Dynamics of Physical Systems
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Lecture - 8
Obtaining First Order Equations

Towards the end of yesterday's class, we learnt on basic mechanism of deriving the first order differential equation. And we said that we will define the new set of coordinates as p i which will be the derivative of the Lagrangian function with respect to q i dot.

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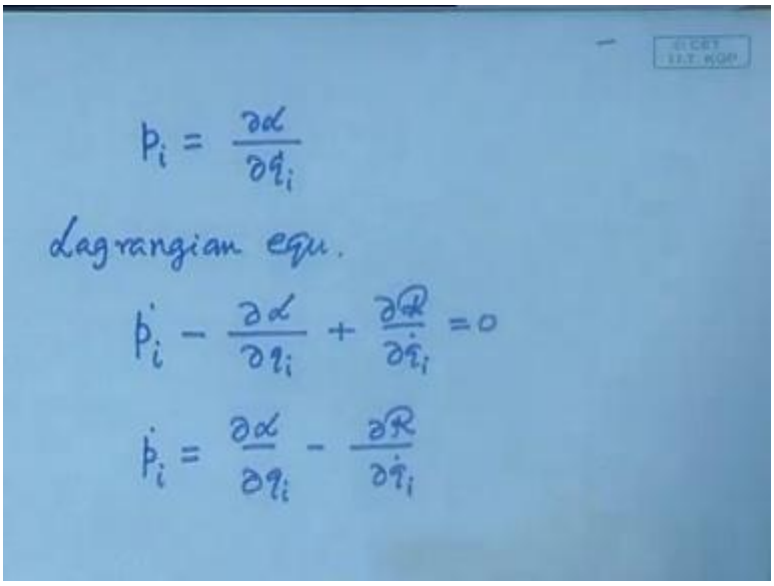

So, that is how we defined it and then if we define this, then the derivative time derivative of this term is nothing but, p i dot, so your Lagrangian equation becomes, equation become p i dot minus derivative of the Lagrangian with respect to q i plus derivative of the Rayleigh with respect to q i dot is equal to 0. We would normally express it as p i dot is equal to derivative of the Lagrangian with respect to q i minus derivative of the Rayleigh with respect to q i dot.

So, you see, here is the dotted term and the right hand side will have to be expressed in terms of the un dotted terms. So, if you want to derive the first order differential equations of any system, essentially you would need to write, q i dot is equal to something and p i dot is equal to something.

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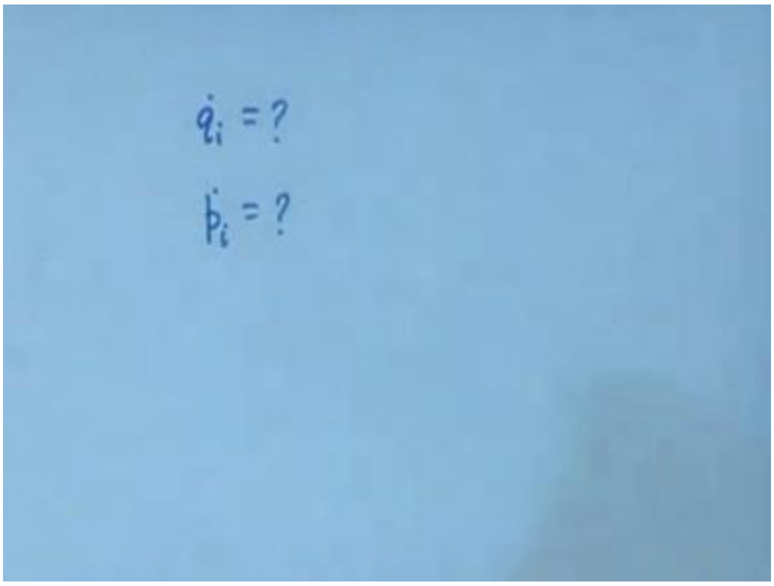

Now, p i dot is equal to something, that is obtained from this equation and q i dot is equal to how do you get it, how do you get it. You will actually get it from ((Refer Time: 02:43)) this, let me illustrate how. Let us take, let us start with you have already done the spring pendulum system you have already the Lagrangian written up, so let us start from there.

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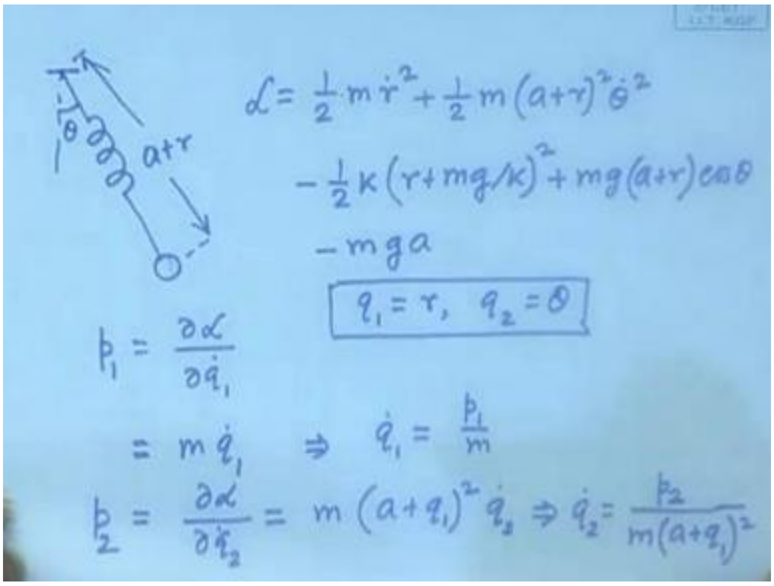

So, the spring pendulum system was, and we had defined the…

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A plus r, where a was the constant and r is the variable, and in it is term, we had written already written, the Lagrangian as, this I will not derive all over again, I will simply write, it was half m r dot square plus half m a plus r square theta dot square minus half k r plus m g by k square plus m g a plus r cos theta minus m g a, that is how it was. So, this is the Lagrangian function and then we know, that the p i is the derivative of the Lagrangian with respect to q i dot, which in this case r and theta.

So, we will write p 1 is equal to derivative of the Lagrangian with respect to q 1 dot, so q 1 was r, so it was, it will be convenient for us to in order to proceed, we will say, q 1 is equal to r and q 2 is equal to theta. So, this is, let us write things in terms of q 1 and q 2, so this is,

Student: ((Refer Time: 05:24))

M.

Student: ((Refer Time: 05:28))

Q 1 dot.

Student: ((Refer Time: 05:31))

So, that is the in the first coordinate the momentum and in the second coordinate the momentum is, it has to be done with here, so it is m,

Student: ((Refer Time: 05:58))

A plus q 1, a plus square into q 2 dot, so notice that you have independently obtain the momentum in the two directions. Now, writing the differential equation, so this equation will give q 1 dot, the dotted term in the left hand side is equal to,

Student: ((Refer Time: 06:31))

That is it, and this will give is, so you see, out of the four equation that you need to derive, two were simply obtained from this, now to the other two. The other two would be, where p 1 dot is in the left hand side and p 2 dot will be in the left hand side, so that will have to be obtained as this equation ((Refer Time: 07:16)). This equation in this case the R term is not there, so only this much.

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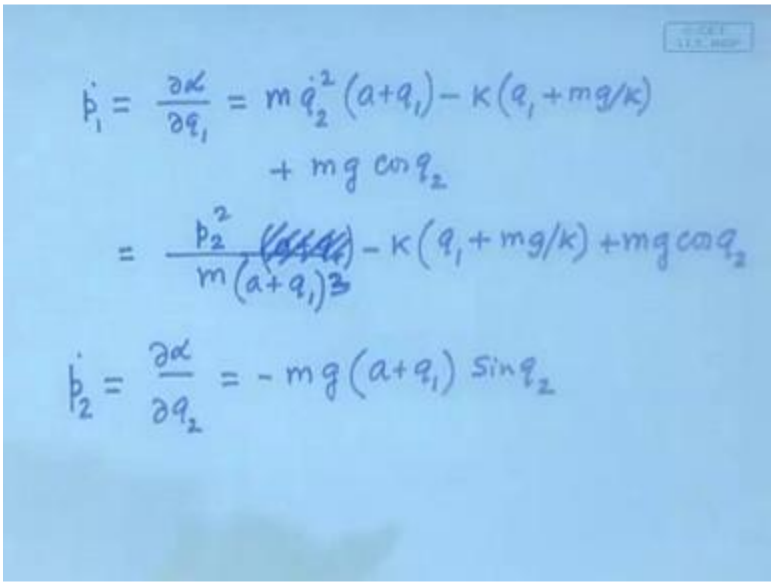

So, we will have to write, P 1 dot is the derivative of the Lagrangian with respect to q 1, which is something that we had already written up, so it is…

Student: ((Refer Time: 07:38))

One thing, now, it has to be done in terms of…

Student: ((Refer Time: 07:44))

In the right hand side, things have to be expressed in un dotted terms, in the right hand side things will have to be, without the dots, so just keep that in mind. So here, we have already written down the derivative of the Lagrangian with respect to q 1, so write it in the right hand side, so I am yet uncomfortable of this term, because here i have a dotted term, which i do not want, in the right hand side, so that can easily be eliminated by substituting this ((Refer Time: 08:55)).

So, that gives finally, the equation, so it will be, this q 2 dot is p 1 by m, so 1 m will cancel off, it is p,

Student: ((Refer Time: 09:13))

2 square here by m, and then it will continue a plus q 1 minus k, all the other terms are without dots, so…

Student: ((Refer Time: 09:31))

Sorry, yes.

Student: ((Refer Time: 09:49))

Cube.

Student: ((Refer Time: 09:54))

No, no.

Student: ((Refer Time: 09:58))

No, no, it will remain yes, so this is m g by k plus m g cos q 2, so we have obtain one equation, one first order equation without any dotted term in the right hand side, have you seen the technique. So, the first point was, that in this problem we had the Lagrangian already derived, we say that let q 1 be r and q 2 be theta, and then your p 1 is derivative of the Lagrangian with respect to q 1 dot, which yield at this, so that directly yields q 1 dot, as one first order differential equation.

P 2 was this, that directly yields the q 2 as the first order differential, q 2 dot as the first order differential equation. P 1 dot, then is obtained from this equation and in this specific problem, this term being 0, it is only this much, which we wrote down. We had already derived this right hand side this one, but here a q 2 dot appeared, which should not appear in the right hand side of a differential equation, so we substitute q 2 dot from there and then we got the final equation.

Similarly, p 2 dot is, it is the derivative of the Lagrangian with respect to q 2, which is you have already have it, it is minus m g.

Student: ((Refer Time: 11:46))

a plus q 1.

Student: ((Refer Time: 11:49))

Yes, it does not have a dotted term in the right hand side, so we are through. Do you see the four equations, so this is how, the first order equation need to be defined, if you are obtaining it from the Lagrangian equation. Let us do one problem, one circuit problem so that you are little more comfortable with it.

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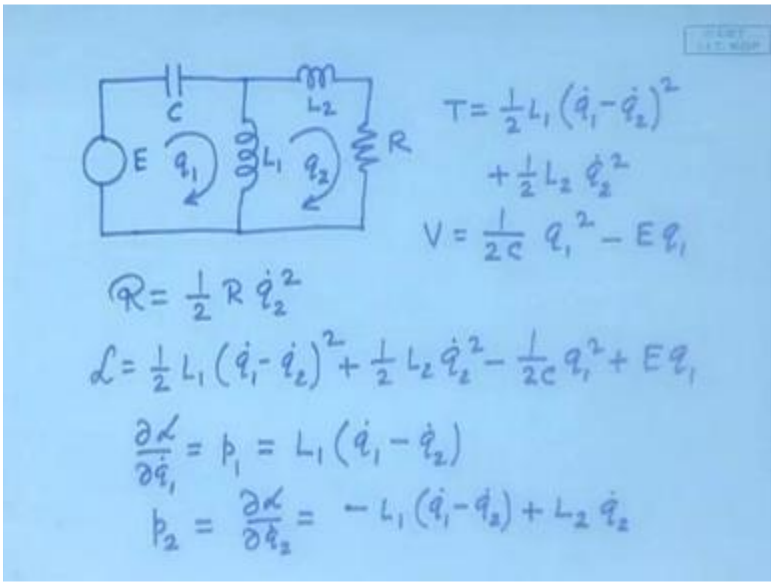

Let us see, let us circuit like this, notice that if we did not derive or did not define the ((Refer Time: 12:47)) p i is this way, then all this will have to be obtained at hock. And, getting rid of the dotted terms in the right hand side would not be, so simple as this, that is why this is a systematic procedure of doing that that is why the position and the momentum, these are taken as the conjugate variables always, so here is E, here is C, L 1 L 2 and R.

How ((Refer Time: 13:22)) go about it, first we will define the charges flowing in the loops, which are q 1 and q 2, these are the position coordinates. Now, you would say, that my T is the energy stored in this one, half L 1 q 1 dot minus q 2 dot square, plus the energy stored in this one, which is plus half L 2 q 2 dot square, the potential here V is the 1 by twice C q 1 square, also due to this, it will be minus since it is in the direction of q 1, E q 1.

The this time there is a Rayleigh term, so in the last problem it was not there, it is only in q 2 direction, so it is half R q 2 dot square. So, from here, your Lagrangian is half L 1 q 1 dot minus q 2 dot square plus half L 2 q 2 dot square minus 1 by twice C q 1 square plus E q 1, so you have it, you have it, so you can easily write the Lagrangian equation, which will now, take the form, can you just obtain it fast, the Lagrangian way. Then I will obtain it, as a first order differential equation, done so keep it in your copy.

Now, notice what we have done on the sheet, here we are writing, that the derivative of the Lagrangian with respect to the q 1 dot is p 1, which is nothing but, this you have already definitely done it. Similarly, p 2 is derivative of the Lagrangian with respect to q 2 dot which is this, now in this case, the q 1 dot and q 2 dot are not directly given, so they are mixed up. But you can see, these are two equations and you can already very, very easily extract q 1 dot and q 2 dot from these two, in terms of p 1 and p 2, cannot you, do that.

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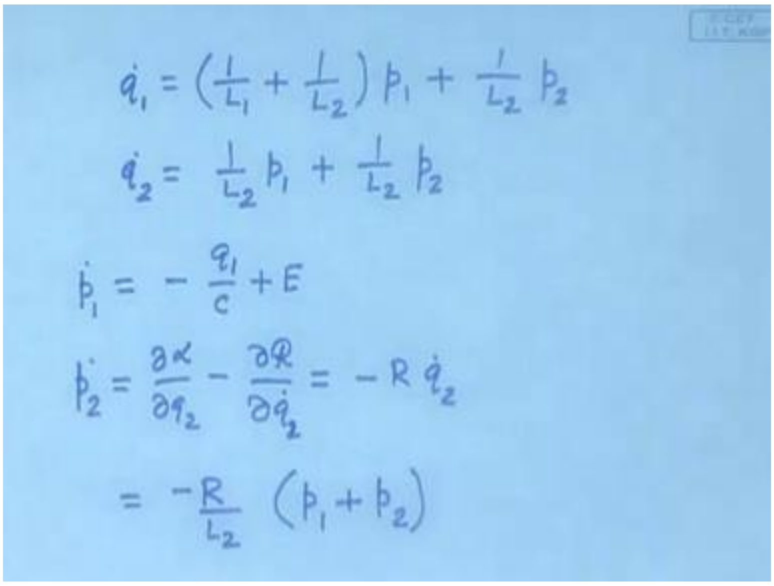

So, you will get q 1 dot is equal to 1 by L 1 plus 1 by L 2 p 1 plus 1 by L 2 p 2, and q 2 dot is equal to 1 by L 2 p 1 plus 1 by L 2 p 2, check, from here sorry, from here ((Refer Time: 18:33)) from these 2 equations, we extract q 1 dot and q 2 dot. Having done so we will proceed to write down, the equations for p 1 dot and p 2 dot, so in this case, they will have to be obtained from this equation, because the R term is there. So, p 1 dot is equal to derivative of the Lagrangian with respect to the q i q 1, that will come to be ((Refer Time: 19:33)) these two terms.

So, it will be minus q 1 by c plus E, now we have to add the derivative with respect to q 1 dot, which is not there, so this ((Refer Time: 19:54)), P 2 dot is derivative of the Lagrangian with respect to the q 2 minus derivative of the Rayleigh with respect to q 1 dot, q 2 dot is, q 2 is not there, so this terms derivative goes to 0 and this term derivative remains, so it is minus R q 2 dot. But, now you have the q 2 dot in the right hand side, that will have to be substituted, so just put it in here, so it is equal to minus R into all these this stuff, R by L 2 you can write, is it pretty simple stuff.

There is no complication this business, only if you adopt the right technique, and the ((Refer Time: 21:03)) of the technique was to properly define the additional variable. Now, let us ask this question, what did we do, we started from the Lagrangian formulation where we had, the second order differential equations, then we additionally defined, the conjugate momentum variables and then using that we derived the first order equations.

But ultimately if, the first order equations is, our objective, why not derive them in the first order, why not go through the convoluted route of first deriving the second order equation and then going to the first order equation. There exist, a technique to do that applicable under most situations, so let us see that is exactly what is known as the Hamiltonian formalism.

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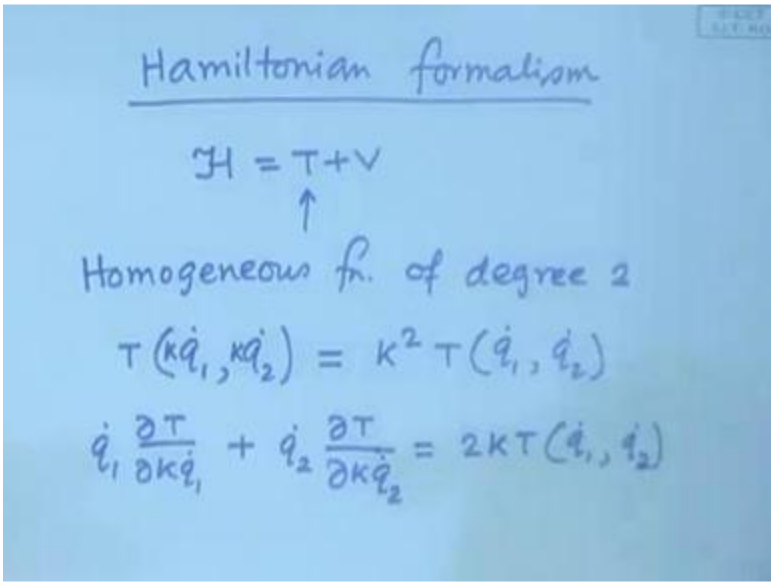

In the early 1830's Hamilton proposed this formalism and that is why it is known as the Hamiltonian formalism. In which, in the Lagrangian formalism, we define the Lagrangian function, what was it, T minus V, in the Hamiltonian form of function, formalism we will define the Hamiltonian function. And in order to understand, what the

Hamiltonian function is at this for more systems, if you read the books in classical mechanics it goes in a different way.

But, we will go through a route that is more intuitive for the engineers, because we understand what a kinetic energy is, we understand what a potential energy is, and therefore, we understand what the total energy in a system is, so the total energy system is nothing but, T plus V. So, we will say that I define a function, I will write it as script H is T plus V, and then we will try to formulate the, Hamiltonian try to formulate the equations in terms of the Hamiltonian function, but presently we have define, what is known as the total energy function, as yet I am not talking about the Hamiltonian function.

So, this is more intuitively clear to us, that I am talking in terms of the total energy in the system T plus V. Now, there is a specific property of the T term, T is the kinetic energy, it has a specific property, that it is in mathematical term it is called the homogeneous of the degree 2. Now, what does it mean by the homogeneous function of degree 2, what it means is that if I ok, it was T would be a function of what, suppose it is two dimensional system, it is function of q 1 dot and q 2 dot, it is a function of q 1 dot and q 2 dot.

So, it is normally and q 2 dot, so there are two variable velocity terms q 1 dot and q 2 dot and this fellow is a function of that. The question concerns, what kind of a function is it, it is that kind of a function, so that if I multiply the q 1 dot and q 2 dot both by some number k that means, I just scale it up and down, by some number k. Then the T,

Student: ((Refer Time: 25:22))

Yes, it will be reflected as k square terms times T, so any function that obeys these property is called a homogeneous function of degree 2. And there are some mathematical rules, that this homogeneous functions of degree 2 follow and we will use that so what is it, suppose I, now differentiate this equation with respect to k, what do you have, we will first say, I am differentiating this with respect to k, then first I will have differentiate with respect to k q 1 dot and then k q 1 dot with respect to k, so what we have, we have q 1 dot times derivative of T, with respect to k q 1 dot, that is the first one chain rule.

So, the second term will be derivative of T with respect to k q 2 dot, so I am differentiating with respect to k using the chain rule, it is a function of two things, so I first differentiate with respect to this and k q 1 dot with respect to k, that yields q 1 dot. And similarly, the next one is equal to in the right hand side, what do you have, twice k T. Now, this is again a property of the homogeneous function of degree 2 this equation, and k being arbitrary, we could choose any k still this equation will be valid, if we choose k is equal to 1, then also this equation will be valid, choose k is equal to 1.

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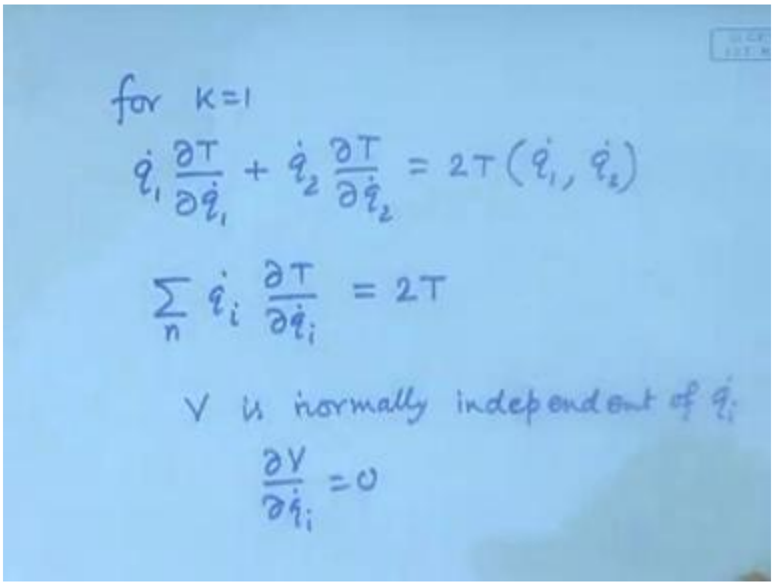

So, for k is equal to 1 it will become q 1 dot, derivative of T with respect to q 1 dot, plus q 2 dot, derivative of T with respect to q 2 dot, will become equal to twice T q 1 dot q 2 dot. So, if this is true for two variables, we had assumed q 1 dot and q 2 and q 1 these are the two variables, if there are N number of variables it will be the same, after all you have to do is to, expand this in a longer chain, so in general we can write, that this left hand side will be, summation of q i dot, how many n, and this side is twice T.

Now, this result is known as the Euler's theorem that goes by the name of Euler, now notice, that this term T, so we already know, that the V term is normally independent of potentially it is not dependent on the velocities, at least normally. We will come across situations, where it can be and in those cases we will have to exercise some caution, in deriving the differential equations, but for many system, most systems this is true. If this is true then if this is so then we can write the Lagrangian here, that is what we did earlier.

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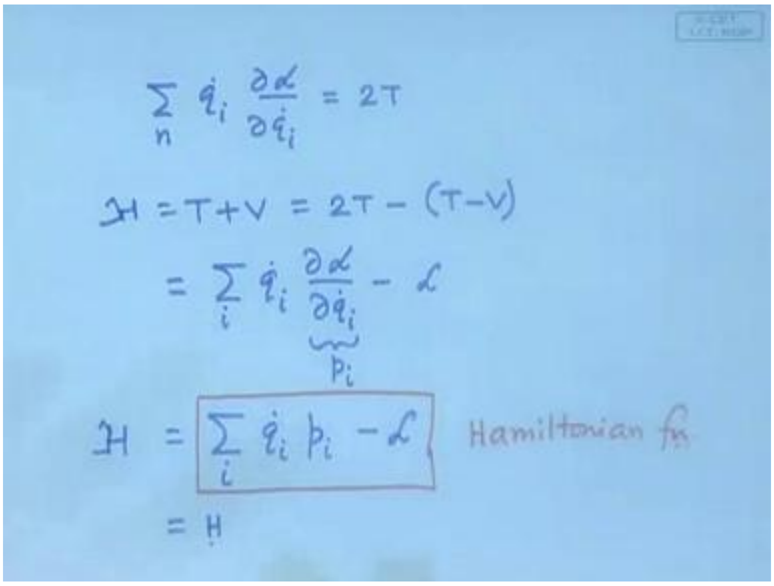

So, this yields summation over n, q i dot derivative of the Lagrangian with respect to dot is equal to twice T. So, this result we will use, just keep it in hand for now, now we had said that we have we want to deal with, the total energy function H, which was T plus V, We will write it as twice T minus T minus V, twice T we know, what it is, T minus V we know what it is. So, substitute you get, some over I, q i dot derivative of the Lagrangian with respect to q i dot, that is what is the twice T term minus the Lagrangian itself.

And, this term also we know, what is this, the pi, this term is nothing but, pi, so we can write, q i dot p i minus L in the right hand side, now it is so happens, that this term, this term in the right hand side is known as the Hamiltonian function. Hamilton actually define this function and then showed, that it is actually the same as the total energy function T plus V. But we had started from the total energy, because for us it is a more intuitively appealing quantity and we showed that the opposite, the total energy is nothing but, this function, but either be it is a same thing.

Now, notice this, we have this equation as the total energy.

Student: ((Refer Time: 32:40))

No not yet, there has been no use of the Rayleigh function at all, because Rayleigh function does not contribute to the total energy. Energy is nothing but, kinetic energy plus potential energy, so Rayleigh function is entirely different affair, it does not contribute to the total energy content of the system, just two components. We will introduce the Rayleigh in the right place, do not worry, we are not ignoring that so this is the Hamiltonian function.

Now, so we have two things in the left hand side we have the H function, and in the right side we have the Hamiltonian function. And in our nomenclature, we had called this term as ((Refer Time: 33:37)) something like script H, because this in concept represent the total energy and the Hamiltonian H is normal H is this. We just showed that the script H that we started with is equal to the Hamiltonian H, now what is this, this total energy dependent on, total energy is dependent on the momentum the position and the time.

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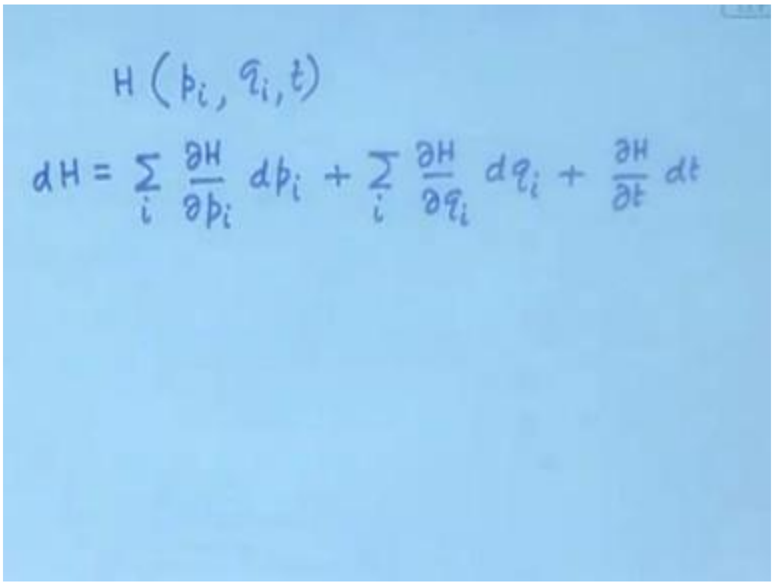

So, we can write the total energy H is a function of p i q i and time, that stands to reason, the momenta dependent on the, the kinetic energy dependent on the momenta and the potential energy dependent on the positions, and things can be dependent on time also. If that is so then if you take a differential of it, we can write d H as, if it is a d, it is dependent on three things, then we can expand it by the chain rule to obtain some over i, it will be the first partial derivative of H with respect to p i.

Then d p i plus a partial derivative of H with respect to q i d q i plus, this time it will not be a summation, these two are summed over, this will be just the term, which is the partial derivative of H with respect to t and this will be d t. So, this is how, this term expands, we have seen that this is also equal to this term, so we can also expand this term in the same way, let us see, what it yields, let us keep it here.

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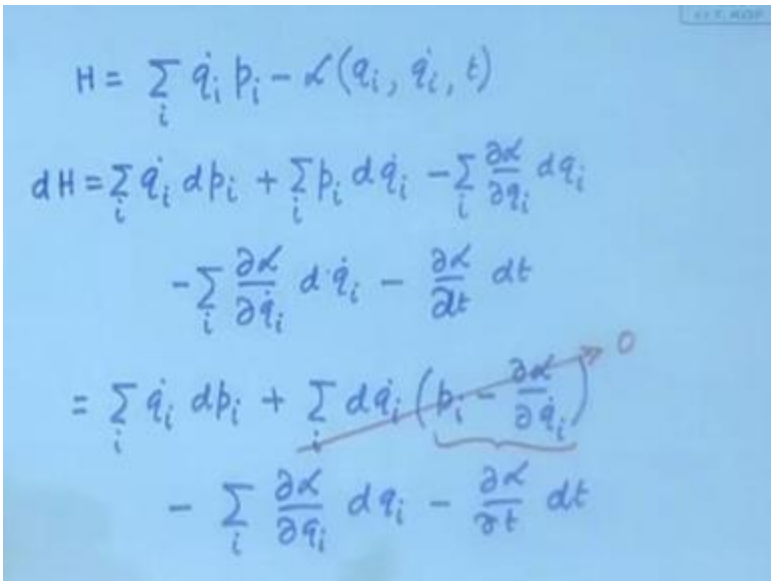

We have seen that the H is also i q i dot p i minus the Lagrangian and now I want to write it as H, how will I write it. First, you have this term here, it is dependent on q i, dependent on p i, dependent on l, so we have to write, taking the derivative of this it will be q i dot d pi and that will be summed over i plus, again pi d q i dot, that is also summed over i. These two yield a derivative of here and now this Lagrangian, Lagrangian we know is a function of q i, q i dot and time, we have already seen that.

So, it will have to be written as minus derivative of the Lagrangian with respect to q i, it will be d q i also summed over i, minus derivative of the Lagrangian with respect to q i dot d q i dot. Again summed over i and time, it will not have to be sum summed over, it is derivative of the Lagrangian with respect to t sorry, that completes the story then notice there are two terms involving d q i dot and we can take them common.

So, if you take them common you have, the first term q i dot d p i, plus first let us take them common, it is summed over i d q i dot and then inside the bracket it will be p i minus delta Lagrangian of that is q i dot, then the rest of the terms remain. Now, notice what is this?

Student: ((Refer Time: 39:21))

Will be 0, yes because this is nothing but, this, so this whole term goes to 0, simplifies our affair, so it is only this, this and this, so write it down.

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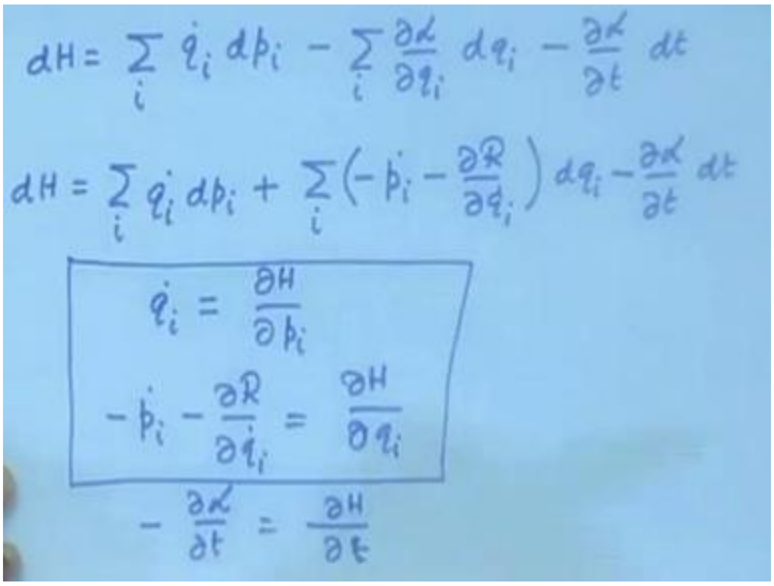

D H, now yields i q i dot d p i minus, now we already have ((Refer Time: 40:16)) this equation, and from here we know this term. We had already obtain this, and from here we know this term, so we let us write down, your derivative of the Lagrangian with respect to q i is, now here we are including, can you see that

Student: ((Refer Time: 40:41))

We are including this yes, so it is p i dot plus dot, so let us substitute it here, you have d H minus this term, it will be, let us start with plus and i, it will be minus p i dot, minus derivative of the Rayleigh with respect to q i dot, it will be d q i minus Lagrangian del t d t, well, well, well. Now, you see, we had we have obtained this, from the right hand side and this was another equivalent expression, we had obtained, in this case, we had stated that H is dependent on p i q i t and we had expanded this.

And in this case, we had written that there is also the Lagrangian function and therefore, we had expanded that in one case we had obtained this, another case we had obtained that. Now notice, that they are almost the same, only here is a term, they summed over i p i term, summed over i q i term, summed over i t term, obviously this term and that term should be the same, this term and that term should be the same and this term and that term should be the same.

Do you see, how simply it becomes, all these mathematics ultimately then yields, three very simple results, the first thing is q i dot will be equal to then this, minus p i dot minus will be and normally in books you will find, these terms written in the left hand side and these terms written in the right hand side. But I have chosen to write it this way, because ultimately, what we want to extract are these two, we will do that but let us understand this, this is the differential equation first order, this is the differential equation first order, this is not a differential equation, this only say that if the Lagrangian function is independent of time, the Hamiltonian function will also be independent of time.

And, if they are dependent on time, they are dependent this way that is all, but this is not a differential equation, therefore it does not really define the dynamics. We are primarily then interested in these two equations.

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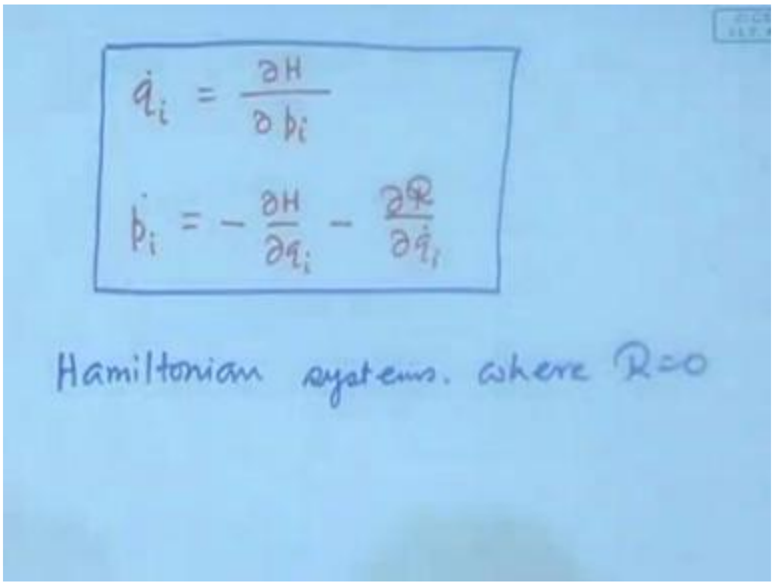

And these two equations immediately yield, q i dot is equal to derivative of the Hamiltonian with respect to p i and p i dot is from here, I want I am trying to extract it, it is this minus, that is it, that is what the Hamiltonian equations are first order nice shape. We will illustrate, how to apply this two physical systems, definitely that is what we are doing always, we derive a formalism and illustrate with it large number of examples, do not worry about it.

But notice one thing, that if this term is ignored that means, if some system does not have this term for example, this spring pendulum and stuff. Then it takes a very nice symmetrical form, can you see that very nice form, q i dot is nothing but, the Hamiltonian derivative of the Hamiltonian with respect to p i, and pi dot is nothing but, the derivative of the Hamiltonian with respect to q i with a negative sign, that is all. Very nice set of equation that is why, this has intrigued physicists for a long time and many physicists had devoted their life time studying this, nice set of equations.

And, if this is ignored, what you have that kind of systems where, such equations will be valid are called the Hamiltonian systems. So, the Hamiltonian systems are that is it, whether the Rayleigh term does not exist, which means there is no dissipation or friction, then you have got Hamiltonian system. Remember such systems are actually very important, supposing one a scientist is studying the motion of mars obviously, that is a Hamiltonian system, the friction dissipation is very small.

If one is studying the motion of electrons in a atom, Hamiltonian system, we do not have any concept of friction there. So, these things are not really imagination, this being ignored actually happens, but for our purpose, since we are primarily interested in engineering systems, we will include this, except for a few exceptions, which are also you know not really coming from engineering, but our exercise to understand such systems, this spring pendulum and all.

But for most engineering system there will be some kind of friction or dissipation, so we will include this. Remember in other books you will not find the equations with this term, only I am including it, so that you can derive this, equations using this formalism, so we have this equations and then we need to understand, how to obtain the equations with this. Let us go back to the circuit example, that we take took today, where was it, circuit example was here, here was the circuit example we took...

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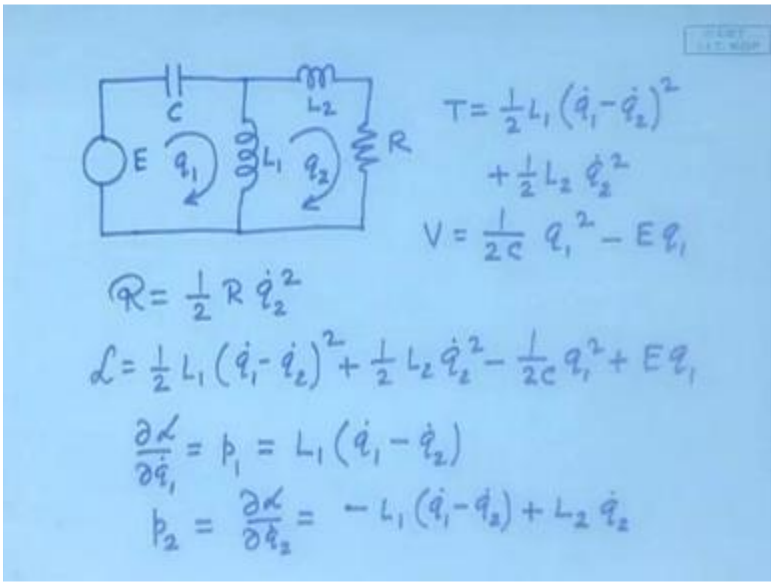

So, here we have already written down the T, we have written down the V and we have written down the R. Notice, that in the Hamiltonian formalism, we need to write down the right hand side, and right hand side means, H as derived in as has taken derivative in terms of p i, H has taken derivative in terms of q i.

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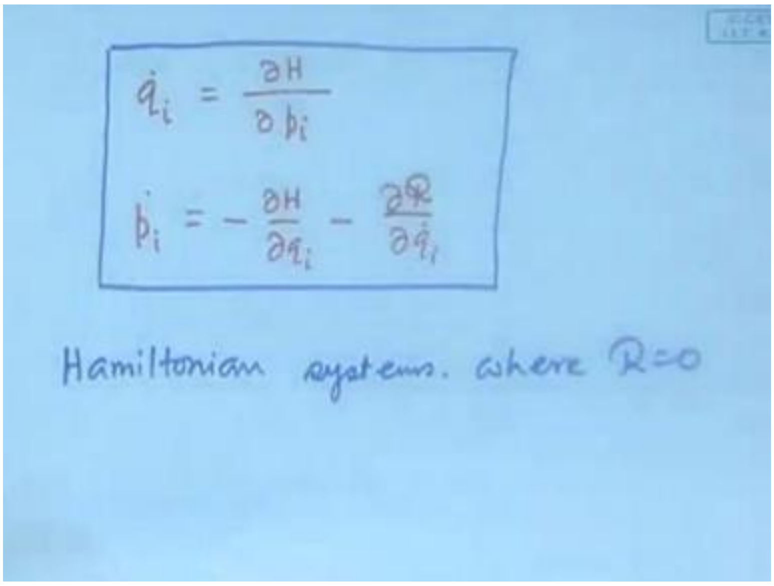

So H has to be expressed as, it has to be expressed as p i, the problem is, that if I simply write it as T plus V, it is expressed in terms of q i dot not as pi, so the first task will be, to express it in terms of p i, so that is what you have to do first. In this case, in this particular example H is T plus V, and let us write down T plus V first, using the expression for the T and the V ((Refer Time: 49:49)), that we have already written. We will say, half L q 1 dot minus q 2 dot square, plus half L sorry, this is L 1 this is L 2, q 2 dot square and the other terms will be now plus, remember T plus V, 1 by twice C, q 1 square minus E q 1.

Now, we need to express this in terms of the pi's, now we had already obtained this, and this, so we can substitute, so if you substitute this here, we get, where is it. This term is p 1 by L 1, this term, so we can easily express, half L 1 p 1 by L 1 square, plus half L 2, this is p 1 plus p 2 by L 2 square plus the rest remains the same, twice C q 1 square minus, we had done, so we will just simplify it, to twice L 1, because. L 1 goes square here, P 1 square plus 1 by twice L 2, p 1 plus p 2 square plus 1 by twice C q 1 square minus E q 1. So, that is how we have derived H in terms of the p's and q's, the rest is algorithmic, the rest is algorithmic.

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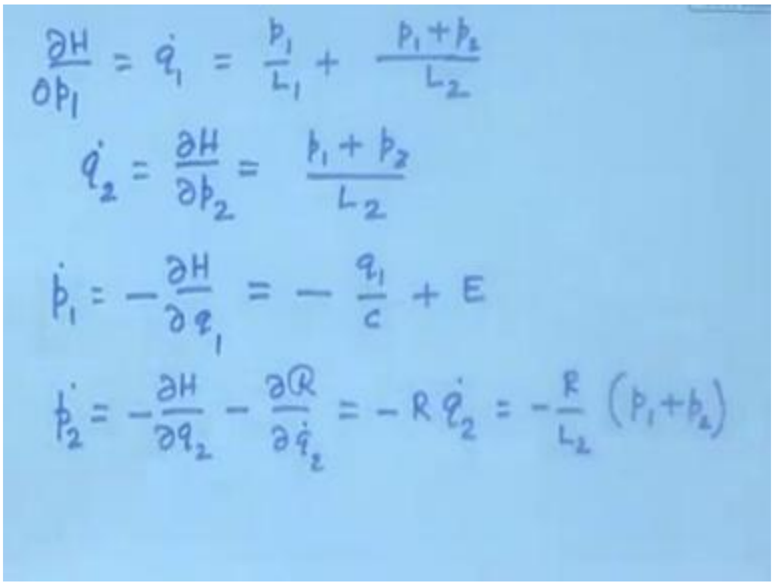

We will say simply that the derivative of H with respect to p 1 is nothing but, q 1 dot simple, and this immediately yields p 1 by L 1 plus p 1 plus p 2 by L 2. Similarly, q 2 dot is derivative of the Hamiltonian function with respect to p 2, which is p 1 plus p 2 by L 2, just check. Now, p 1 dot is minus derivative of the Hamiltonian with respect to q 1, which is these two terms appear, it will be minus q 1 by c plus E done, P 2 dot is well, in the sorry, I did not include the Rayleigh term, but in any case the Rayleigh term, does not depend on q 1 dot, so even if we should have written it here, we are not wrong.

And this is minus derivative of the Hamiltonian with respect to q 2 minus Rayleigh with respect to q 2 dot, that becomes, this is 0 this remains minus R q 2 dot and then you substitute minus R by L 2 p 1 plus, so that is how you write down the differential equations. Simple procedure can be applied and can be applied to most systems, most in the sense that we had made some assumptions, while deriving the Hamiltonian equations. If some system violates them, then we have to exercise caution, otherwise this simple procedure works, we will continue in the next class.