Networks, Signals and Systems

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Lecture -36

Fourier series (Contd…)

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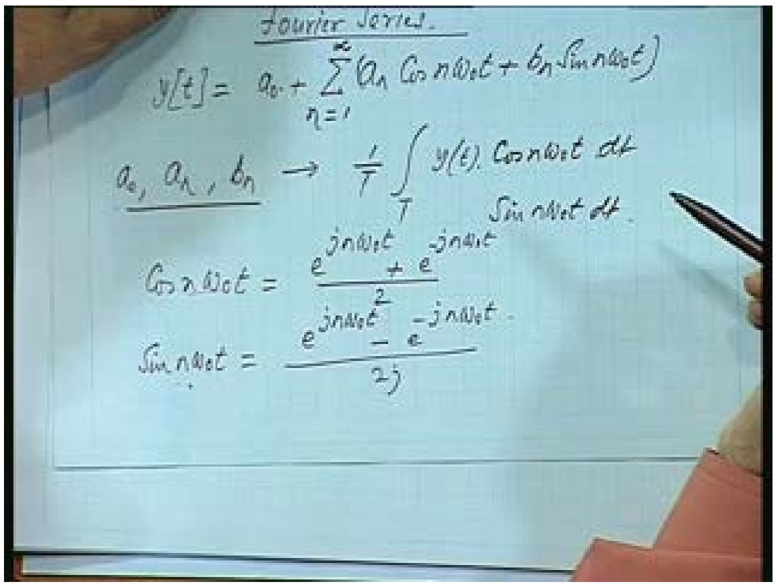

Good afternoon friends, we are discussing last time Fourier series. Fourier series represents basically is a representation of periodic function in terms of cosine and sine functions of different frequencies. The frequencies are multiples of one fundamental frequency, so we wrote y(t) as a_0 plus summation a_n cosine n omega naught t plus b_n sin n omega naught t where n varies from 1 to infinity.

Now this representation gives me the coefficients a_n and b_n including a_0 as 1 by t integration over t this function y(t) multiplied by the corresponding orthogonal function say if I want to evaluate a_n it is to be multiplied by cos n omega naught t, if it is b_n then to be multiplied by sin n omega naught t and then integrated over this period and if it is a naught then a naught is multiplied by just a constant. So we have to just integrate this and that gives me the coefficients a_0, a_n and b_n the views the properties of orthogonal functions.

Now one may express cosine n omega naught t and sin n omega naught t in terms of exponential functions. Similarly, sin n omega naught t yes, e to the power j n omega naught t minus e to the power minus j n omega naught t by 2 j. If we express the functions sin and cosine functions in terms of the exponential functions and substitute here what you get is y(t) as a_0 plus a_n for cosine n omega naught t I will substitute this, a to the power j n omega naught t plus a to the power minus j n omega naught t by 2 plus b_n into e to the power j n omega naught t minus e to the power minus j sorry n omega naught t divided by 2 j. If you allow me to segregate the exponential terms a_0 I can write this is 1 to infinity a_n, b_n by 2 j and a_n by 2 so I will write minus j to n by 2, a to the power j n omega naught t plus a_n, a to the power minus j n omega naught t by 2 b_n minus this quantity divided by 2 j.

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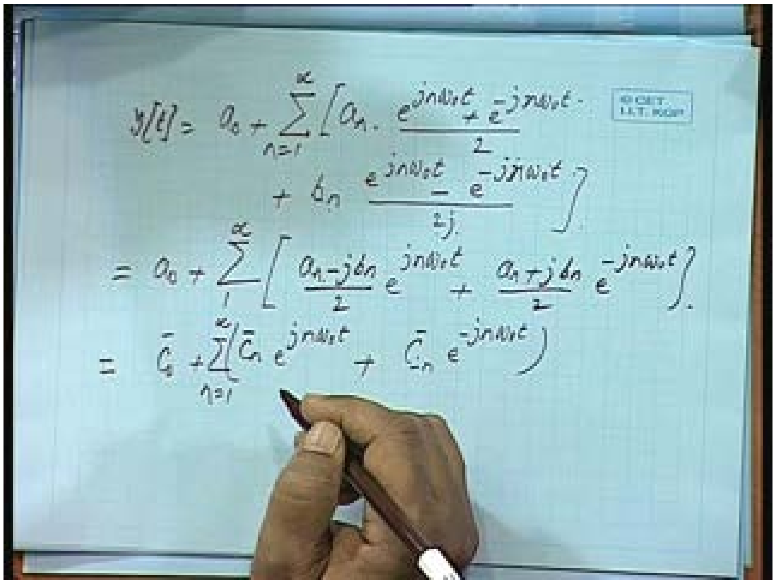

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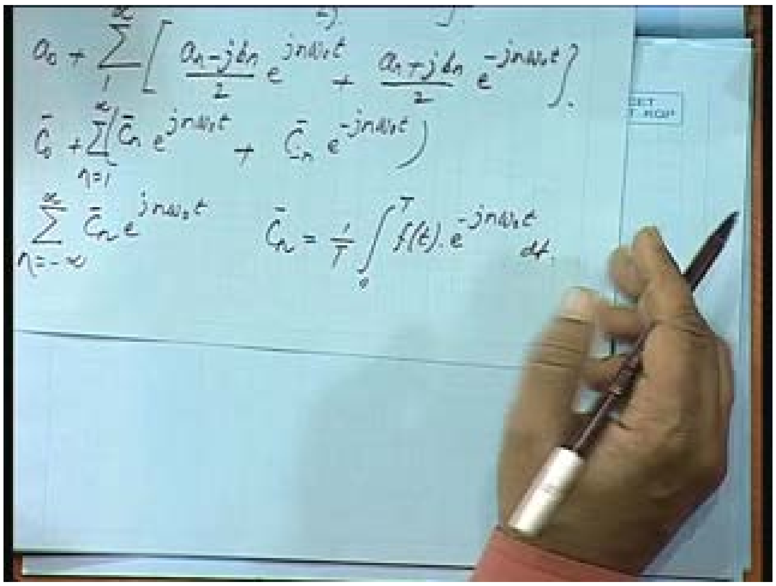

So if j is brought up if we plus j b_n by 2 e to the power of minus j n omega naught t okay. So we can express in terms of 2 exponential terms and the coefficients will be just complex conjugates one may write this as C_0 plus C_n I will write C_n bar as a_n minus j b_n by 2 if this is C_n bar then this is C_n bar minus I will put minus 1 to infinity, is it all right. So one may write this as C_n e to the power j n omega naught t, n varying from minus infinity to plus infinity where both of them are taken together where c minus n means a_n minus j b_n by a_n plus j b_n by 2 there is complex conjugate of this where C_n, C_n bar sorry, C_n bar is 1 by t, 0 to T f(t) e to the power minus j n omega naught t dt, do you agree.

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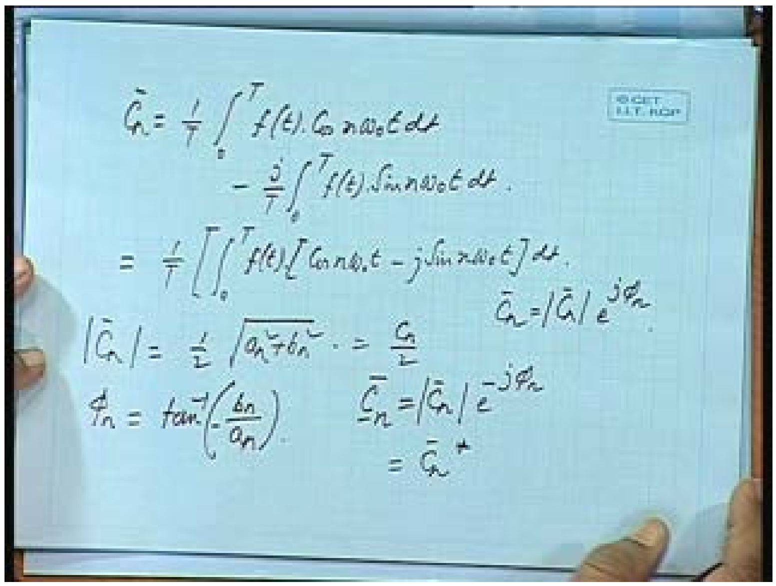

Let us see how we get this C_n bar this C_n is a_n minus j b_n by 2 so if we write in terms of those co-efficients it is f(t) cosine n omega naught t dt minus j by j into b_n by 2, so j by T, 0 to T, f(t) sine n omega naught t dt is that all right. So that gives me 1 by T_0 to T f(t) cos n omega naught t minus j sin n omega naught t dt. So this is nothing but e to the power minus j n omega naught t so this is what we have written f(t) into e to the power minus j n omega naught t dt, C_n bar magnitude will be half of you can see a_n squared plus b_n squared under root by 2 that is old C_n by 2 if you remember when we expressed these quantities together by C_n earlier we had written plus phi n in this form where, C_n square is a_n square plus b_n square under root here it is that divided by 2.

So old C_n by 2 is C_n bar magnitude and phi n here if you can write C_n as C_n bar magnitude and e to the power j phi n then phi n is 10 inverse b_n by a_n is negative sign c minus n is magnitude e to the power minus j phi n it is nothing but C_n star. Now in a Fourier series representation you are taking an infinite sequence physically it is not possible to consider an infinite sequence always so we somehow we have to truncate somewhere, so what is the error involved when you go for a truncation. Now any series f(t) if you express any periodic series I mean any periodic function if you express in terms of Fourier series a_n cos n omega naught t plus b_n sin n omega naught t.

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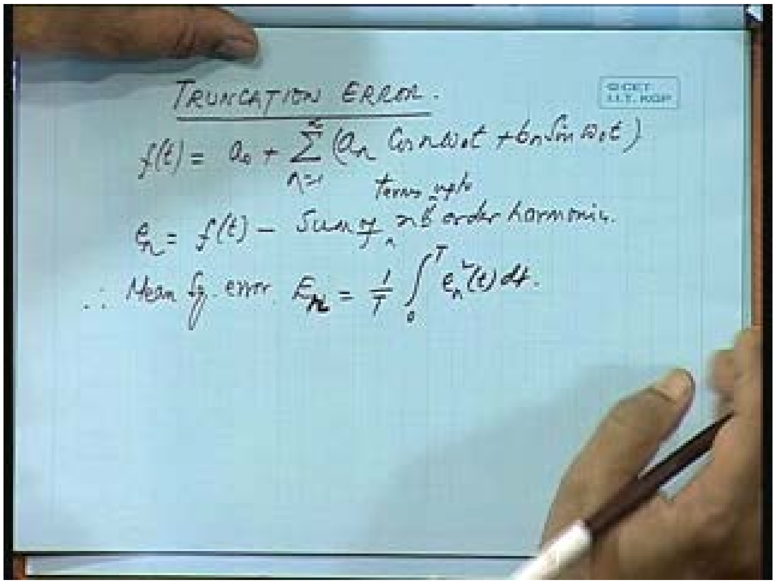

Suppose you truncate it after 20 terms that means we concern only the first 20 terms and then after that we drop that the magnitudes of these coefficients a_n and b_n may be very very small. So you can drop them then how much is the error incurred, so error if we consider from nth harmonic onward nth harmonic onward then what is the error? Error will be will take the original function minus the aggregate of the terms up to that nth term, up to that 20th terms I say. So that much error if you take the square of this error all right that will give me some measure of we can take the mean square error say as a measure of performance representation.

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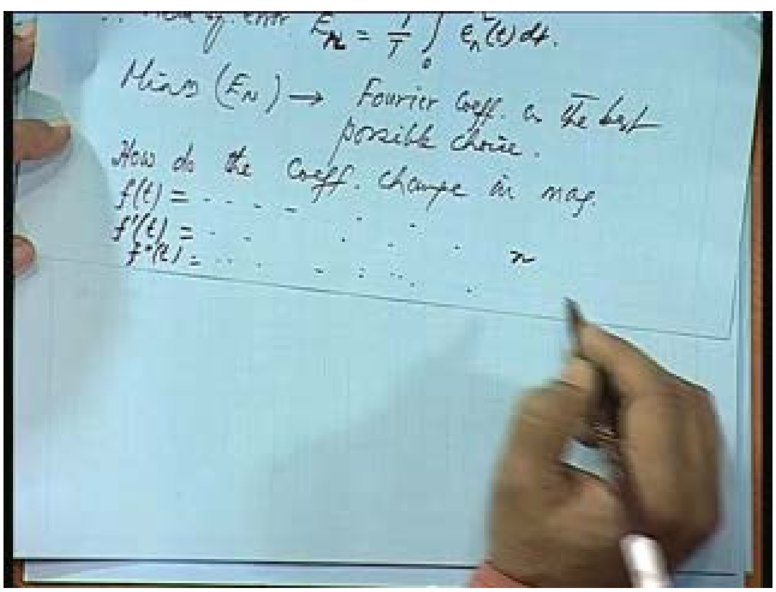

So e_n will be f(t) minus some of say nth order harmonics up to nth order some of terms up to nth order harmonic okay. For mean square error, if you call it capital E_n, E_n it is up to nth order we are considering it is 1 by T, 0 to T this error e_n squared dt. Now is there any other representation which can give me, which can give me up to 20th term a better representation of the signal that means if you consider up to 20th term some other series, some other functions, some other combinations, can you get a better series, better approximation, so that error will be minimum. It can be shown that if you minimize this error, minimization of this error gives me that the series that you have considered in this sum, this truncated series, the coefficient of the truncated series is the coefficient of the Fourier series, original Fourier series that means if you consider Fourier series if you truncate it the truncated series is series is the best possible series that can express the function given function in a least square sense, in a least square sense of the error.

So minimum e_n gives me the Fourier coefficients as the best possible choice. Now next we take up another issue how do the coefficients change in magnitude as we go down as you go down in the derivative functions that means if f(t) is a function given by a series like this f dash t is derivative f double dash t and so on. If you consider these are also periodic functions, if we consider these functions how will the coefficients a_n's and b_n's in this respective series, how will these coefficients vary with n. Let us take f(t) once again a_n cosine n omega naught t, let us consider only the first term then what will be f dash t it will be n omega naught a_n cosine n omega naught t similarly, for b_n terms.

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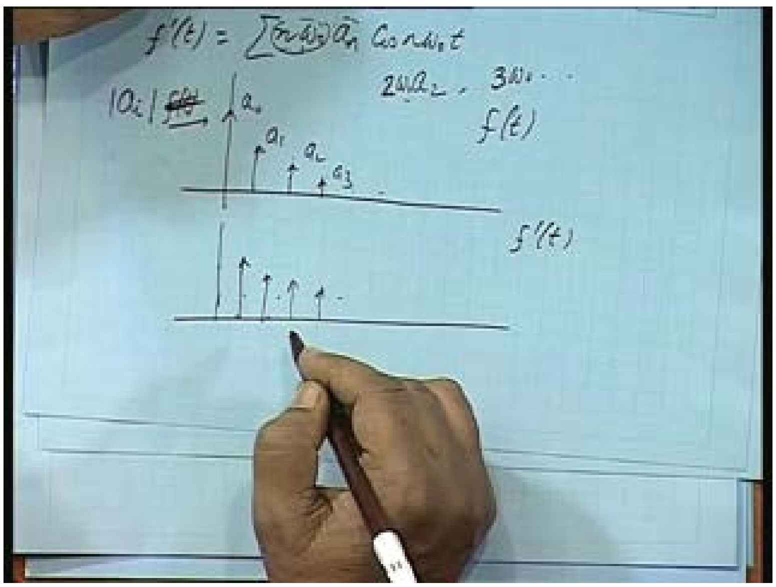

Now if the earlier coefficient had been a_n now it is a_n into n omega naught that means as n increases as n increases the coefficient the importance of the coefficient is enhanced it is increased. For example, if f(t) for f(t) the coefficients are say let us consider only a a co-efficients I will consider a_i magnitudes, so a_0, a_1, a_2 and so on a_3 suppose these are the coefficients and this is for f(t) then for f dash t the same coefficients is a_0 cannot be present in this case it will not be present a_1 you say this much a_1 gets multiplied by n, n is 1 then a_2 will get multiplied by 2 times omega naught, a_3 will be multiplied by 3 times omega naught and so on. So depending on the value of omega not and also they are multiplied by the harmonic number, so these gain importance how harmonics do have an enhanced importance in f dash t if we go further down then it will be increasing further that means it will contain, it will be rich in harmonics as you go to further and further derivatives you will have harmonic treats signals. Let us see what that means, let us take square function square like this .

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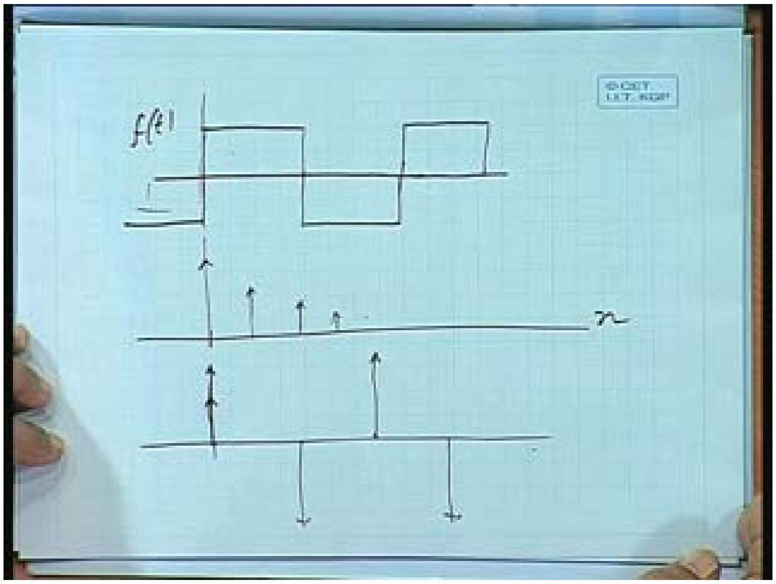

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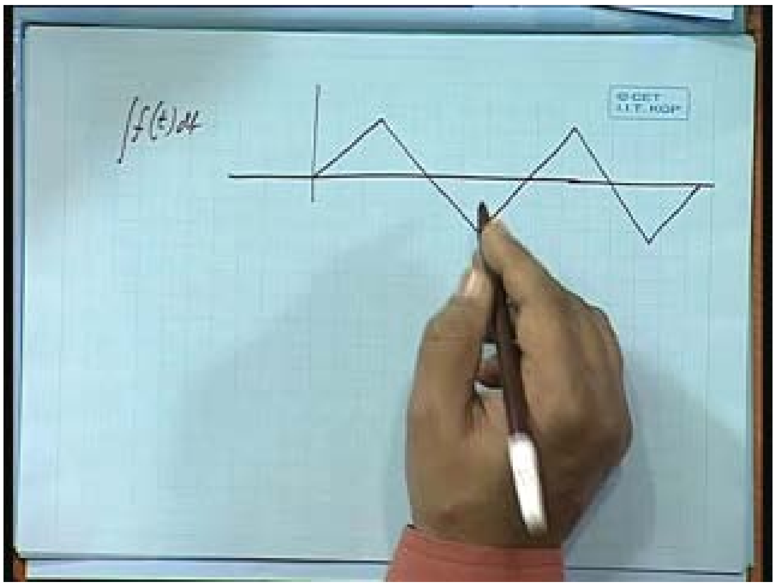

Now it may have say coefficients may be b_n in this case it will be it is an odd function. So it will be the con consisting of only sin terms suppose these are the coefficients and so on then what is derivative of f(t), derivative of f(t) will be delta function here delta function here, delta function there and so on actually it is coming from this point, this point.

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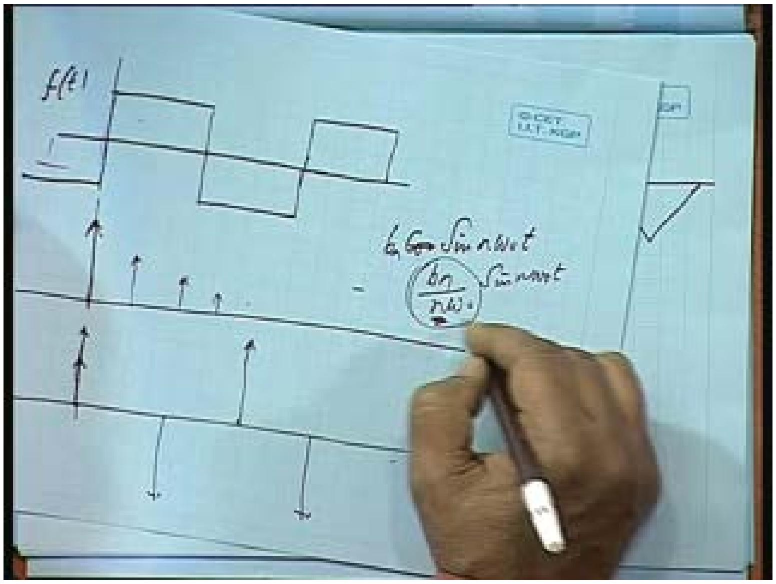

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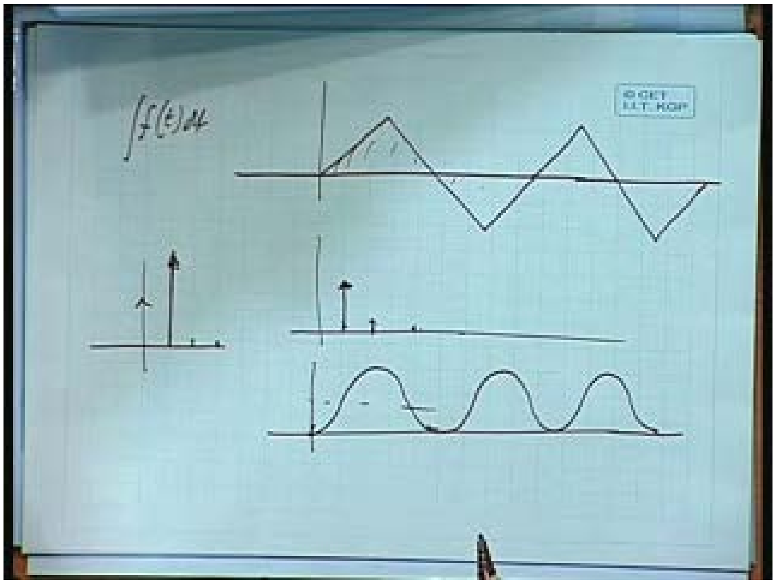

So there will be of same magnitude okay that means you are finally coming to only impulses and impulse as you know there is a very sharp change that is taking place, a sharp change means there will be large number of harmonics other than and if we indicate this if you would have gone just 1 step higher f(t) integral then what would have been the function like it is a triangular function and so on. So what will be the harmonic comp component for this whatever harmonic component, components we had for this f(t). Now it will be say cosine or sine n omega naught t b_n this is to be indicated, if I indicate this I get this. So in the series if I indicate this it will be b_n by n omega naught t okay, okay sin n omega naught t.

So as n increases the magnitudes of this coefficient will be decreasing, so coefficients for these will be decreasing very fast okay. If you indicate it further it will be say as we indicate this it is a parabolic function we can see it is very smooth very much like a sin, sin f, so smooth function means it contains less of harmonics so the fundamental will be very high it may have a dc value also because now it is indicating and it indicating this you will get positive area and then that starts reducing it becomes 0 here it never becomes negative.

So it is always having some average value, so it will have this and then this is further increased compared to this their relative magnitude will be much much smaller. So as you keep on in as you keep on indicating you have less number of harmonics. As you differentiate we have more number of harmonics that means the harmonics that we are earlier neglecting cannot be neglected as you go to derivative functions, this is very important.

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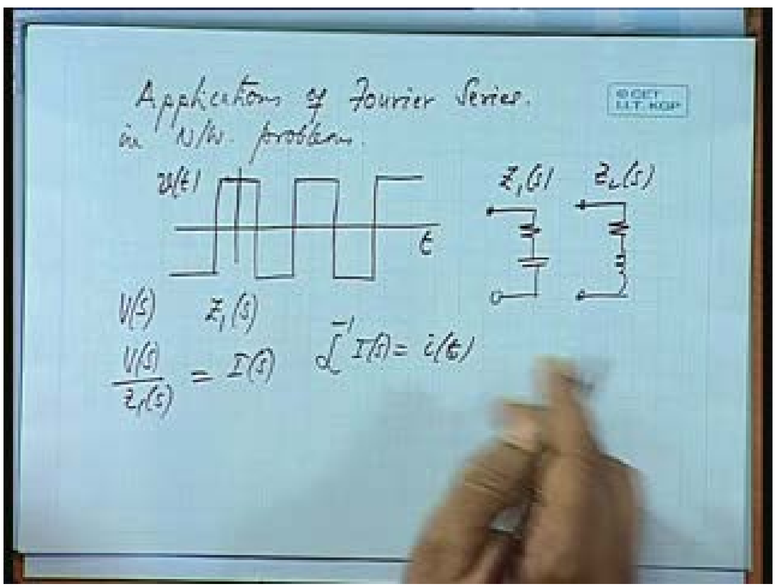

Now we will consider some applications of Fourier series in network functions, in network problems. We excite network by a square row like this. Say network may consists of r and c or a simple network we can concern r and l there are 2 networks say Z_1 and Z_2, what will be the response if this is the voltage applied, periodic voltage applied across these 2 networks what would be the current like. Now one way of solving this is calculate V(s) that is the Laplace transform of this, calculate the impedance in the S domain that is Z_1(s) or Z_2(s) then V(s) by Z(s) will give me corresponding I(s) take Laplace inverse of I(s) that will give you I(t). Now we are interested in the steady state value that means at steady state when the transients have die down what would be the magnitude of this current I(t), what will be the what will be the expression for I(t).

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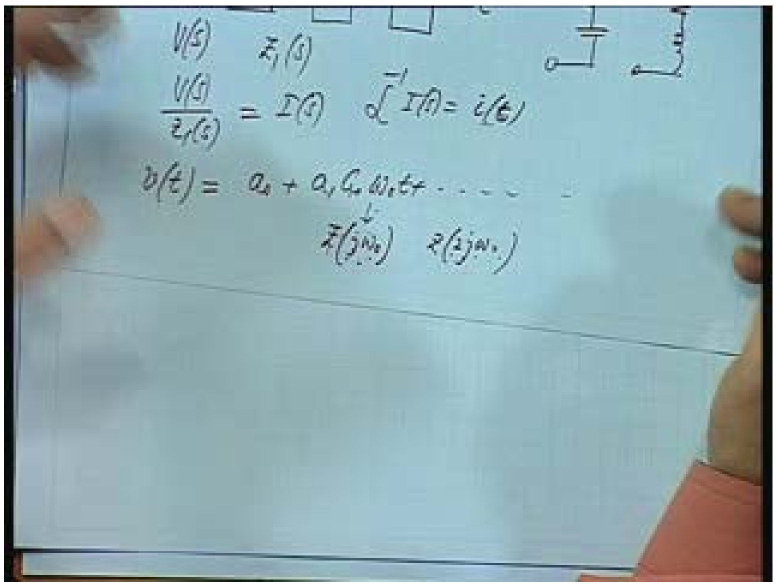

Now instead of going by this Laplace transform and then taking inverse we may, we may find out the Fourier series the Fourier components for this periodic function, so v(t) we can express in terms of a_0 plus a_1 cos omega naught t and so on and in this particular case b_n terms will be 0 otherwise b_n terms will also be present then for each frequency we calculate what will be the corresponding Z j omega okay. For different frequencies at j omega naught 2 j omega naught, 3 j omega naught etcetera we calculate the complex impedance Once you know the complex impedance the voltage components divided by the corresponding impedance gives you the component of current at that frequency. So now we can apply superstition theorem take all the currents together and then you can add them together okay.

So let us take this very example v(t) we can write as cosine t minus 1 third cosine 3 t plus 1 by 5 sin 5 t and so on. Let us take the first voltage of magnitude v_1 which will be you may take rms value or you may take maximum value because finally we shall be writing the expression for current V_3 will be 13, 1 by 3 and angle is minus, so minus 180 degrees, V_5 will be 1 by 5 angle this is plus 0 and so on. So what will be the current I_1 in the rms sense in the maximum value sense we are not taking the rms value if we take the rms value divided by root 2. So I_1 will be V_1 by Z_1 if I call it Z_1 similarly, I_3 will be V_3 by Z_3, where Z_1 , Z_3, Z_5 mean by this we mean the value of the impedance, complex impedance at the first harmonic, fundamental third harmonic, fifth harmonic and so on and the corresponding frequencies are known omega 0 is 1 omega this is 3 times omega, omega naught, 5 times omega naught and so on.

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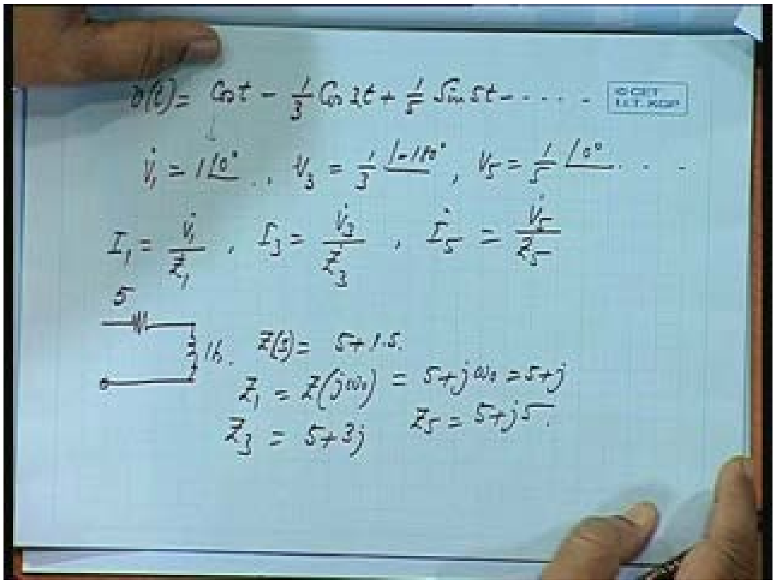

So in this case suppose you have a resistance of 5 ohms and then inductance of 1 Henry, so Z_1 okay Z(s) will be 5 plus 1 into S, so Z_1 will be Z at j omega naught okay omega naught is 1 cos 1 into t. So 5 plus j omega naught 1 into S, so 1 is inductance value so it is multiplied by 15 plus j omega naught and that is nothing but 5 plus j omega naught is 1, Z_3 will be similarly 5 plus 3j this is 3j, Z_5 will be 5 plus 5j and so on. So correspondingly currents will be V_1 by Z_1 which is 1 angle 0 by 5 plus j if you permit me this is 1 plus 5 plus j.

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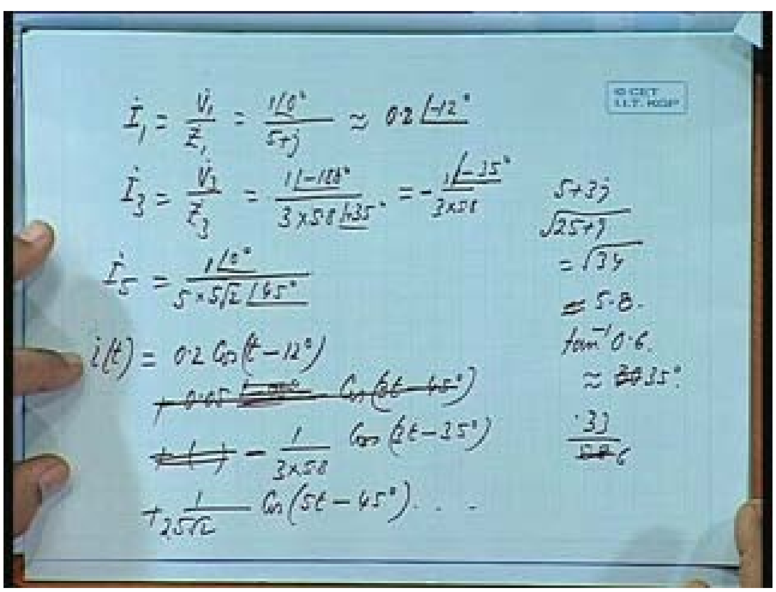

So root over of 26, so approximately 1 by 5 is 0.2 and then angle of tan inverse 5, 1 by 5, so about 70 bout 12 degrees minus I_3 similarly, will be V_3 by Z_3 and V_3 is 1 third minus 180 and Z_3, Z_3 will be 5 plus 3j. So 25 plus 9 under root, so under root 34, so that is approximately 5.8 okay so 1 by 3 into 5.8 and an angle of tan inverse 3 by 5. So tan inverse 0.6, so that equals to 30 degrees somewhere 35 degrees as shown okay. Let us say 35 degrees.

So I just taken a rough value we can evaluate this 1 by 3, 5.8 and this 35 will be go up similarly, I_5 will be 1 by 5 angle of 0 degree and this is 5 plus j_5, so 5 root 2 and an angle of 45 degrees okay. Therefore i(t) can be written as supervision of these components will give me 0.2 cosine t minus 12 degrees plus 1 by 3, .33 by 5.8. So approximately 6, so .05 very crude approximation little more than .05 okay and angle of minus 45 degrees sorry .05 cosine 3t minus 45 degrees similarly plus 1 by 5 root 2, so 1 by 25 root 2 whatever be that value sorry 45 degrees will come here, here it is 35 degrees, mind you this will be 180 degree will contribute to a negative term. So I may put this as minus whatever be this value 1 by 3 into 5.8 and an angle of minus 35 degrees okay.

So it will be minus this value cosine 3t minus 35 degrees plus again it comes back to 0 angle so plus this quantity 1 by 5, 25 root 2 cosine 5t minus 45 degrees and so on. So this will be the total sum of the current, so this is a very handy method for computing the components the current the time domain for periodic functions then excited by periodic function under steady state conditions.

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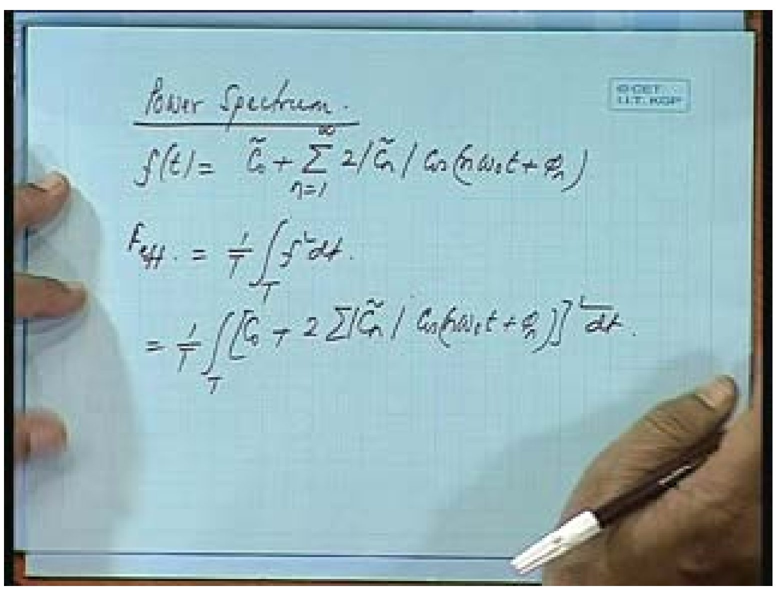

Now we define Power Spectrum, we see frequency spectrum that is when we arrange the Fourier coefficients in terms of against frequencies we get lines spectrum. Now if we take the power effective power for each component, each component then that gives me what is known? What is known as powers spectrum, so f(t) is say C_0 plus 2 into C_n cosine n omega naught t plus phi n you can write like this sine and cosine terms put together will appear like this and C_n was basically half of a_n plus j b_n or a_n minus j b_n. So f effective when we talk about effective value it is basically the hitting value when we pass a current through a rev a resistance then how much is the effective value of the current we measure the effectiveness in terms of the amount of heat dissipated.

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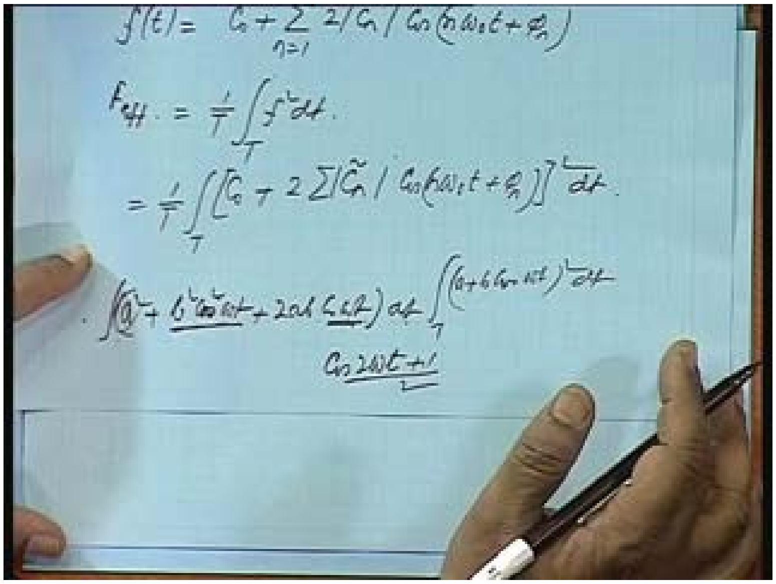

So square of that function over this interval T and then take the average that is F effective so that is equal to 1 by T over T C_0 plus twice C_n squared okay C_n cos n omega naught t plus phi n whole thing squared into dt okay. When you take an integral of this kind a plus something like a plus b cosine omega t squared bt over a period t one term you get is a squared the other one is b squared cos squared omega t the third one is twice ab cos omega t indicate. This will give me a squared constant this is b squared cos square omega t I can write this as in terms of cos 2 omega naught t okay plus 1 by 2. So I can substitute that and then twice ab cos omega t into dt.

So cosine omega t dt indicated over 1 period is 0 similarly, cosine 2 omega t will also give me 0 because they are orthogonal functions. So average value over 1 time period is 0, so we are making use of that property and finally it boils down to in this case this constant squared plus 2 into this constant square so it will be C_0 squared plus 2 into sigma C_n square. This is the maximum value of the cosine functions C_n you can also express in terms of rms value, rms is equal to I_{max} by root 2. So $I_{effective}$ squared will be $I_{effective}$ 1 squared plus $I_{effective}$ 2 squared plus $I_{effective}$ 3 squared and so on that is if you are having harmonic rich signal you want to find out the power spectrum, you take each harmonic and corresponding effective values square them off and addition of that will be the overall effective values squared.

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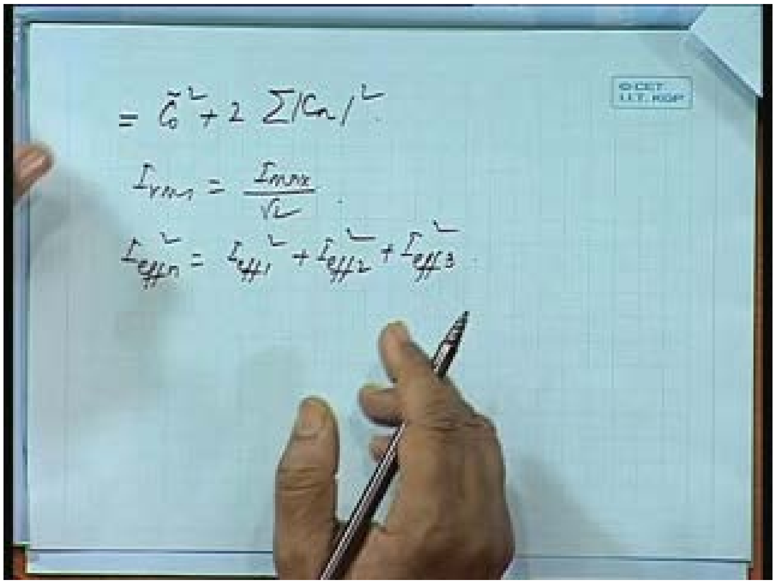

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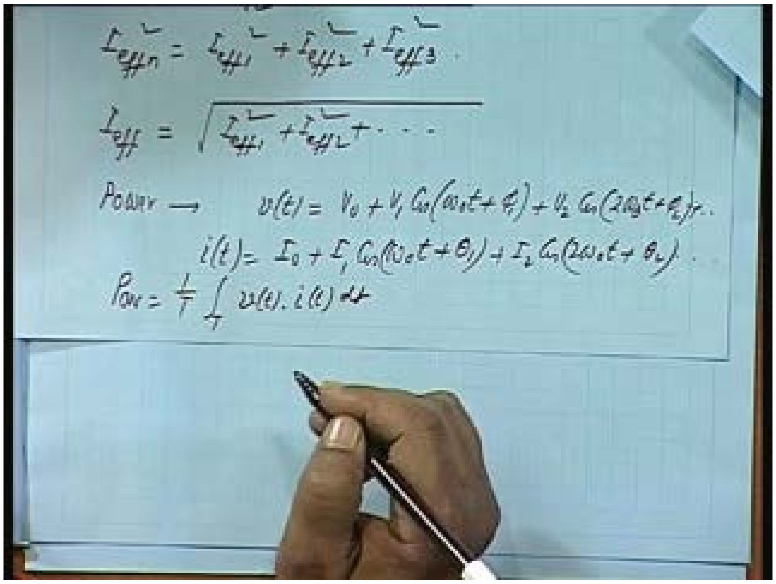

So $I_{effective}$ will be square root of $I_{effective}$ 1 square plus $I_{effective}$ 2 square and so on. In a network, the power will be suppose we have got expression for v(t) as V_0 plus V_1 cosine omega naught t plus phi 1 plus V_2 cosine 2 omega naught t plus phi 2 and so on and i(t) as I naught plus I_1 cosine omega naught t plus theta 1 plus I_2 cosine 2 omega naught t plus theta 2 and so on. Then average power is 1 by T indicated over T, v(t), i(t) dt and this will be you can see if we indicate this with this, V_0 with I_2, V_2 with I_1 these terms will be giving, giving me an effective value 0 because there are different frequencies, we are taking the product cosine functions of 2 different frequencies then taking average over 1 period. So it will be 0 by orthogonal property.

So cosine omega naught t will be coming with this cosine, cosine omega naught t terms cosine 2 omega naught t will come with cosine 2 omega naught t term and so on. So finally the effective power will be V_0 with I_0 whatever is effective power V_1, I_1 we see the power is V_1, I_1 cosine of the angle between the 2 phi 1 minus theta, theta 1 phi 2 minus theta 2 and so on, so that will be the effective value of power. So P average is equal to P average 1 plus P average 2 and so on, the average of all these powers, individual components that will be the average power.

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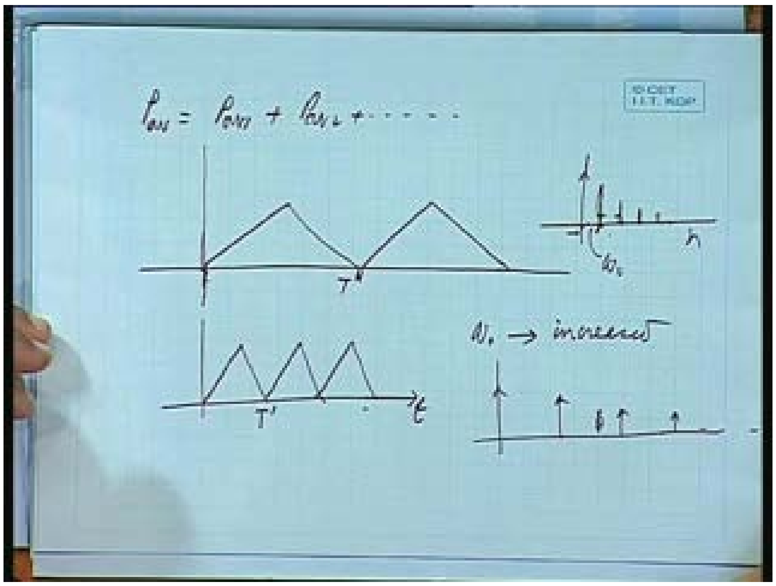

Now this is all about Fourier series, we have talked about periodic functions. Now can we stretch it a little beyond this. We have considered periodic function say like this the period was t and in the frequencies spectrum we got lines, this gap is omega naught between 2 successive frequencies that are present here, the gap is omega naught, suppose we consider the power frequency 0, 50 hertz, 100 hertz, 150 hertz and so on, these are the frequencies present and that frequency corresponds to this T, 1 by T is that fundamental frequency.

Now suppose we have this T magnitude is reduced this period is reduced that means corresponding frequency has been increased that means this gap has been increased. So the spectrum will be like this okay. On the other hand if we stretch it, if we stretch T then this will be shrinking, the gap will be narrowed down. So these lines will be coming closer. So let us consider a function which is not periodic we call them aperiodic function, suppose we have function like this. One may assume this to be a part of the periodic function whose period is very very large that means as if this function is going to repeat like this like this but this period is very very long, it is a part of very long period wave that means this function I have assumed it repeats after a long time, may be after 1 year, 2 years. So this is 1 year or 2 years that means it remains 0 for large time it is as good as considering this aperiodic function as it is.

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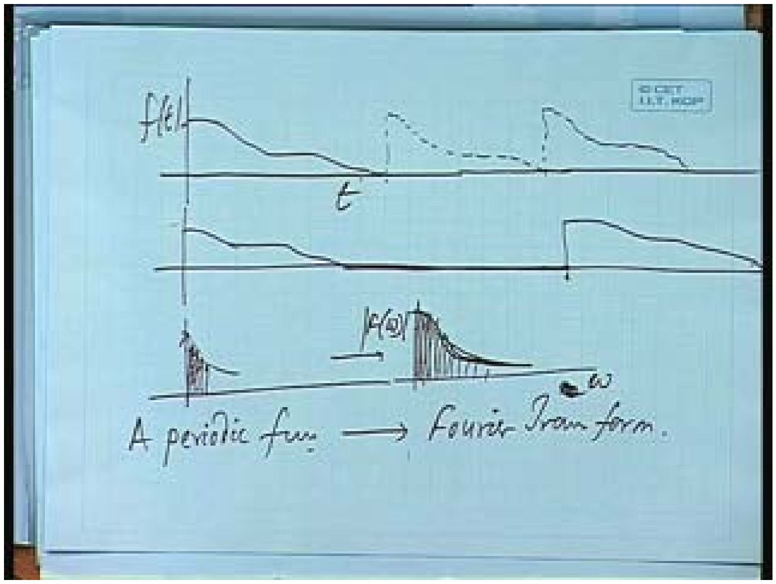

So if we assume this to be part of aperiodic function of very large period then what happens to the frequency spectrum, frequency spectrum these lines are coming very very close, is it not. These lines will become closer and closer because omega naught is reduced very much and in the limit, in the limit we shall be getting a continuous frequency spectrum. These values if there, if these components become very very close to each other then it will generate continuous function like this a continuous function like this, for an aperiodic function f(t) for an aperiodic function in the frequency domain the representation is no more n because they are very very close. So we cannot count them so they are as good as a continuous function.

So f omega against omega will be continuous function. So an aperiodic function it is a Fourier series, we call this as Fourier Transform. A Fourier Transform is basically frequency domain representation of an aperiodic function in this all possible frequencies are present that means an aperiodic function can be can be thought of and aggregate of can be thought of and aggregate of a large number of frequencies, infinite number of frequencies of different magnitudes. There is a phase also associated with each magnitude, so an infinite number of sinusoids will compose this function all right. So this is known as Fourier Transform. We shall not going to the details of Fourier Transform in this class, we shall restrict ourselves to Fourier series.

Now some of the interesting series that we come across in practical systems will be some of the periodic functions other than a sinusoids is rectified sine wave, half wave rectified sine wave then full wave rectified sine wave then also get fired pulses. These sinusoidal functions are fired by gate, so they are all chopped the error signals of this kind generated form power electronic drives, this is periodic, this is also periodic, this is also periodic sorry, all these are examples of periodic waves that will come across in electrical engineering, in practical situations and they give rise to harmonics if decompose them into Fourier analysis into Fourier series we get large number of harmonics or saw tooth wave, sometimes we generate these type of periodic functions also in the laboratories for various studies.

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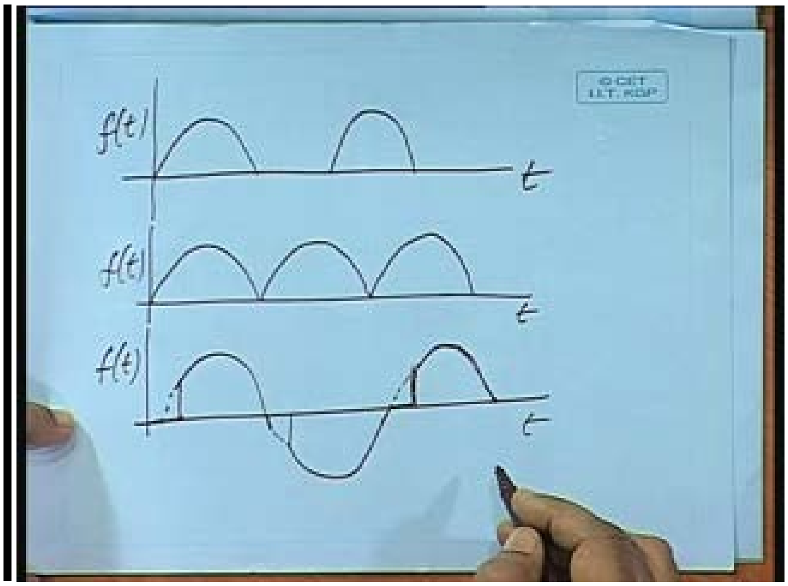

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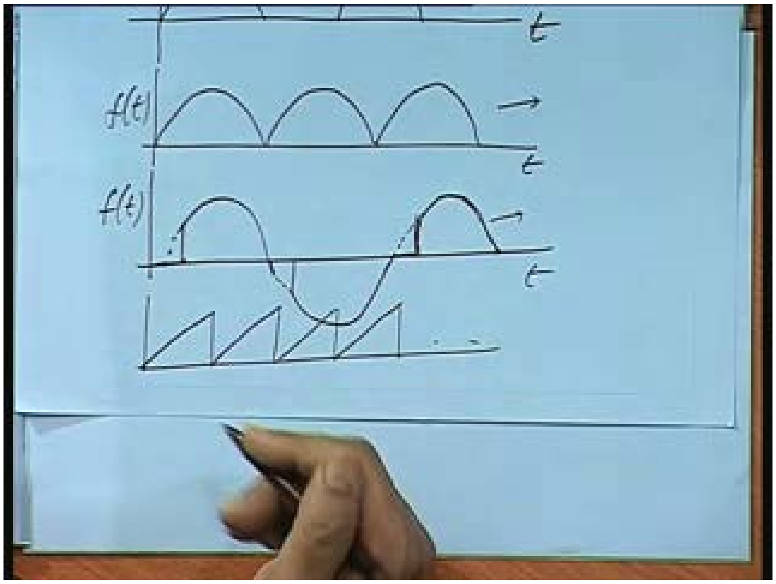

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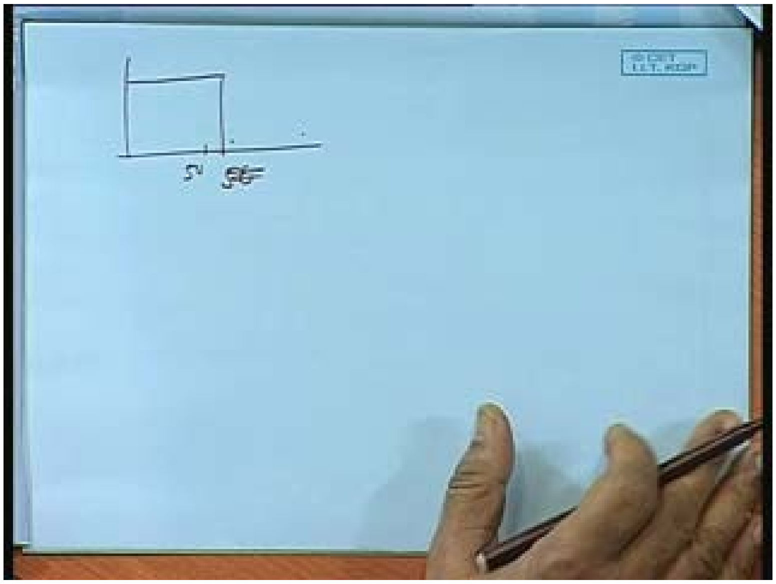

So all these will give rise to harmonics and harmonics disturb the system. If the system gets too many harmonics then there are many equipments which are to be protected against harmonics. So we have to design filters to suppress those harmonics. So low pass filters are provided in a large number of equipment to suppress these harmonics. So you should have low pass filter which will allow only 50 hertz, only 50 hertz and nothing beyond all right.

So the harmonics that will be generated here are 50, 100, 150 and so on. So they are all to be suppressed for the protection of equipments this is a very important area of application of Fourier series in networks, electrical networks. It is also in mechanical systems sometimes say we have a reciprocating motion a diesel engine or any IC engine or many drives where we have a reciprocating motion. So that causes some kind of a vibration all right it is a periodic function but it is not a sinusoidal function. So any periodic function we can decompose into large number of harmonics so mechanical systems say a shaft is rotating at 500 rpm because of some eccentricity it is having a vibration all right or it is may be having a ram which is moving corresponding to that speed all right which is giving you a non-sinusoidal motion.

So you can decompose that into harmonics, so multiples of 500 rpm that is 1000, 1500 those frequencies are also present and they will give rise to some kind of a vibration. Some times to protect the equipments we need to suppress those vibrations high frequency vibrations, so we need mechanical filters, shock observers these are the mechanical analog of electrical filters. We use dampers then mass, spring and dash pot, these are the 3 equivalent items mass corresponds to inductions, a dash pot corresponds to resistance and spring correspondence to a capacitance in electrical network.

So with the help of these 3 elements you can always design a filter to suppress some of the harmonics and to protect the equipments from vibrations because there may be some frequencies at which the system frequency, the systems natural frequency may be very close to some of these harmonics. Then the system will start resonating, so to stop resonance of mechanical systems you have to suppress those higher harmonics.

So that is why we protect many of mechanical systems from from resonance, resonance phenomenon with the help of such shock observing this such filters, mechanical filters, shock observers are also basically doing the same thing. So this is all about Fourier Series and applications, thank you very much. We will take up mechanical systems in the next class.