will be a 40 reveal per decay fall all right, is that okay? At omega n that means 2 roots are coming.

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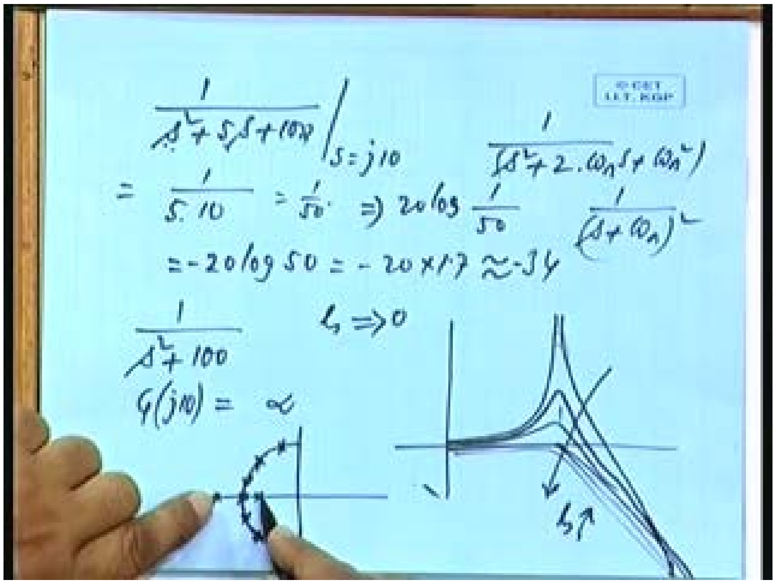

If you remember for different values of zeta if you calculate the roots of this quadratic okay will take it up in the next class, how the roots travel, how the roots are changing for different values of zeta the complex pair of roots will be moving like this and then this is a time when they will become equal if you increase zeta beyond this the roots will be a real but they are separated now 2 distinct roots will be getting okay. So will take it up in the next class and then will see how to make the face plot from the game plot.