Dynamics of Physical Systems

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Lecture - 06
Using the Lagrangian Equation to Obtain Differential Equations
(Part - III)

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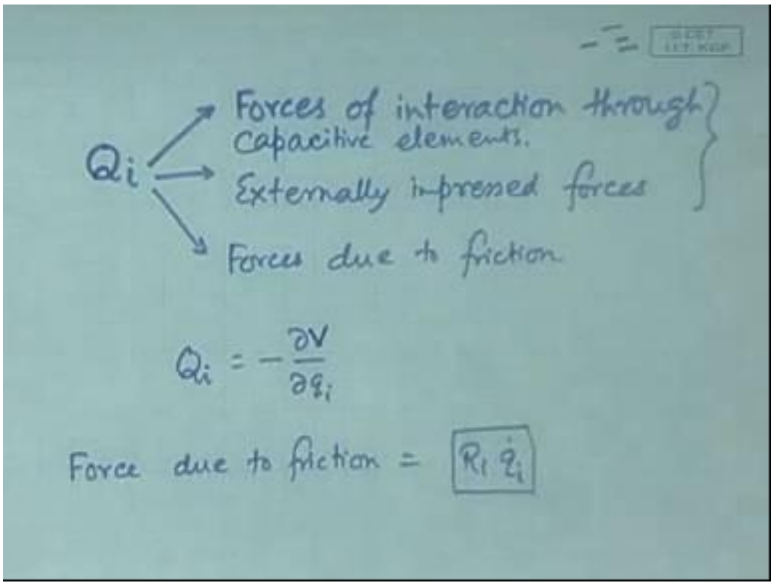

Earlier when we were developing the equations for the Lagrangian' s formalism. We said that the generalized forces were expressed as Q i and that has three components, three possible types of forces. One the forces of interaction between mass points through capacitive elements, so of interaction through capacitive elements. The second possibility is externally impressed forces, so and third possibility is the forces due to friction. Out of that these two we say that these two can be obtained from a suitably defined potential function, so that is how we proceeded.

And when we deal so we wrote the Q i is minus, so that is how we proceeded. So, this was just to recapitulate where we were, so far we have only taken into account these two types and we have not yet accounted for the forces due to friction. So, that is what we will try to do today. So, the objective today will be somehow to include the forces due to friction into the Q i term, but there is a problem here.

The problem is that while the other forces could be obtained from a potential function by partially differentiating the potential function with respect to a the generalized coordinates Q i. Obviously that cannot be done for the friction why, because the frictional force say I have something moving against say this one, is moving against a surface like this.

Then the forces due to friction if it is not a dry friction, then the frictional force is proportional to what the velocity. Therefore there is no way that you can express that as a function of V. Because, after all what you are what is it is what is dependent on it is dependent on the velocity not the positions, so that is the problem. So, in general if the coefficient of friction is say r and the speed of motion is Q i dot, then I know that the force will be R i q i dot.

So, force due to friction where obviously, i is the i th generalized coordinate i th direction in that direction the velocity is q i dot and on that we have to multiply the frictional coefficient. So, this is the expression, obviously that cannot be obtained like this. But, still we would like to we have already seen there are many advantages of the Lagrangian formalism which we do not want to give up. So, somehow this feat has to be obtained that we would not include this into the Lagrangian formalism.

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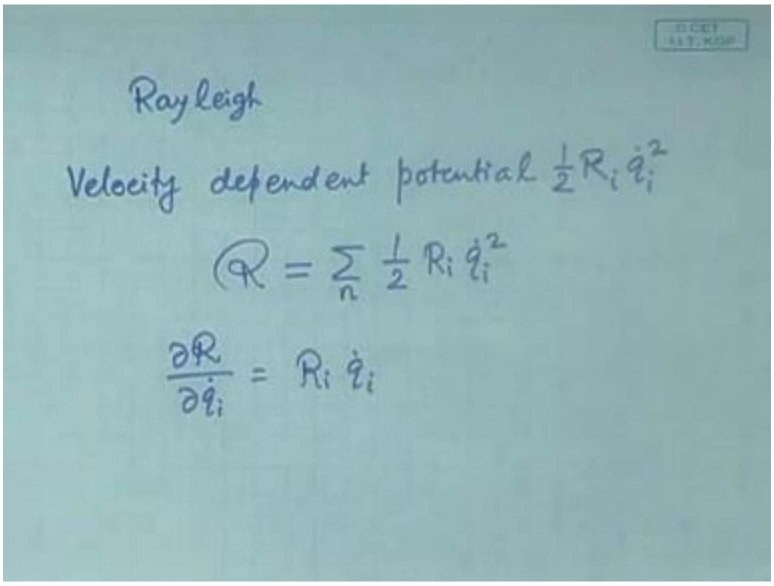

Rayleigh proposed that, this can be done by defining a new type of potential which is obviously velocity dependent. So, velocity dependent potential, potential that would be expressed as half R i q i dot square, along each of these directions. So, total Rayleigh potential is I will express the R Rayleigh potential, in order to distinguish it from this force coefficient R by some kind of a script R. That is equal to summation over a number of generalized coordinates.

That was the proposition of Rayleigh, therefore this is called a Rayleigh potential. So, what are the properties of this potential, this potential is firstly not dependent on the coordinate, it is dependent on the velocity. The second property is that if you differentiate this with respect to q i dot what you get, that is what you wanted. We want to obtain wanted to obtain the force and that is what we have obtained. Which means, that this Rayleigh potential when partially differentiated with respect to the generalized velocities give the force.

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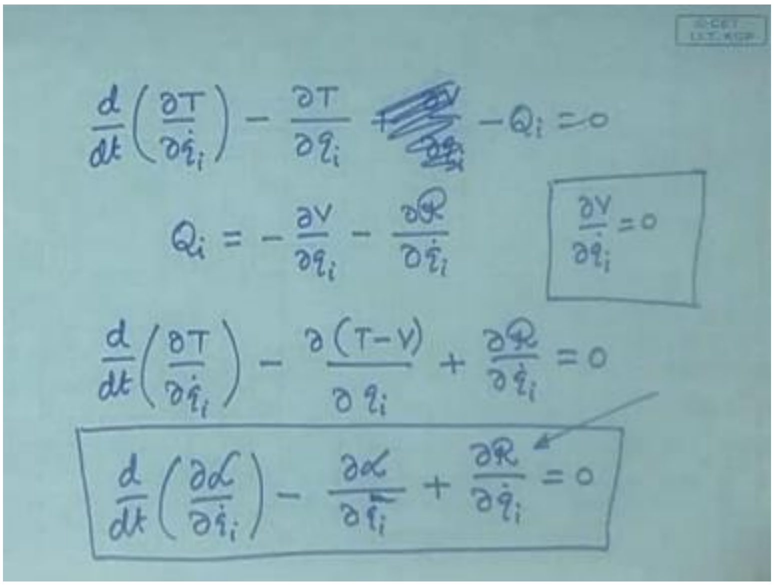

Now, recall how we proceeded to obtain the Lagrangian function at some stage, if you go back to the page in which you have written down, where I was deriving the equation, there was one stage where you have d d t of partial derivative T with respect to q i dot minus partial derivative of T with respect to q i plus partial derivative of V with respect to q i. At this stage it was minus Q i equal to 0, and then we had substituted this by this, so this is where the generalized force term appears. And so long as there is no dissipative term, no friction term, no resistive element in a electrical circuit, this term only contains.

It can be obtained from by differentiating the potential function V, that is how we proceeded. But now, we find that in the new framework where we need to include the resistive things, then you Q i earlier was expressed as this only. Now, it has to include in addition, so if this is the new term, that means this was due to the conservative forces and this is the non conservative forces.

Then these together give you the Q i term, substitute it here you have d d t of, here we have the and here will be ((Refer Time: 09:43)). And then we had argued that since the v does not depend on q i dot, therefore I can also write this as the Lagrangian term. So, we will continue with that argument, we will say here is the Lagrangian minus here is the Lagrangian, this is q i plus the Rayleigh term with respect to q i dot.

So, this will now be the modified Lagrangian equation that takes into account, the dissipative forces here this, this is the additional thing. By the way before we proceed I would draw your attention to something that we have said, which sort of one tends to take for granted. We said that this to this is possible only when, I will write it here only when, because actually it was the partial derivative of T with respect to q i dot. But, we had replaced it by in order to simplify the notations by the Lagrangian with respect to q i dot.

And that assumes that the other term is 0, now naturally these will be valid when this is true. I will site you some examples where this is not true there can be situations like that. Now, when that is observed, then you will have to take this equation into consideration not this, just this pointer just keep in mind. Fine let us now see a few examples of how this could be done, this can be done in a practical system.

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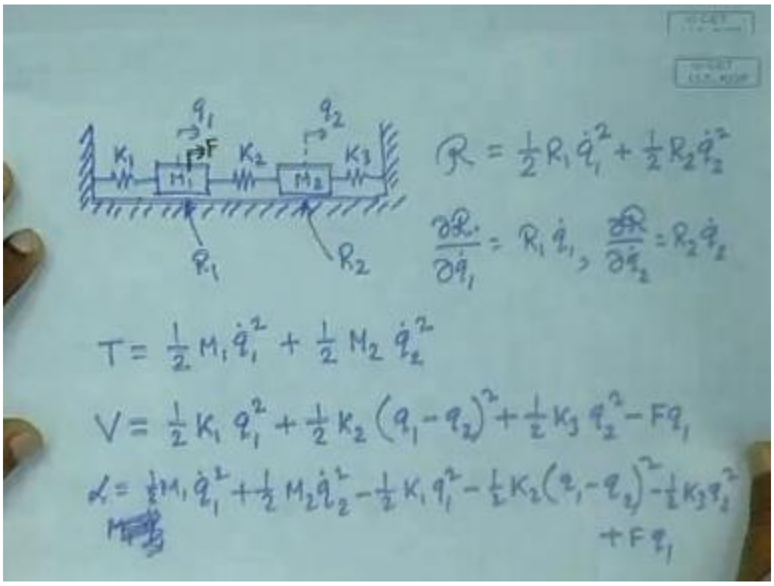

First consider, we had already considered this system a two dimensional oscillator, there are two position coordinates q 1 and q 2. We had said this was K 1 this was K 2 and this was K 3 and this is the valve, this is the valve and this is the base. Earlier when we are solving this problem, we said that these two surfaces are perfectly frictionless surfaces, but now let there be friction. Let us see how would you do it. Yes…

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Here.

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So, his question let me repeat for the sake of the microphone. His question would be was that, since if you differentiate it you get this, therefore should it not be plus, is that the question yes. Now, you notice that ultimately we want to get the signs right, signs right means here is the equation in which the force must appear in the proper direction. Now, in order to have this force appear in the proper direction, since this was negative, therefore I had to have this as positive, this as negative.

So, that ultimately we get it positive, this will be clear, when we actually solve the problem, and then you will be able to check whether this particular force term has the right sense or not. If it has then this is if it does not have it is wrong, let us check that. So, we had come to this problem and since we had already done this except for these two, suppose here the frictional coefficient R 1 and here the frictional coefficient is R 2.

Then we had solved the problem except these two, so just recall that, and then let us add otherwise we will unnecessarily lose time writing things. So, we had already written the kinetic energy, the potential energy, they remain unchanged. Only what is added is the Rayleigh term, which is half R 1, R 1 is acting along the q 1 dot direction, so q 1 dot square plus half R 2 q 2 dot square. So, that the dou R dou q 1 dot is R 1 q 1 dot and dou R dou q 2 dot is equal to R 2 q 2 dot.

Then after that we just recall what was our expression for the T and V you already have it, let me write down, so that we can derive the whole thing. T was half this was M 1, this was M 2, M 1 q one dot square plus half M 2 q 2 dot square that was the expression for the kinetic energy. The potential energy expression was, potential would be for this spring, this spring, this spring and no gravitational potential. So, it was half K 1 q 1 square plus half K 2 q 1 minus q 2 square plus half K 3 q 2 square.

This would be the potentials due to the three springs, at that time we had considered there is a application of force where was it on this mass. So, there was the force, that force will be since it is in the same direction as q 1, therefore I would have to write it minus F q 1, that is how it was. And now you have included this new term and now do the proper differentiations and write these equations, do this. This will have to be written in both q 1 direction as well as the q 2 direction.

So, is it visible yes, I can see that in the q 1 direction I will write the Lagrangian differentiated with respect to q 1 dot will be only this. So, M 1 q 1 double dot as the first term, the second term would be this, where I will be interested in, let me write down the Lagrangian else it becomes problematic. This minus this half M 1 I will write it later, so that was the Lagrangian function and Rayleigh function is there.

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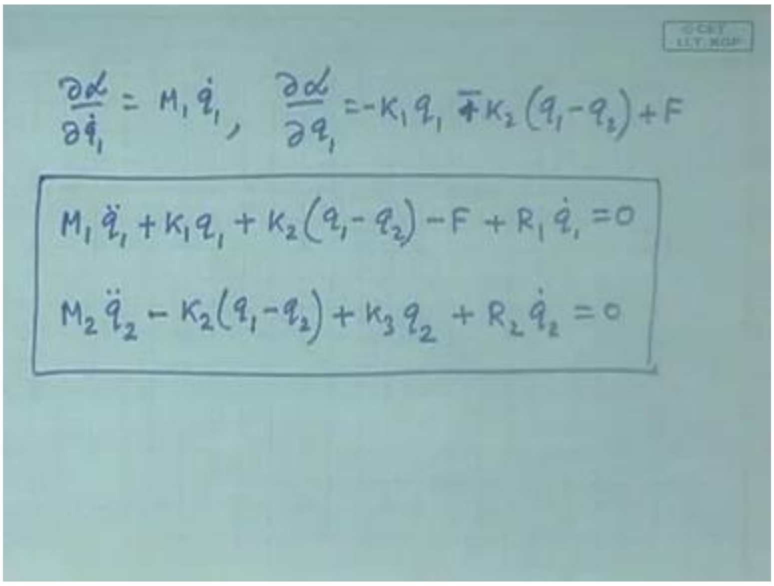

So, if you now differentiate in the first direction you will get, in the q 1 direction you have this, K 1 q 1 plus K 2 no. And the Rayleigh terms derivative was this ((Refer Time: 20:12)). So, my final expressions becomes M 1 q 1 double dot plus K 1 q 1 plus K 2 q 1 minus q 2 minus F, then I have to include this term plus R 1 q 1 dot equal to 0. Now, let us see is it right, first let us write down the second equation also.

The second equation would be M 2 q 2 double dot minus K 2 q 1 minus q 2 plus K 3 q 2 this will not be there, then plus R 2 q 2 dot. So, you see these are the expressions and let us get back to the picture can you see. Force, if you take the other side force is acting in this direction and this is moving in this direction, so force is equal to mass into acceleration here. Force will have F in this direction minus K 1 q 1 pulling in this direction K 2 q 1 minus q 2 in that direction.

And this fellow, so if you bring it to this side F will be positive and this fellow is negative, so this is acting in the direction opposite to the force. This fellow is moving in this direction, so this is also acting in this in this direction, so that has got the sense. So, these are the final two equations for this coordinates.

I would later consider how to handle this, tackle this, how to solve those things we will handle later. For at this stage we are only obtaining the differential equation, so this is one good example. Let us for this system can you visualize what will be the electrical equivalent, we have already seen there will be two resistances. So, we have already seen that the representation of this would be the two masses will be represented by inductors.

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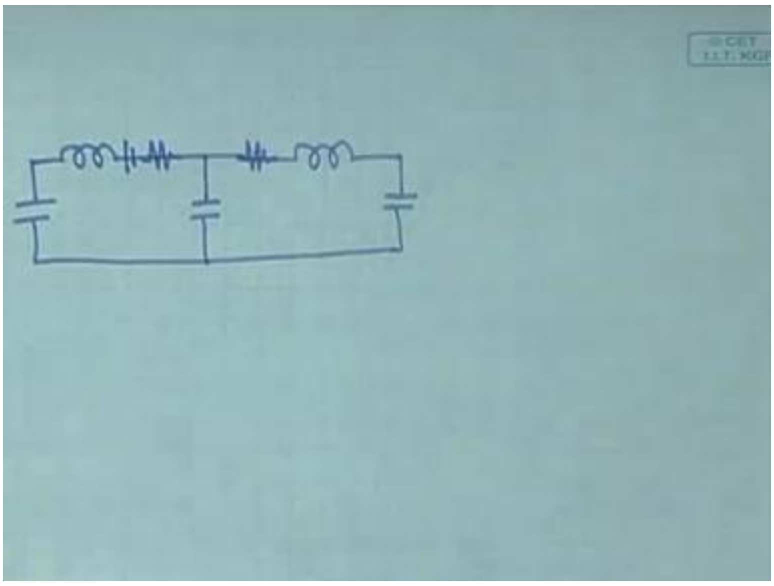

The springs will be represented by capacitors and this force will be represented by battery, and there is a resistance that shares the same motion with this mass the inductor. So, you have, so this is the equivalent of this mechanical system. If you obtain the equations for this electrical system you will got exactly the same equations, only the K and 1 by C will have to be interchange, that is all. And let us solve one typical electrical circuit problem to illustrate how this may that helps in obtaining the electrical circuits.

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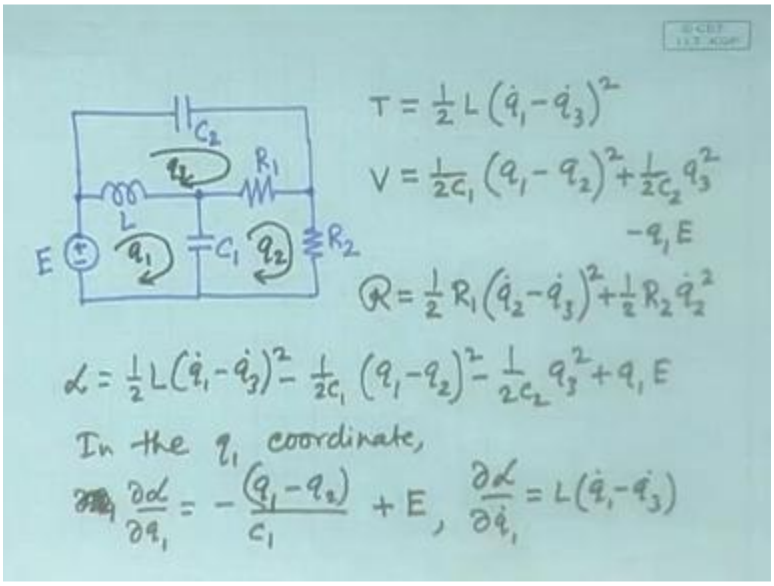

Let there be a resistance here, let there be another resistance here, let there be capacitance here and let there be a voltage source here, the differential equations of this system that is what we would like to obtain. This is our E, this is L, this is C 1, this is C 2, this is R 1, this is R 2, first how would you decide what are the generalized coordinates. The minimum number of potentials, potential coordinates that we can identify, no not potential charge.

Because the equivalent of the position is charge, so I want to identify what are the independent directions of flow of charge. And from the first circuit theory course you have learnt, that if there are there is a circuit like this with you have to first identify the windows or mesh. And the minimum number of variables necessary would be the number of meshes, so it will be very convenient to identify the three meshes like this.

Then, we will say this is our q 1 direction of flow, this is our q 2 and this is our q 3. Once we have done that we can then proceed to write the kinetic energy, and the potential energy of this circuit. What will they have first they kinetic energy T, kinetic energy will consist of the energy stored in the inductor and the energy stored in this inductor. And the charges flowing are q 1 and q 2, therefore the current flowing through this is q 1 dot minus q 3 dot, so this is what produces the stored charge.

So, you have half L q 1 dot minus q 3 dot square and that is the only place where kinetic energy could be stored, the potential energy is the energy stored in the two capacitors. So, let us take this one first, it will be half 1 by 2 C, C is in the denominator 1 by 2 C this is C 1, charge flowing is q 1 minus q 2, so q 1 minus q 2 square. In this one it is C 2 it is q 3 plus there is something here, this is the externally applied force which is acting in the positive direction of q 1. Therefore, you will have to say minus q 1 E.

So, that is the potential energy and the Rayleigh function would be here, for this one it would be half R 1, then I have to identify what are the currents flowing through this. It is q 2 dot minus q 3 dot, q 2 dot is flowing this way and q 3 dot is flowing in the opposite way, so q 2 dot minus q 3 dot square. And this one plus half R 2 q 2 dot square, the moment you have written this the rest is trivial, so all you have to do is to differentiate these three functions.

So, let the write their Lagrangian, it is I will write fast half L q 1 dot minus q 3 dot square minus 2 C 1 q 1 minus q 2 square minus 2 C 2 q 3 square plus q 1 E and Rayleigh is this. So, the first direction q 1, in the q 1 coordinate, what do you have minus q 1 by C 1q 1 minus q 2 by C 1, is L into q 1 dot q 3 dot.

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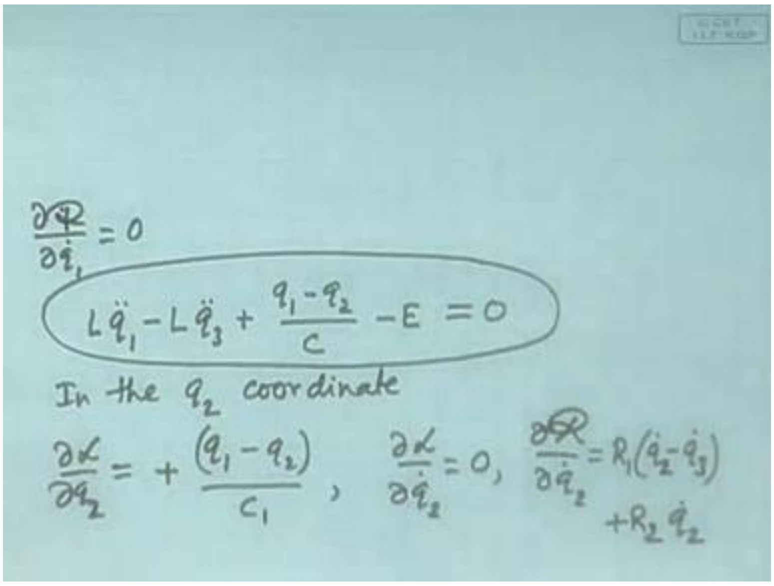

And your Rayleigh function is what is your Rayleigh function, yes it is independent of q 1 0. So, now if you write the equation it will be, so the equation is d d t of this which means L q 1 double dot minus L q 3 double dot. Then this one negative of this one q 1 minus q 2 by C minus E, this term Rayleigh term is 0, therefore this is ok. Let us do it the direction of q 2 also, because that is that illustrates the use of the Rayleigh term.

In the q 2 coordinate, what we have q 2 is this one we are differentiating this one, we are can you see yes, this one we are differentiating with respect to q 2, so you have plus and nothing. Lagrangian with respect to q 2 dot is 0 and delta R with respect to is, it was here yes it will be, it will be R 1 q 1 dot minus q 3 dot minus. And the other term will also remain, now if you write down the equation it will take the form. This is 0, therefore d d t of this does not exist.

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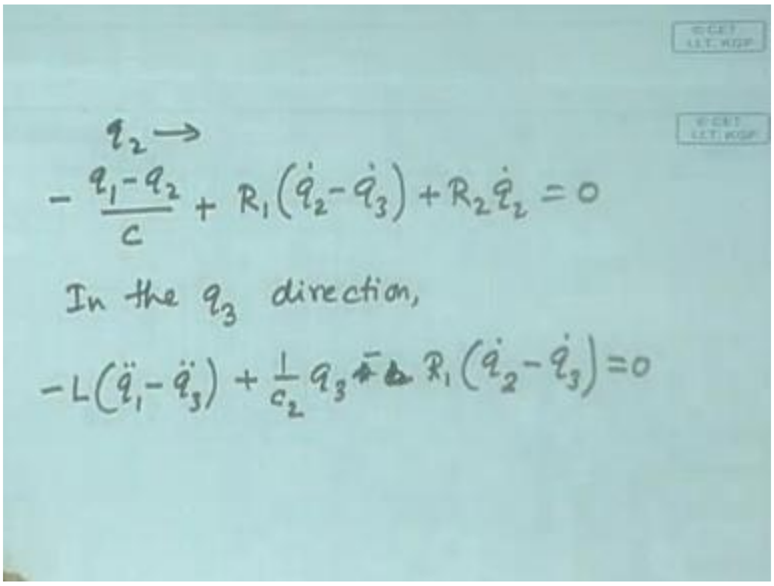

So, you will have minus of q 1 minus q 2 by C plus R 1 q 2 dot minus q 3 dot plus R 2 q 2 dot equal to 0, similarly you obtain the equation in third direction. And it will actually take the form L q 1 double dot minus q 3 double dot plus 1 by C 2 q 3 plus R 1 q 2 dot minus q 3 dot. So, three equations are this, this and...

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Where?

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This one, let us check sure yes, wait this term since you are differentiating this there will be a minus term here and the third term will also minus. See in this case no, I should actually write explicitly otherwise this is creating problems.

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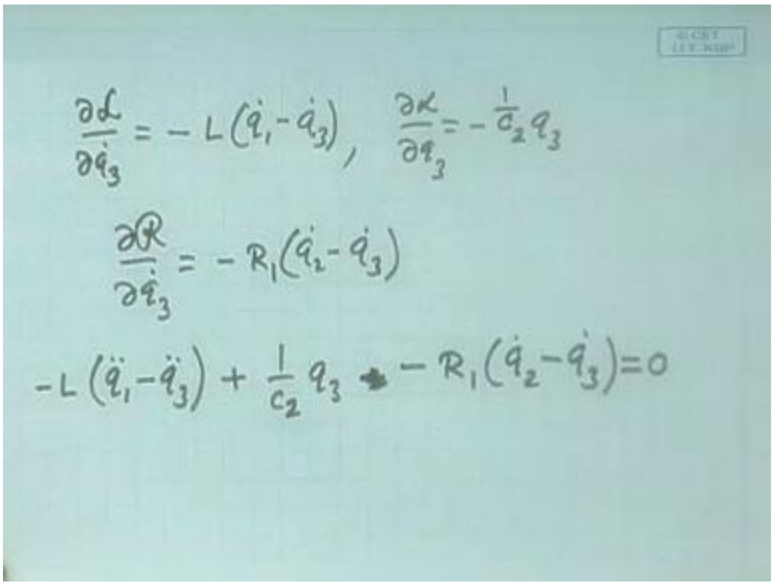

So, the Lagrangian with respect to q 3 dot is, notice the Lagrangian it will be L times this, but now this has to be differentiated, so it will be minus. So, minus L q 1 dot minus q 3 dot, the derivative of L with respect to q 3 is q 3 only here, minus q C 2 q 3. And, then here again it will lead to a minus, so R 1 q 2 dot minus q 3 dot and the second it will lead to a minus sign. So, minus R 1 q 2 dot minus q 3 dot and this is 0.

So, now the equation becomes minus L q 1 double dot minus q three double dot derivative of this with respect to time this plus 1 by C 2 q 3 plus or minus, minus R 1 q 2 dot minus q 3 dot. So, that is the equation, then so you have the three equations, where are they three equations, this is the q 2 equation, this is the q 3 equation. And where was the q 1 equation here, now notice one thing what is this equation actually, notice carefully what does this say look at the circuit, no.

Can you see these two now ((Refer Time: 38:57)) no, any way look at this circuit and you have already written down, so tell you what does it say L q 1 double dot minus L q 3 double dot, so it is actually L d i d t, where i is q 1 dot minus q 3 dot. Here is the voltage across the capacitor q by C and here is E, so this is nothing but, the KVL. Which means though we have not derived the Lagrangian equation using the Kirchhoff’s laws, it is actually consistent with the Kirchhoff’s laws.

So, when we obtain a set of differential equations we can also start from the Kirchhoff’s law and do it, no problem. And you will not be asked forced to do it this way, but you will find that for many systems since I do not have to worry about the direction of this for example, because it is squared. I do not really care which direction is positive, which direction is negative, it is far more advantageous in terms of writing down the equations correctly.

And later we will also illustrate a more suitable method for specifically devoted to electrical circuits, but you can easily see that the set of equation that we develop using mechanical systems as example are equally applicable to electrical circuits. And they are not contradicting the Kirchhoff's law which to be correct. Similarly, you can check that the second equation will be actually the KVL equation Kirchhoff's voltage law equation in the q 2 loop and the third one is along the q 3 loop.

Only one pointer, that is I do not know if whether you know this. That we had started by the assumption, that we can define a minimum number of necessary coordinates by simply looking at the windows and put in the q 1, q 2, q 3 along them, this is in general true. But, there are situations where this is not true, there are situations where this procedure does not give the minimum set of coordinates. That gives you can always define that way, it would not be wrong.

But that will not give you the minimum set of coordinates and how to write down the minimum set of coordinates that is a problem, we will take up later. Just a pointer that this works and what you get is, but that is not the minimum thing, let us carry on. So, we have learnt how to use it can you just quickly write it for this system.

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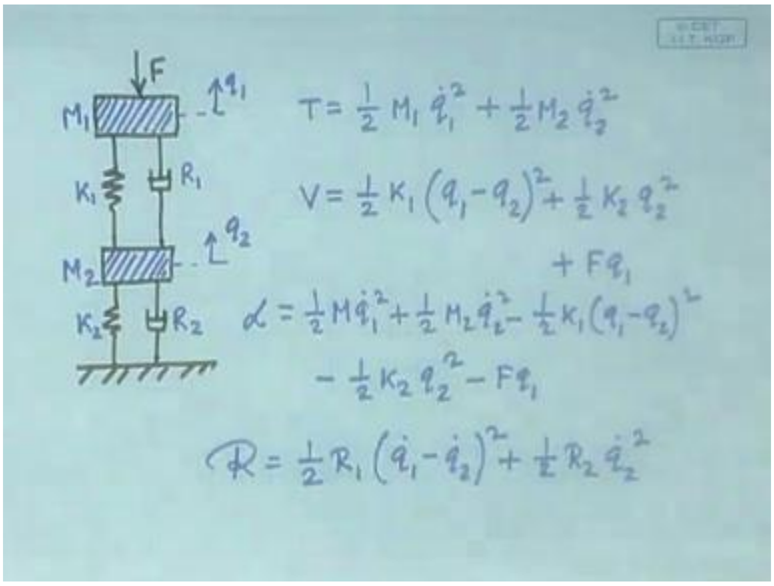

Say there is a mass, there is a spring, there is a damper, there is another mass, there is a spring, there is a damper and now it is connected it is resting on the ground. And now you apply a force, can you write down the differential equations for this system. Do this first you will need to define the coordinates. How will you define a coordinates, position coordinates of these two masses. And if you want to define the position coordinates of these two masses, you have to specify what is their 0.

So, you cannot simply say that here is my position, and then this is say q 1 and this is say q 2. You can say that, but the moment you will have to say where is my 0, now let that 0 position be where these two springs are relaxed, no energy stored in the springs. So, the q 1 and q 2 will directly give the amount of energy stored in the spring. So, you can easily see that if you do not apply any force let it rest, then two springs will come to a compressed position at rest.

And that is not what I am defining as the 0 why, because in that case how much has it been compressed initially and all that we have to consider. We say that let these two be measured from their zero position, where in the zero position there is no energy stored in the springs. Quickly M 1, M 2, so let this be M 1 and let this be M 2. Now, if you proceed to write first you will write the kinetic energy which is somewhat trivial half M 1 q 1 dot square plus half M 2 q 2 dot square, potential will be the potential stored.

For now let us ignore gravity because ultimately if you consider gravity, it will lead to constant terms that will be differentiated away, so who cares I do not want that. So, V will be due to the two springs and the force, so it will be half, let this be K 1 and let this be K 2 and this is be R 1, R 2. Half K 1, then how will you write no q 1 minus q 2, q 1 minus q 2 is the amount of displacement experienced by this spring, so q 1 minus q 2 square plus half K 2 q 2 square.

Now, here you see q 1 is up and F is down, so we will say plus no just q 1 is acting in the direction of q 1. Then your Lagrangian becomes I will just quickly write, so that we can differentiate fast what I will write. Now, the Rayleigh function it will be what is the relative velocity experienced by this R, it is q 1 dot minus q 2 dot. So, it is half R 1 q 1 dot minus q 2 dot square and how much is the it is only q 2 dot, for R 2 it is only q 2 dot So, it is plus half R 2 q 2 dot square.

From here do I really need to the rest is trivial, the rest is trivial and therefore, I will leave it to you know how to do it. See you did you notice that, the actual thing that you have to do is to write these two nothing else. The rest is so algorithmic that you do not really need to bother about it. You only have to do the differentiations right, which I suppose you learnt in school, so this is rather simple way of doing it. By the way this actually represents physical cushioning system, cushioning system where cars and motor cycles they have a cushion.

So, that you do not directly feel the vibration of the these are somewhat like that, you may have seen this arrangement around the back wheel of the motor cycles, you get a cylinder in which the frictional spring arrangement is there. So, exactly this kind of arrangements in order to dampen or cushion your the riders motion. Now, let us come to another issue that you might notice that you have, where was the circuit I was dealing with circuit. It is not necessary we can again do the circuit, there may be a number of inductances in the circuit.

And if there are number of inductances, so there may also be mutual coupling be between the inductances. What happens, then if there are two mutual inductances, then there are two inductances there is a mutual coupling between them. Then how do you, how does it get reflected in the equations and through what, there is a question let us deal with. So, let us take one example a circuit simple one.

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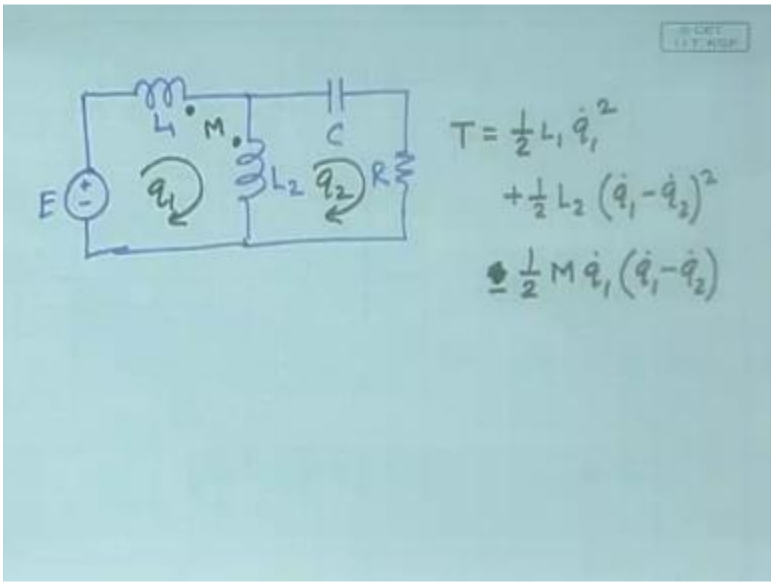

This is the positive direction of the voltage E, here say an inductor and here say another inductor. And the rest of it can be connected to resistance and capacitive branches. So, you have L 1, L 2, C, R, but now in addition to this there is a mutual coupling between them. Now, whenever there was a mutual coupling between two things what you do, you always put these blobs what does it mean. It means that if the current is flowing into the blob, in these two directions, then they would be additive, magnetic fields would be additive.

If the currents flow in the opposite directions that means, in one it goes through into the blob and the other one it goes this way that means, it goes out of the blob, then they are subtractive. After all that addition or subtraction does what it adds to or subtracts from the total kinetic energy that is stored in these two. So, this change should be reflected in some kind of a change in the kinetic energy. So, the kinetic energy if this is not there, the kinetic energy would be written as half, let us designate the two coordinates.

Then half L 1 q 1 dot square plus half L 2, here it is q 1 minus q 2, q 1 dot minus q 2 dot square, in the absence of this mutual inductance. Now, if the mutual inductance is there, if the mutual inductance is additionally there, then you see due to q 1 it is going out of this blob and the q 1 minus q 2 is going into the blob. So, how will you designate this, the amount of additional energy stored in the inductor is half M, the two i, i 1, i 2 you know that.

So, the additional kinetic energy would be plus half mutual inductance half M q 1 dot into q 1 dot minus q 2 dot. Now, here I will first put plus minus, and then do the choice this is the amount, and then we will do the choice. Now, you see because I put the dots here and here, q 1 goes out through the dot and q 1 minus q 2 goes into the dot. Therefore, these two currents are subtractive, if it is subtractive I have to take the minus sign, the rest remains the same.

All you need to do is to do this, now if I change the position of the dot say, this one is moved to here obviously, you can see that this will become additive you have to take plus. And in when you are actually doing this, in problems then you have to take that into account somewhat minutely. So, that many people I have seen that making mistakes in the signs here, because this is see if these are two independent coordinates there is no problem.

But, because these are two related q 1 minus q 2, that is why I do not why, but students tend to make mistakes. So, be careful about that if you are taking q 1 minus q 2 it is going into, if you taking q 2 minus q 2 it is going out, then you can continue with this. Mechanical equivalent of a mutual inductance, no mechanical equivalent of a mutual inductance tough question, But, there can be, but there can be and that this question I will come to about a month later, when they required background has been built up.

At this moment I will not get into these, because that will require a lot of things to be told I will come to that later. But, keep this question in mind and when the time comes raise it again and we will see where it comes. So, what is the mechanical equivalent of a mutual inductance, we will come to that.

Thank you very much today.