Dynamics of Physical Systems
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Lecture - 9
Application of the Hamiltonian Method

So, in the last class we had seen that the Hamilton's approach provides a nice way of writing the first order differential equations.

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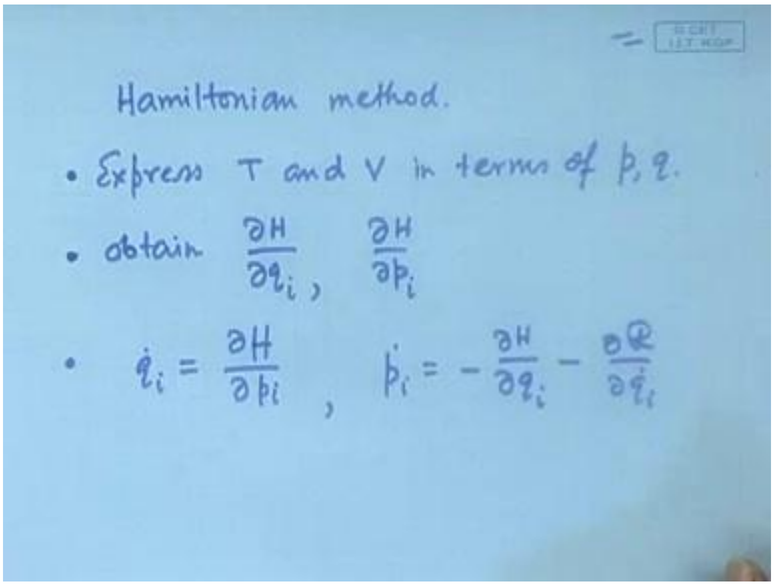

And, in that method the Hamiltonian method, we write first express T and V in terms of, of what p and q. In the Lagrangian method, we wrote it in terms of q and q dot, so there lies 1 difference, then you obtain the, partial derivatives H is just T plus V, then write the Hamiltonian function H as T plus V. Then, you obtain delta q and once you have done so or because there will be different directions do not talk, there will be different directions and so you will have to take the derivative of the Hamiltonian function with respect to each of these.

And then finally, once you have done that, the equations would nicely be expressed as q i dot is equal to and p i dot is equal to minus, so this is the essential technique. Now, you have seen that in some cases, these equations will yield dotted terms in the right hand side, which again you have to substitute from here. in order to express in terms of undotted terms, so that is the essential technique. But, now so far we have, solved problems that are you know, more of the circuit or sort of physics oriented.

And that might, erroneously give you an idea, that this method is these methods, that we have discussed so far, Lagrangian and Hamiltonian. These are not all that much useful in engineering systems, so today let us, attack an engineering system right away, most of you are electrical engineers, so let us do a electrical engineering problem, a DC machine driving a load.

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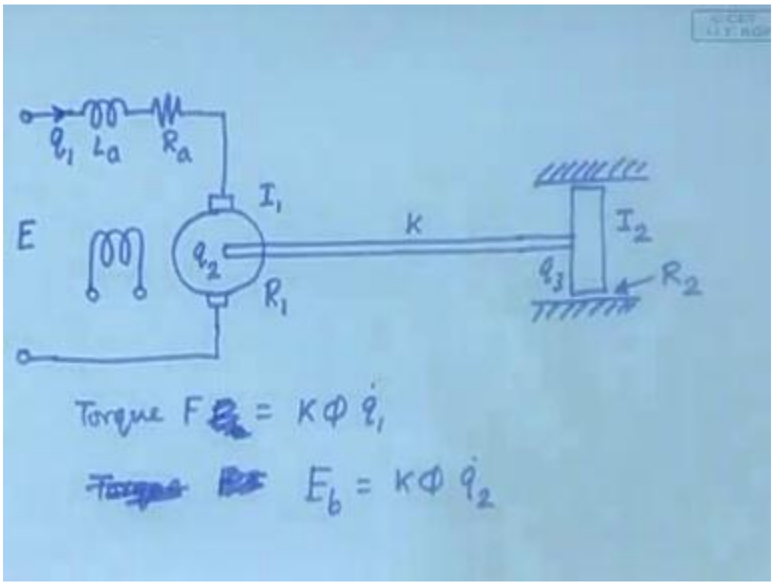

So, what is the structure of the DC machine, you have the armature, the armature is fed by means of a supply a DC supply. There will of course, be some resistance and inductance of the armature winding, so, let us take that out, so that we can, directly see that they are in the circuit, so that is the armature circuit. Assume, that the field is constant we will later assume that the field is also variable, but to begin with, let us assume that the field is constant, which means it could be a permanent magnet machine or a separately excited machine.

And you have applied a voltage say E, to this and this is what the inductance of the armature, resistance of the armature. Now here, this fellow rotates and there has to be a mechanical system to, so from here, there is a shaft and the shaft is moving something say a disc, with some mass against this. Now, this armature has a mass or moment of inertia, assume that to be I 1, here there will be a bearing and bearing will have some friction, so assume that to be R 1.

Similarly here, this disc will have a mass or moment of inertia call it I 2, and these will have another bearing friction called R 2. There is another degree of freedom, you must notice, that these 2 are not moving exactly in step, there has to be a some kind of a springiness of the shaft, so that is represented by a spring constant K. If it is not there, if it is rigid, then they will not, have their own independent degrees of freedom., so in order for these 2 to have their independent degrees of freedom, this fellow should be slightly tortionable, so there will be a tortional spring here.

So, suppose this is the system, how do we represent it, in terms of a set of differential equations. First let us, identify what are the variables, this the electromechanical system, the main advantage of whatever we are doing is that, you know the methods of writing the differential equations for electrical systems, you know it for the mechanical systems individually, but when you attack a electromechanical system, then these methods are very handy, because they treat them in the same way, so what are the state variables, what are the the variables in the system.

We are trying to identify the minimum set of coordinates, tell me what are they here there would be a current flowing, in our notation we will consider a charge flowing here. The electrical system, will it need any other variable, no 1 will suffice, how, how about the mechanical system. We will need the position of this armature and the position of this that uniquely identifies, so there will be a position coordinate q, say let us equal q 1, q 2 and q 3, so these are the position coordinates.

Position means, angle essentially and q 2 dot will be the angular velocity, q 3 dot will be its angular velocity. Now, notice that the electrical system and the mechanical system, how are they coupled, I am not talking about the physical coupling, I am talking about the mathematical coupling between them, how are they coupled. The electrical system applies a torque on the mechanical system, and the mechanical system responds by applying a back EMF on the electrical system, that is how they coupled.

And, what are there, these couplings, the back EMF E b is K phi I a is q 1 dot, and the torque, we will not write it as T, because T we are representing the kinetic energy by, so we will write it as say F is so this is the torque. Sorry, no this is my F is K phi I a is the torque, back EMF E b is K phi the speed of rotation of this one, so now these 2 K phi's are. in fact, the same, these 2 K's are in fact, the same. That is what, if you assume that the power input into the, motor is equal to the power output into, out of the motor.

If you ignore the losses, then these 2 are actually equal, else there will slight difference, which you can ignore for now. So, these 2 are the relationship of the interaction between the mechanical subsystem and the electrical subsystem, now one convenient way of handling these would be, to assume to handle the electrical subsystem and the mechanical subsystem separately. Why because then the electrical subsystem, will be simply a circuit with, which sees a voltage which is the back EMF. You can do it that way, a mechanical subsystem is just a the mechanical system, which sees a torque, you can do it that way, first let us do it that way, and then we will handle it as integrated system, so for the electrical subsystem and the mechanical subsystem.

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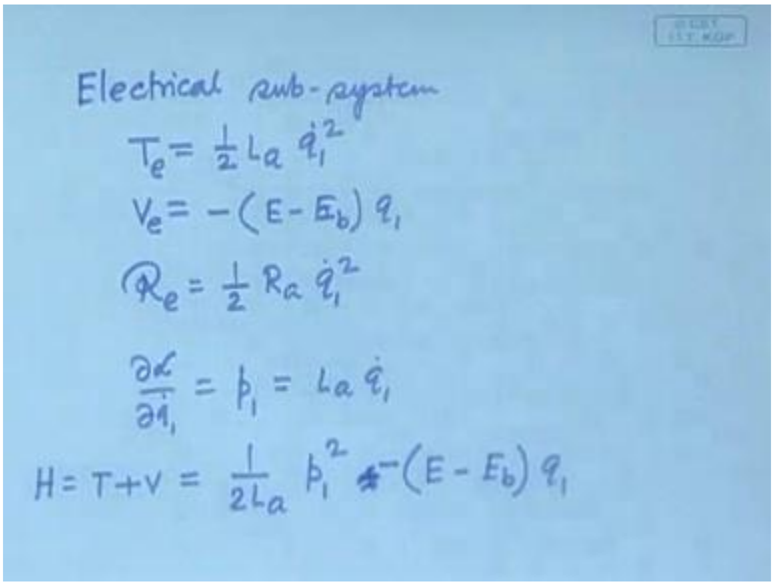

So, electrical subsystem, what will be the kinetic energy, what will the potential energy, so we will put the subscript e. ((Refer Time: 11:09)) Half q 1 dot square L a, so this is half L a q 1 dot square, that is the kinetic energy, the potential energy is q 1 is a direction, in which E is added E is applied, but it is acted on in the opposite direction by back EMF. So, the total voltage effectively applied on this, circuit is E minus E b and that as acts in the direction of E q 1.

So, we will have to put a minus, E minus E b sorry, E b into q 1, the Rayleigh term of the electrical subsystem is half R a q 1 dot square, so done. From here, we can write the momentum, because momentum is, is what Lagrangian is T minus V, yes, so no only the Lagrangian. So, the p 1 is the derivative of only the Lagrangian the Rayleigh does not appear, so this is nothing but, the momentum see, L is equivalent to the mass q 1 is equal to the velocity, so it is the electrical momentum, so this is the momentum.

So, now we can write your Hamiltonian function as T plus V is equal to, so what is T plus V, T we substitute it here, so that we can express in terms of p, it will be, it will come twice, so 1 by twice L a in the denominator p 1 square, this term minus V, so is becomes plus E minus E b q 1.

Student: ((Refer Time: 14:01))

Sorry, so T plus V it is minus, you are right, so this is the Hamiltonian function, so if you write the Hamiltonian function, then the rest falls in place immediately.

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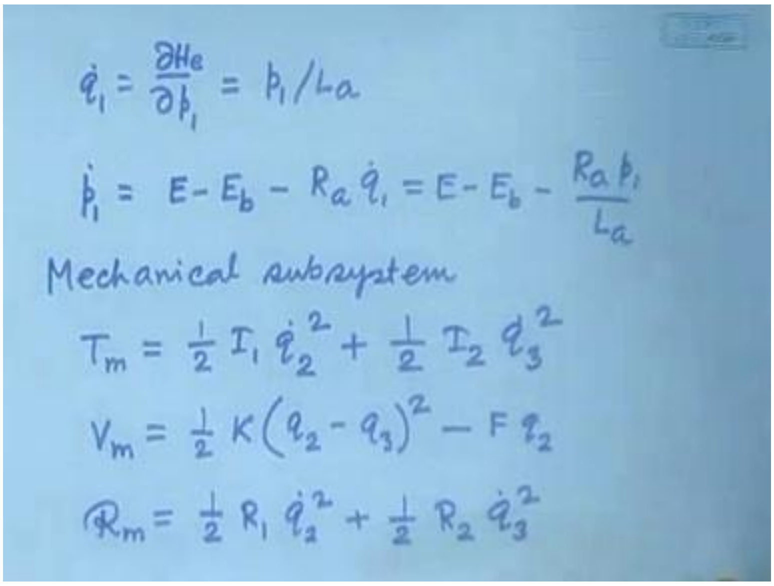

q 1 dot is the derivative of the Hamiltonian with respect to I will write e with respect to p 1, that is derivative of the Hamiltonian with respect to p 1 is…

Student: ((Refer time: 14:38))

Yes, P 1 by L a, the other one, P 1 dot is equal to ((Refer Time: 14:53)) we have to use this, so take a derivative of this and this, it is ((Refer Time: 15:03)) with respect to q 1 it will be this term, minus of that term, so E minus E b, because it was negative here, minus

Student: ((Refer Time: 15:23))

R a q 1 dot, but dot I do not want in the right hand side, so we can substitute, E minus E b minus R a p 1 by equation done. Now, in the mechanical subsystem the T, Kinetic energy of the mechanical side, will consists of the 2 bodies their own kinetic energies, so it is half I 1 q, q 2 yes, q 2 dot square plus half I 2 q 3 dot square. Now, the potential energy would be, potential energy would be due to the K the K term here and there would be the torque applied, that will be acting in the direction of q 2.

So, we have to take him to that into account, so it is half K, it is q 2 minus q 3, q 2 minus q 3 is the torsion in that spring, that is squared plus F acting in the direction of q 2, so minus F q 2. And the Rayleigh term is the two friction elements, so half R 1 q 2 dot square plus half R 2 q 3 dot square is that right. So, the three terms we have written down, now we have to express T plus V, in terms of p and q remember that, so we have to find out, what is p or p 2 and p 3.

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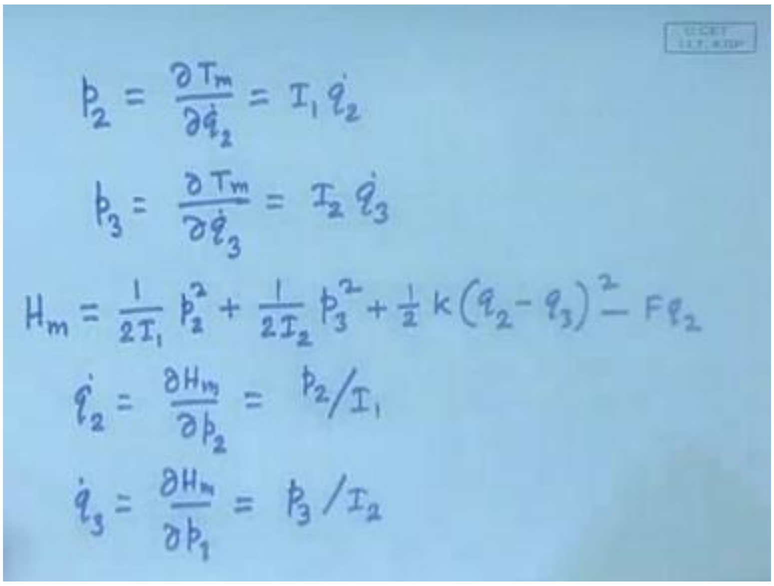

So, p 3, p 2 is basically the derivative of the, this term, derivative of the Lagrangian we had written, because the let derivative of the potential with respect to q dot was 0, so we can write directly like this q 2 dot is I 1 q 2 dot, that is p 2. Similarly, p 3 is the derivative of the this thing in terms of q 3 dot is I 2 q 3 dot, Obviously because these are the two angular momenta, now we have to express this in terms of that, so we will have to write, H of the mechanical side is ((Refer Time: 19:03)) we have to express these, which means, these terms we will have to write in terms of the ((Refer Time: 19:09)), that we have just derived.

It will be 1 by twice I 1 p 2 square plus 1 by twice I 2 p 3 square plus the potential term, plus half K q 2 minus q 3 square, then minus no F q 2, K is only one, there is only one.

Student: ((Refer Time: 19:59))

No, wait, wait, wait, we will do that later, first let us, obtain the mechanical systems stuff independently and then we will substitute. So, this is the total Hamiltonian function, so we can now write q 2 dot is derivative of the Hamiltonian with respect to p 2, what remains only these, so p 2 by I 1. Similarly, q 3 dot will be trivial is simple p 3 by I 2 right from here, so these three things are relatively easier, let us go to the momentum terms.

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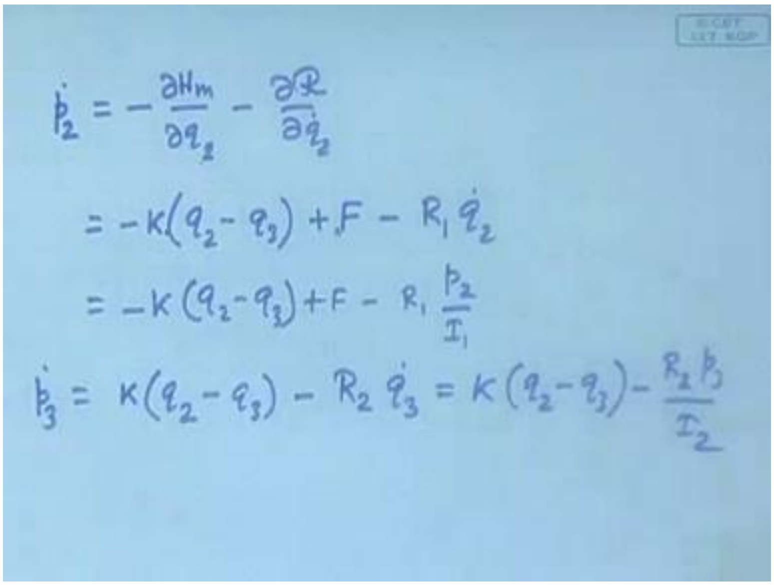

P 2 dot that will be, let us first write and then evaluate H m by del q 2 minus del R plus what, the first term when differentiated with respect to q 2, ((Refer Time: 21:30)) yields two terms here. So, you write down them carefully, K minus K q 2 minus q 3, so plus F yes F and this term minus R 1 q 2 dot, and then this has to be substituted, so we will write minus K q 2 minus q 3 plus F minus R 1 q 2 dot is p 2 by I 1 done. The next equation would be p 3 dot is, is what, again just use this with 2 substituted by 3. So, you have the first equation this remains, so this remains only it will be K q 2 minus q 3 and then…

Student: ((Refer Time: 22:55))

Yes, and this Rayleigh term will remain, that will be minus R 2 q 3 dot, which you can now substitute as R 2 p 3 by done. So, at this stage, your certain things remain, where are the equations, where are the equations? Finally, here are equations E b remained, and we need to substitute that, F remained we need to substitute that. So, just substitute them, from whatever we had written earlier and ((Refer Time: 23:47)) put them back, the moment you put them back, you will need to substitute this, because there are dotted terms.

So, once you substitute, can you write down the equations now, this term will have another term, this term will have to be substituted from there, you do that, I will leave. So, this is how, the equations for the DC motor could be obtained, you might argue, that now here we, took the two things separately, what if you take them as one single electromechanical system, cannot you do that, yes you can do that.

But in doing so, in when you do this this kind of problems in exams, you might as well argue that I will do, it as one single electromechanical system, but there has to be some cautions exercised, let me illustrate that, because otherwise, you might commit problem there. Let us do it, as when we treat the whole thing ((Refer Time: 24:56)) as a single electromechanical system, there the variables remain the same, but there has to be, some problems regarding the interaction between the electrical part and the mechanical part in terms of these, let us see. Let us do it by, the Lagrangian method first not going by the Hamiltonian rule, because that is another method I thought, you have to learn how to use it.

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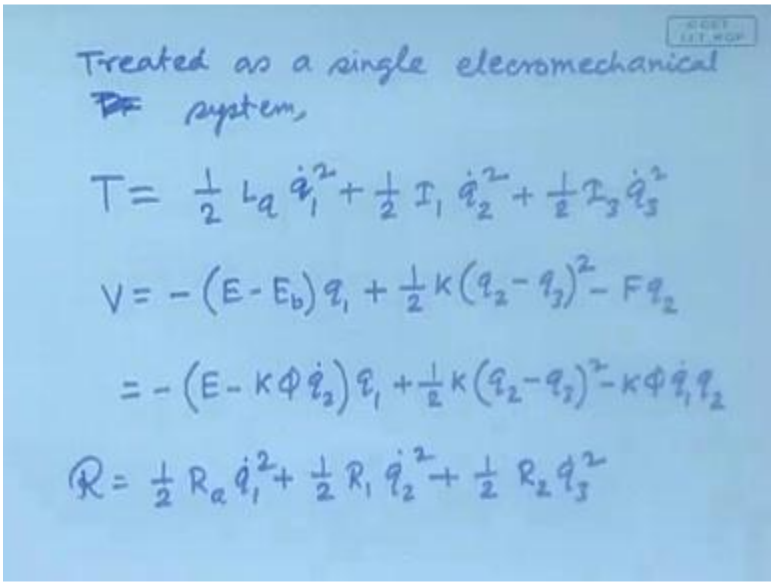

So, in this case the T, I will write treated as a single electromechanical system, then T is it will be the same the three kinetic energies, half L a q 1 dot square plus half I 1 q 2 dot square plus half I 3 q 3 dot square done. Potential energy V, here no it is, what are what is the total potential energy, potential energy will include the back EMF, potential energy will include the torque and that is what we need to treat carefully.

So, first the electrical part, it is minus E minus E b into q 1 it appeared, now plus half k q 2 minus q 3 of the spring minus F q 2 this is the torque. So, notice there was this term. and there was this term, that naturally appeared, they are inside the system still they appeared, now we have to do something about that, so let us first substitute, what we know they are, substitute it, you get minus E minus K phi, E b is q 2 dot q 1 plus half k q 2 minus q 3 square minus F is k phi q 1 dot, K phi I a q 2, leave it like that.

And the total Rayleigh function is half R a q 1 dot plus half R 1 q 2 dot square plus half R 2 dot q 3 dot square. Now, notice one problem, the problem is that in the whole derivation we had assumed, that the v is independent of the velocities, and they are dependent, now V is now dependent on the velocities. And therefore we cannot use exactly the same formulation. But still we can use the similar formulation, because we had proceeded up to 1 point by without assuming that, and at some point where we had introduced that assumption at what point.

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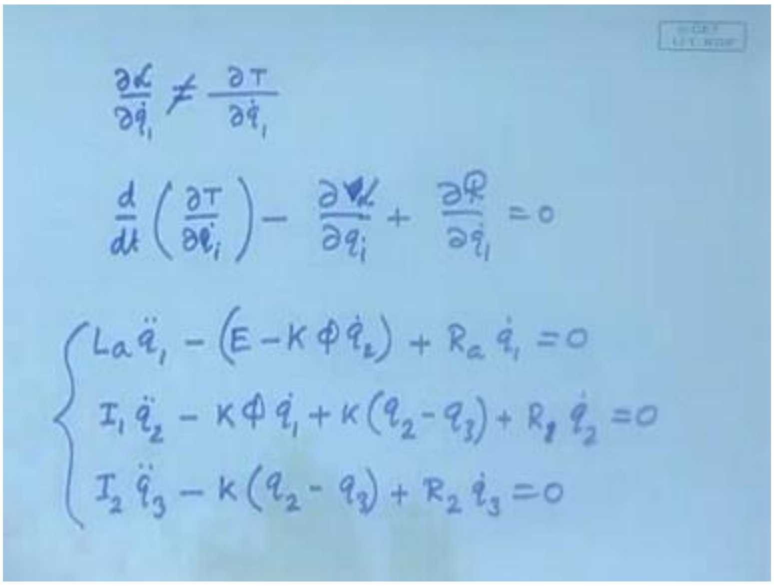

When we had said that we know what the delta of the Lagrangian of the q 1 dot is, we said that is equal to, that is what will not happen here, where did we use it at one stage we had obtained the Lagrangian equation as d d t of the derivative of T with respect to q, q 1, q i dot minus derivative of the V with respect to q i, plus the derivative of the Rayleigh with respect to q i dot equal to 0. This we said that if we can write, this as the no here it was the Lagrangian, here it was Lagrangian.

So, there was no problem here, the problem was here, we said that since V is independent of q i dot we can substitute this as the Lagrangian, this we cannot do now. So, we will have to use this equation, as the basic equation on the base basis of which we obtain the equation, everything will the same, will be the same, go ahead do it. In terms of this we have, we will have the second order equations ((Refer Time: 30:31)) as T first, the in the direction of the q 1 will be this derivative, it will be L a q 1 double dot, first term.

Second term the derivative of the Lagrangian T minus V with respect to q 1, it will be this term only other terms vanish, so it will be minus, E minus K phi q 2 dot, ((Refer Time: 31:14)) yes, plus this term plus R a q 1 dot is equal to 0, that is the first equation. Second equation do the same thing where q 2 is here, q 2 is here, q 2 is here, you will have, I 1 q 2 double dot minus in terms of q 2, what remains here it remains, here it remains. So, write it carefully, minus K phi q 1 dot plus K q 2 minus q 3, plus R, R 1 q 2 dot equal to 0.

The third equation will be, I similarly I 2, I can write almost by similarity with this, q 3 double dot minus, when you do it in terms of ((Refer Time: 32:36)) three what remains, this remains nothing else. So I will have to write it carefully, it will be minus K, q 2 minus q 3 that does the potential part plus the Rayleigh part, it will be plus R 2 q 3 dot equal to 0. So, these three are the Lagrangian equations, but we want to derive it in the first order, and we have already derived it, we want to see that we are doing it.

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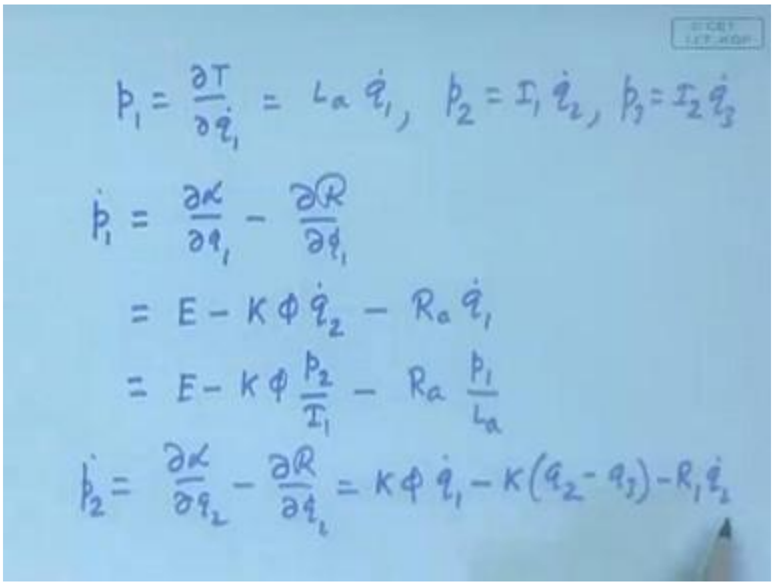

So, we define the potentials, P 1 is equal to what, you have already done that, similarly, p 2 is equal to I 1 q 2 dot and p 3 is equal to I 2 q 3 dot, in terms of this. Now, once you have define that, we can write the Lagrangian equation had taken the form p 1 dot is equal to derivative of the Lagrangian with respect to q 1 minus derivative of the Rayleigh with respect to q 1 dot. So, p 1, this we can write in the right hand side, it will be p 1 dot is derivative of the Lagrangian with respect to q 1, Lagrangian with respect to q 1 this goes T minus V, so this becomes plus and this remains nothing else.

So, it will be E minus K phi q 2 dot minus the Rayleigh term R a q 1 dot, both the dotted terms in the right hand side, so substitute you get, E minus K phi q 2 dot is p 2 by I 1, and this is here, minus R a P 1 by L a, so that is the equation. Notice that equation has come out to the same as the one that we have derived, using the Hamiltonian method, check Similarly, p 2 dot is the derivative of the Lagrangian with respect to q 2 minus derivative of the Rayleigh with respect to q 2 dot is… It will be derivative of Lagrangian with respect to q 2, this term will remain this term will remain.

So, you will have to write it as K phi q 1 dot minus K q 2 minus q 3 minus R 1 q 2 dot, and then substitute these two. So, you more or less understand. how to do it, do you, and you, I would request you to check, that this method is yielding the same set of differential equations, that you obtain by the Hamiltonian method. So, is is the point clear, here in this case, there were some forces acting inside the system, the torque and the back EMF, both were inside the system.

If you take the whole thing as one electromechanical system, it was not that somebody else is applying that, but then those things are also, properly accounted for the moment, you use the Lagrangian equation properly. Only thing is that, that is the character of any electromechanical device, that the torque which is the force is proportional to the current, which is a velocity. So, there will always be velocity dependent potentials in any electromechanical system. And therefore, we have to be careful about such things. In the last class, we had started handling some electrical circuits there you probably noticed a problem, let us do one electrical problem, in order to illustrate that carefully.

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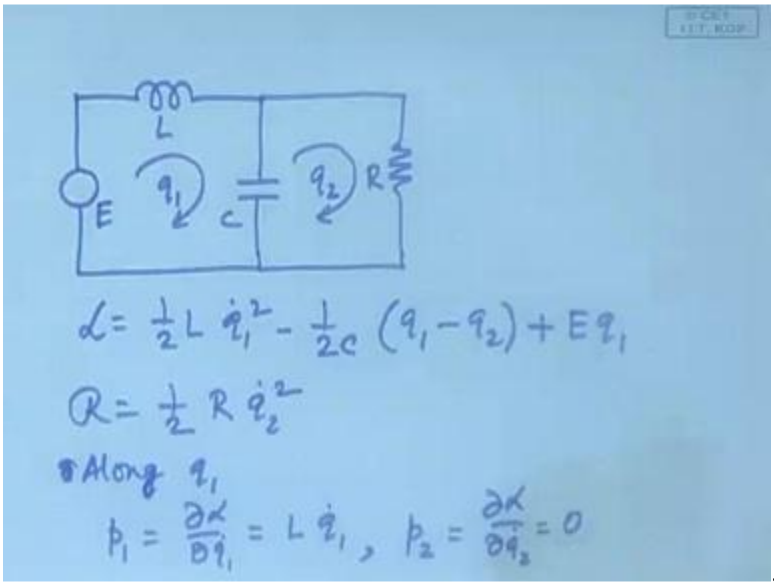

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Here we consider say a voltage source, an inductor, a capacitor and a resistor, simple circuit, how will you do it E, L, C, R will say simple q 1 and q 2 and we can simply write, it fine lets write. You will have the Lagrangian as the kinetic energy half L q 1 dot square, minus the potential energy half 1 by 2 c q 1 minus q 2, then it will become plus, E q 1, that is the Lagrangian function. And the Rayleigh function is half R q 2 dot square, no problem, so we can directly go ahead.

So, along q 1 the equation would be p 1 is the derivative of the Lagrangian with respect to q 1 dot, then L q 1 dot, that is the p 1, now, if you have any difficulty ask me, if you have any difficulty ask me, I am the person to ask. So, you can write p 1, then you can express P 1 is this and p 2 is a problem there, it is 0, if you do it like this q 2 dot is 0, so p 2 is 0, That is obvious, because this line, does not have this particular loop, does not have inductance p 2 is 0.

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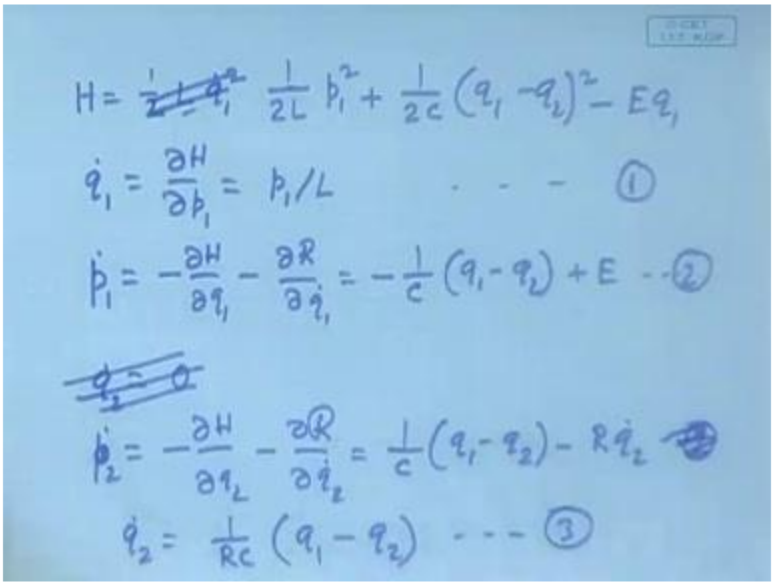

If p 2 is 0, then the Hamiltonian is expressed as half L Hamiltonian would be the potential energy plus kinetic energy, Kinetic energy would be half L q 1 dot square. Now, these will have to be substituted, so we will write it as, it is easy to write 1 by twice L p 1 square, using this plus the potential function it is 1 by twice c q 1 minus q 2 square minus E q 1 done, that is the Hamiltonian function. Then we can write the equations directly q 1 dot is the derivative of the Hamiltonian with respect to p 1 is simple.

Here p 1 by L, where ((Refer Time: 41:44)) I have written the Lagrangian I have not put.

Student: ((Refer Time: 41:50))

You are right, so yes whenever i make such careless mistakes please point out, because that goes into the recording. So, you have q 1 dot is this and then p 1 dot is p 1 dot is minus, minus delta R delta q 1 dot is you can easily write now,

Student: ((Refer Time: 42:37))

Yes minus 1 by c q 1 minus q 2 and plus E, done, right hand side does not have any dotted term, so we are happy. Then along the p q 2 direction, we have q 2 dot is q sorry, p 2 dot is what?

Student: ((Refer Time: 43:20))

Yes something, so 1 by c, q 1 minus q 2 only the sign is different minus, in this direction there is the Rayleigh term, so it will be R q 2 dot. So, what are the ultimate final equations, it is 1, 2, 3, these 3 are the equations.

Student: ((Refer Time: 43:48))

Wait, I will i have not yet talked about the q 2 dot thing, I have written it, but wait, q 2 dot is the derivative of the Hamiltonian with respect to p 2, which has turned out to be 0, which has or in other words, it cannot be evaluated, it could not be evaluated. But still, where is p 2,

Student: p 2 is 0.

p 2 is 0, if p 2 is 0 p 2 is always 0, what is p 2 dot obviously 0, so, if p 2 dot is 0, actually this is not correct, this is all that we can say, it we could not evaluate it. If p 2 dot is 0, we now have q 2 dot, from here we will be able to write, q 2 dot is this goes in the other side 1 by R C can you see, yes q 1 minus q 2, that is it. When you write this it was actually wrong, you could not evaluate it, from what you have, so we have taken a different route and that argument was sound, since you know that your p 2 is 0, p 2 dot also has to be 0 there is no change in that.

And so this term is 0, and therefore from there you can extract, so that gives you the equation 3, which this equation, we had obtain this, now in the left hand side it is p 2 dot, which you argued that it is 0, so if you put 0 here, then you can extract this. See, you needed 3 equations, but in the circuit there are only two storage elements ((Refer Time:

46:12)) and the minimum number of equation, that you really need is equal to the number of the storage elements.

So, we had done all this exercise, but ultimately ended up with 3 equations, which is not very economical, they are all right, they are all correct, but they are not economical. So, the conclusion is that this Lagrangian, Hamiltonian formalism, that we are taking, this often leads to uneconomical set of differential equations. But they are all right, they are not wrong and we will have to learn, how to obtain the correct economical set of differential equations.

You might argue, that this problem happened because we sort of blindly took the q 1 and q 2 as the coordinates ((Refer Time: 47:05)) the two loops. And these two loops independently may not be the minimum or correct set, let us follow that line of argument, Let us see, what where we end up.

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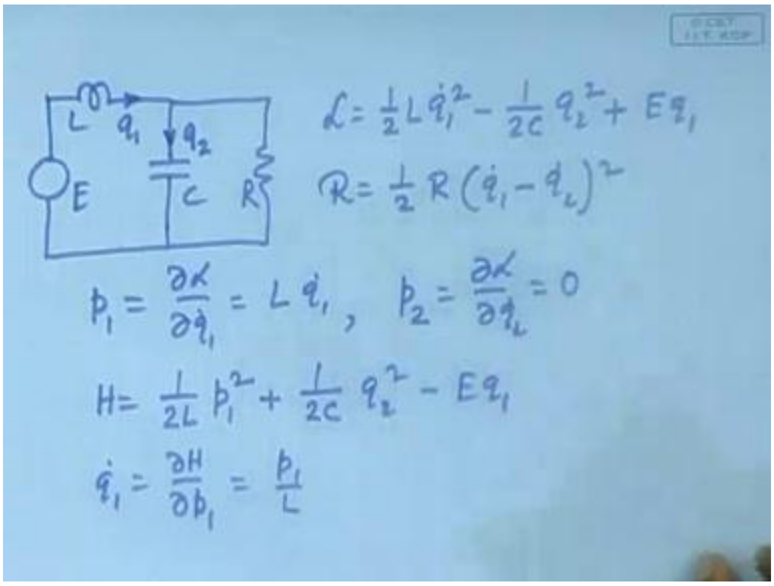

It is the capacitor here and the resistor here, instead of defining the loops, this time what we will say, we will argue that, the current or the charge through the inductor is one state variable q 1, and the charge through the capacitor is another state variable q 2, you can do that. So, we are not using the mesh current kind of argument, we are using a different argument to see whether that helps, here is your E, here is your L, C, R then can you write the Lagrangian and the Rayleigh. The Lagrangian will be in this case, half L q 1 dot square due to this, now minus due to this, it is 1 by twice c q 2 square, now it will not be q 1 minus q 2 q 2 square plus E q 1.

And the Rayleigh equation would be simply now this term, how much current flows through this, q 1 minus q 2, so here it will be half R q 1 dot minus q 2 dot square. Now, therefore, p 1 is the derivative of the Lagrangian with respect to q 1, it will be L q 1 dot, no problem. P 2 will be derivative of the Lagrangian with respect to q 2 is 0, so in this case, also we get it 0, there is no problem about it is still 0.

Student: ((Refer Time: 49:28))

Sorry, q 2 dot is 0, q 2 dot does not appear in this equation, so if p 2 is 0 and p 1 is this and then we can, we can write the Hamiltonian as 1 by twice L p 1 square and the rest remains it will be plus 1 by twice c q 2 square minus E q 1. So, now your you can either write q 1 dot is equal to simply p 1 by L from here, or you can write q 1 dot is the derivative of Hamiltonian with respect to p 1, which is same thing, so it is not really necessary to evaluate these, this automatically follow from the definitions of p 1 p 2. q 1 is this, and I will not bother about the q 2 at this stage, because here there is something wrong.

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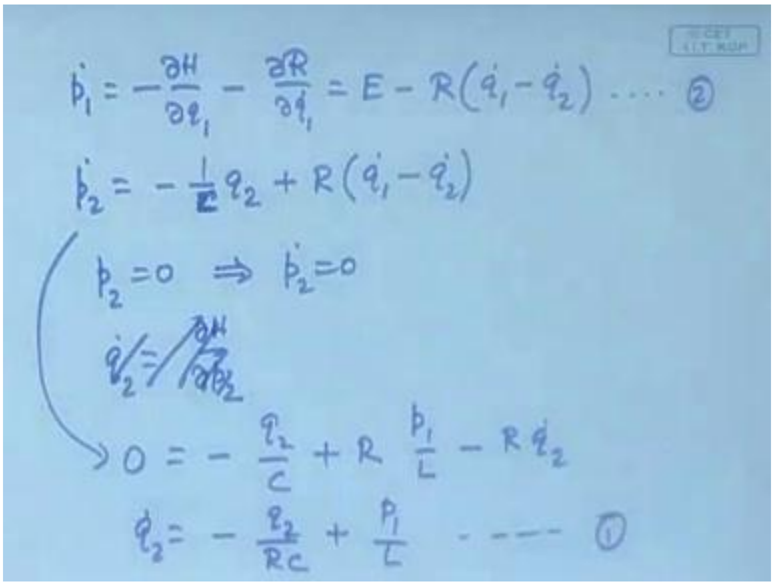

Let us do it for, p 1 dot is minus derivative of the Hamiltonian with respect to q 1 minus derivative of the Rayleigh with respect to q 1 dot is what, ((Refer Time: 51:06)) when you write the Hamiltonian with respect to q 1 only this survives.

Student: ((Refer Time: 51:10))

Only this survives yes, ((Refer Time: 51:14)) write E minus.

Student: ((Refer Time: 51:18))

Yes, the Rayleigh term survives R times q 1 dot minus q 2 dot, and p 2 dot is same thing with respect to q 2, it will be Minus 1 by c q 1 by c q 2 plus R again this term will remain q 1 dot minus q 2 dot. Now, you notice that p 2 is 0, p 2 is 0, this implies p 2 dot is 0, if p 2 is 0 p 2 dot is 0, so you have q 2 dot is what, q 2 dot is, there would be, there will be something in the right hand side wait, we could not extract it from here. q 2 dot is the derivative of the Hamiltonian with respect to p 2, so how do you write it.

Student: ((Refer Time: 52:48))

Yes, this will not give here, you have this will be 0 we know this and therefore, this is obtained. So, we will not use this, we will use this 0 is equal to minus q 2 by c plus R q 1 dot we already know is p 1 by L minus R q 2 dot, so you have q 2 dot is, you can now extract it, minus q 2 by R C plus p 1 by L, so this is how. Ultimately, what are the equations you have, ultimately what are the differential equations you have, tell me.

This is this is definitely 1.

Student: ((Refer Time: 53:55))

This is another.

Student: ((Refer Time: 54:01))

Yes, so you still ended up with 3 equations, so this actually, did not help and we need to still refine this and in fact, the proper methods, you know, that the for electrical circuits, which are pure electrical circuits, we have the Kirchhoff's laws for defining how the currents and voltages are distributed and their relationship. We will show then that can be used in order to formulate the minimum set of equations, but that will also tell us how to formulate the correct variables for this formulation. So, that is what we will take up from the next class, typically for electrical circuits, how do we, obtain the minimum set of differential equations.

That is all, thank you.