**Dynamics of Physical System**
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**Lecture - 4**
**Using the Lagrangian Equation to Obtain Differential Equations (Part - I)**

In the last class, we had basically achieved the derivation of the Lagrangian equation, so what actually did it do, if there are n number of bodies in a system each marked as j.

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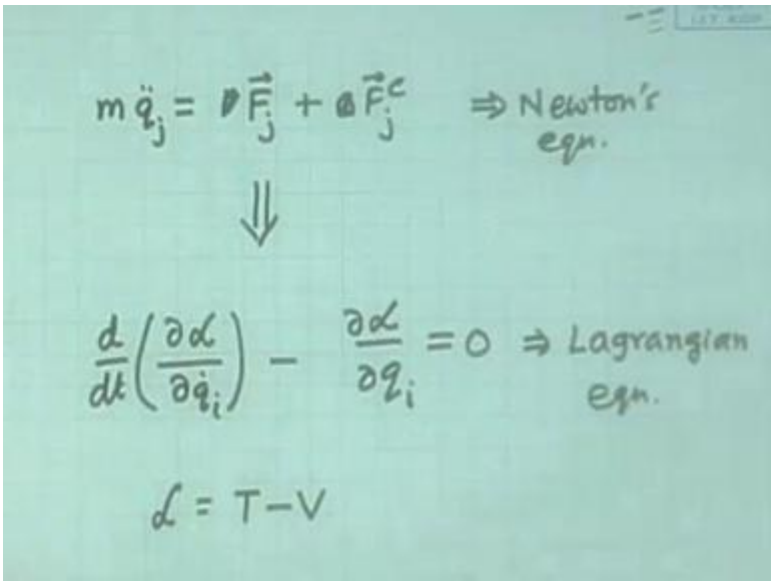


Then for each equation in the Newtonian way would be m q double dot is equal to the applied force plus the constant force F c this would be vector, this would be vector and all that will have the j subscript, that was the Newton's way. Now, you can see that if there are say 25 bodies in a system, you would need 25 such equations and then you will have to you will get the set of differential equations as that 25 set.

But, that 25 could be in a three dimensional space, so how many coordinates do you really need three and how many equations do you really need three, if there are holonomic constants then it even goes to less. So, that was one problem with the Newton's equation and from there we had obtained the Lagrangian equation as d d t of this is the Lagrangian function with respect to q dot here minus Lagrangian function.

So, this is the Lagrangian way and these two forms are mathematically equivalent, because they yield the same differential equation, we had seen that starting from here we had arrived at this. Now, these are each equation will have the subscript i, is i then represents the identifier of the new coordinates the generalized coordinates. So, if there are n number of such generalized coordinates, which is the total number of necessary configuration variables minus the number of holonomic constants, this would be that many number.

Now, let us try to understand how this equation can be used, but first what does this consists of the Lagrangian function is the kinetic energy minus the potential energy that what we said. And in the potential energy we said that at least we have gone up to that extent, where we said that at least we understand how it is for the gravitational potential, we understand how it is for the electrostatic potential. So, that much we understand, but in general that could a body could be attached to other bodies through spring like elements, through other kind of elements, so let us understand that first.

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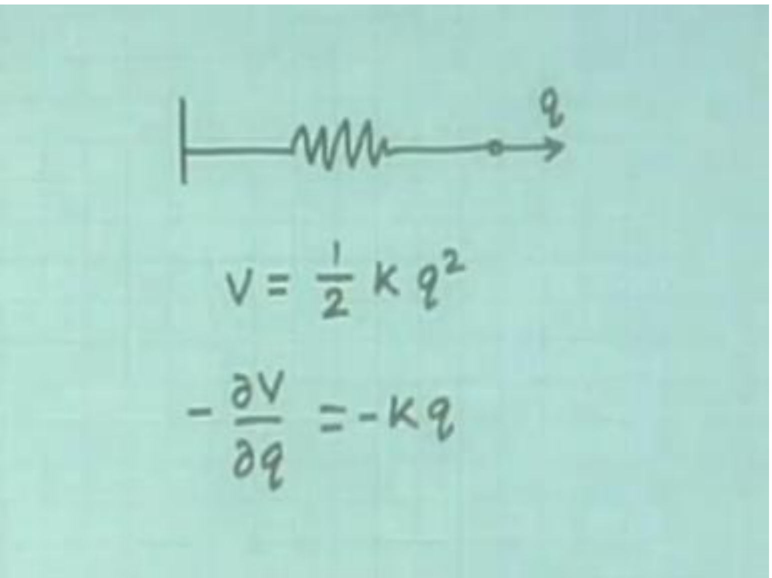


Now, suppose you have a spring and this end is being pulled and from the un stretched position it has been pulled by an amount q, which is the position coordinate, then what is the energy stored in this. So, energy stored is half K q square, now notice that this has the form of the potential energy why, because if this is the potential then is just K q and K q is the amount of force that we expect right.

So, you get the and why is it minus, because you are pulling in this direction and the force is applied in the opposite direction, that is why this minus comes has to come. So, you have the force obtained as the gradient of the potential, so the energy stored in the spring is actually potential energy is that clear. So, if a system has both springs and gravitational energy stored, so the total potential energy would be the potential energy stored in the spring as well as the in the gravitational energy. And this potential energy is a scalar, just a number it has no direction, so if something is stored here, something is stored there, something is stored there, just add them up that is it that gives you the total potential energy that is one advantage.

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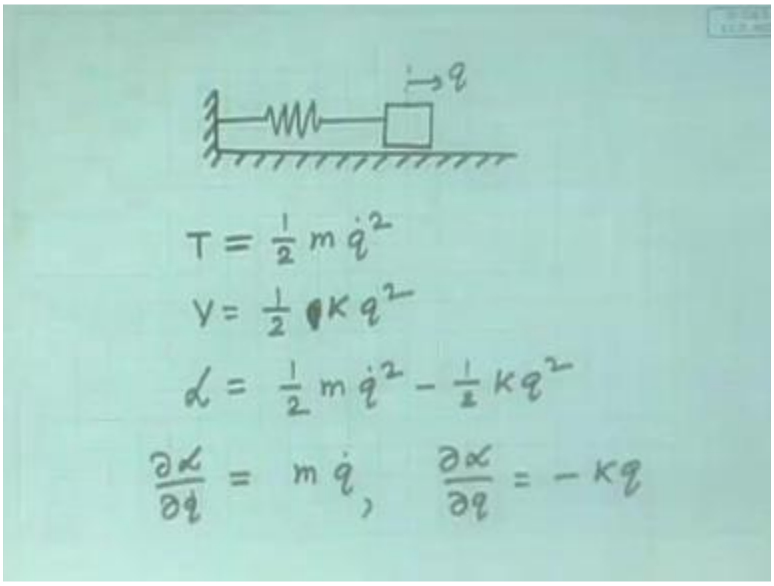


Let us illustrate with a very, very simple system the simple oscillator, how this can be used, suppose there is a spring here connected to a mass, simple no complication how would you obtain the differential equation in this case. Let us start from the simple system yes, we would first say no before going to that let me start from here ((Refer Time: 07:10)) and illustrate how the process of obtaining the differential equation goes.

In case of the Newtonian equation what did you do, for every mass point we identified the forces acting on it and in order to facilitate that we drew the free body diagram, that you probably learnt in the mechanics class. So, we drew the free body diagram then we either wrote force is equal to mass into acceleration or if we write in the D Alembert's way, the free body diagram itself contains the m q double dot, so just balance of the forces.

In this case, what we do is we have already achieved the feet of reducing the number of differential equations using the holonomic constants. So, now if looking at a system we find that we can define a set of position coordinates, position coordinates remember, position coordinates that are consistent with the constants. The minimum number of position coordinates necessary to define the position us status of the whole system unitly that is it.

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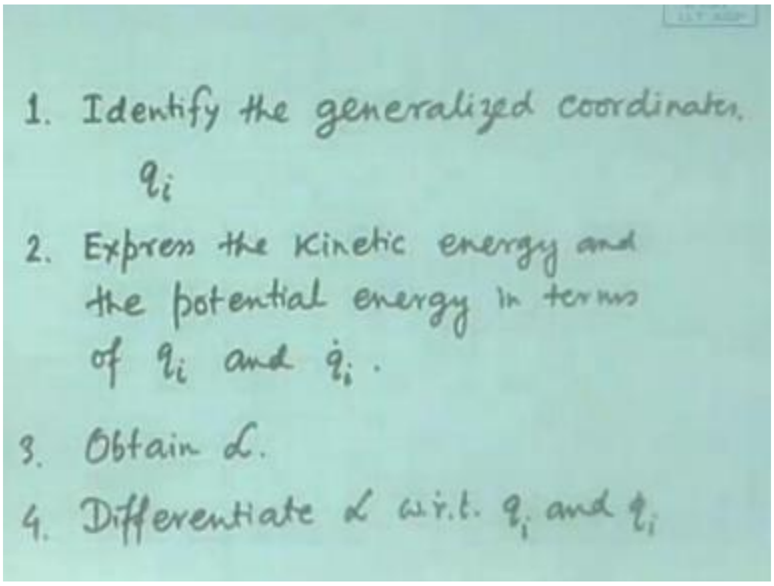


So, the first step in the Lagrangian formulation would be one, identify the generalized coordinates. Now, remember this generalized coordinates there this nomenclature should not confuse you, in case of a system like this ((Refer Time: 09:01)) what can be the generalized coordinates, here is a body in the Newtonian way it has x, y, z coordinates in the Lagrangian way I do not need the y and z coordinates that is it. We will say, that from the un stretched position of the spring I will measure the variable q that is it just one necessary.

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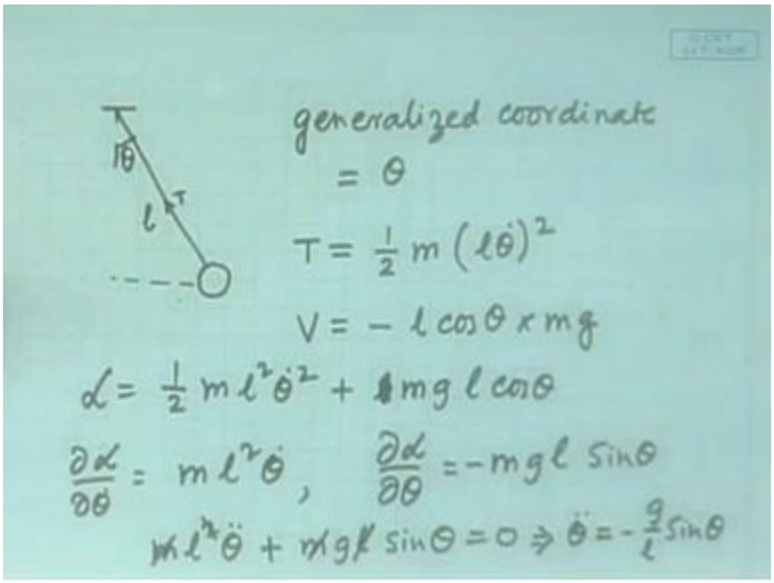


So, that is the generalized coordinate in case of the simple pendulum, what is the generalized coordinate, simple ((Refer Time: 09:42)) suppose it is a planar pendulum, which means that the first holonomic constant forces it to move on a sphere and the second holonomic constant imposes another plane on it. So, it is forced to move on an arc, if it moves on a arc then what is the minimum number of position coordinates can I define, that is consistent with the constant just one and that is simple theta.

So, whatever follows from your common sense that exactly is the generalized coordinate it just goes as the name fine ((Refer Time: 10:21)). So, you have the number of generalized coordinates, normally they would be denoted as q i that is the way we normally define. So, even if something is theta we can call it q 1, q 2 and all that, but sometimes we might in order to make the identification easy, continue with the alpha, beta, theta and whatever, but always remember that this is the coordinate.

Well once we have identified this, then express the Kinetic energy and the Potential energy in terms of, so we have the position coordinates. Now, express the Potential energy and the Kinetic energy in terms of this q i and q i dot, then that is it, then we just substitute in the Lagrangian equation. Let us illustrate start with this problem, ((Refer Time: 12:18)) number one identify the generalized coordinate, we have identified only one yes.

Student: ((Refer Time: 12:25))

No it will matter, his question is if we cannot identify the generalized coordinate will it matter, one clarification might be necessary. In a system there might be some generalized coordinates, that are spherical there might be some generalized coordinate that are linear. And also in electromechanical system there can be some generalized coordinate that are in the mechanical domain, some generalized coordinates that are in the electrical domain it is possible, but you will have to identify, that is the first step.

So, that has to be identified and I will work out here many problems, so that you will after sometime be comfortable with it. Once, you have done that express the Kinetic energy and the Potential energy in terms of q and q dot, so you can easily do that if the velocity is q dot then T is half m, the Potential energy is half K q square simple, so the Lagrangian is T minus v half m q dot square minus half K q square. So, the derivative of Lagrangian with respect to q dot is and the derivative of the Lagrangian with respect to q is minus K q, once you have written this we have to substitute here.

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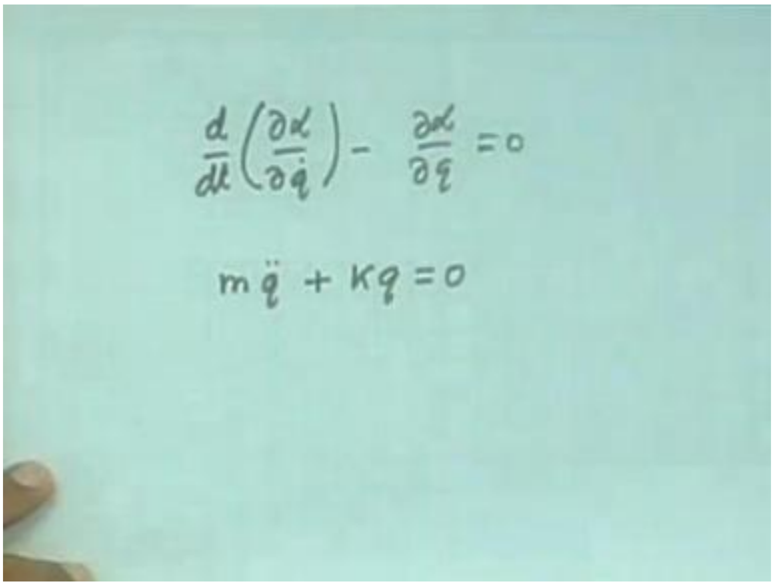

So, let us substitute you get d d t of Lagrangian dou q dot minus substitute this you get m q double dot here, because it is again derivative with respect to time plus K q equal to 0, that is the equation of the system. Pretty simple really, well you might say that as yet the advantage of the Lagrangian form, Lagrangian way of writing equation vis-à-vis the Newtonian form is not very clear, ((Refer Time: 15:34) because it could have done this way, that here if I pull it a bit then the mass is acted on by a force like K q in the direction.

So, K q is the force acting on it and that is mass into acceleration that is exactly what you have written here, yes right from here it may not be all that clear. Only thing is that, in the Newtonian way logically you will have to write the equations for all the directions, but now you know that they are not moving in the all the directions. So, exactly from this one it may not be all that clear, so let us now handle the this problem which we had done earlier, just turn the page where we have earlier obtained the differential equations by the Newtonian way ((Refer Time: 16:29)).

There the tension appear and it had to appear, because that is the force acting on the body it had to appear. But, now in the Lagrangian way we will not bother about it at all why, because that is a constant force and we have eliminated the constant force already that from the formulation. So, we will forget about it, so the generalized coordinate is what theta, the generalized velocity remember one thing, the generalized velocity is simply the derivative of theta, derivative of the generalized coordinate not the actual velocity of the body, the actual velocity of the body is...

Student: ((Refer Time: 17:34))

So, this is l then l theta dot but we are not talking about that, your generalized velocity is simply the derivative of this. Fine, the next step is to obtain the kinetic energy and the potential energy, the Kinetic energy is what is the velocity of this fellow, it is say this length is l, so it is l theta dot is the velocity, so half m l theta dot square. And your potential energy is suppose I measure from here, how much it is below, then if this is theta then this height is l cos theta, so negative of that minus l cos theta m g l, so yes l cos theta m g, m g has to be there.

So, your Lagrangian function is T minus v half m l square theta dot square plus m g l cos theta. So, your derivative of the Lagrangian with respect to theta dot is m l square theta dot, the derivative of the Lagrangian with respect to theta is do it, it will be m g l cos theta minus sin theta. The moment you have this, you can easily is there space here yes you can write down the differential equation, derivative of this with respect to time which is m l square theta double dot minus of this, so it is plus m g l sin theta equal to 0 m cancels off l one l cancels off, so you get theta double dot is equal to minus g by l sin theta that is the equation.

So, that is how we write it and you see that we do not need to write three equations, we do not need to consider the tension it becomes simple and that is the main advantage of the Lagrangian form. If the advantage is salient even for such a simple system for more complicated system, it will be far more salient let us try to illustrate some of that.

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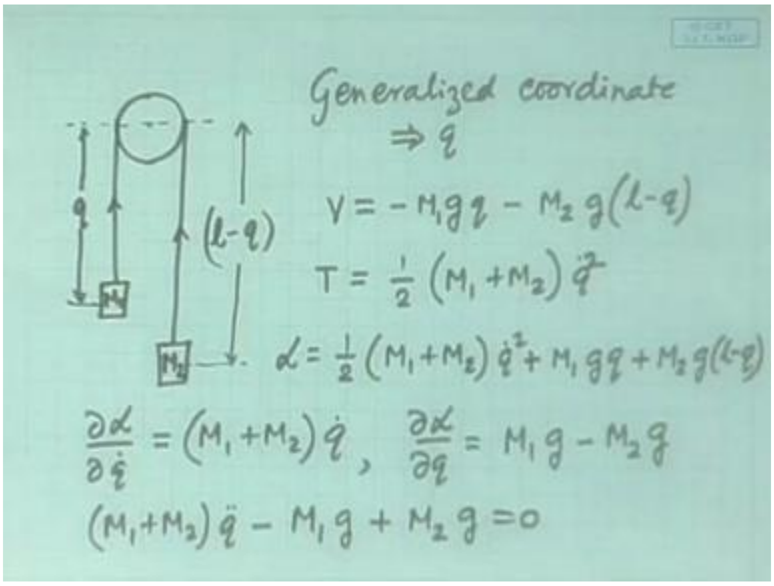


Now, you have while talking about the constants we had considered this kind of a situation, we were considered this. So, let us obtain the differential equation of this, here you would notice that here is a tension, here is a tension and we had argued that this tension does work, this tension also does work, but put together they do no work. Anyway that is not important, what is important is that written in the Newtonian way, you will have to considered this two tensions and written in the Lagrangian way we can write at the beginning forget about them.

So, first question how many generalized coordinates do we need in this case only one, because the moment it goes up it has to go that amount down. So, both this positions are given by just one body was one coordinate and let that be starting from say this position, we can say this is q, if the total length here is say l then this becomes l minus q. So, the generalized coordinate is consistent with the constant is q, now write the kinetic energy and the potential energy.

So, what will be the potential energy, potential energy is the potential of this body and the potential of that body taken together. So, you have s minus let us see let this be M 1 and let this be M 2, so you have minus M 1 g and q for this one and minus M 1, M 2 g l minus q for this one. So, that is the total potential energy and the kinetic energy is kinetic energy they of course, move together.

So, if this fellow moves with a velocity q dot that fellow also moves with a velocity q dot, we can simply add them up. So, half M 1 plus M 2 q dot square, so the Lagrangian is half M 1 plus M 2 q dot square plus M 1 g q plus M 2 g will it go no l minus q, now let us differentiate. So, is M 1 plus M 2 q dot and the derivative with respect to q is let us see this term M 1 g this fellow cancels off this fellow vanishes, so minus M 2 g.

So, these are two parts one the fixed part another the variable part that is what remains, so ultimately since there is space here we can write, the derivative of this with respect to time M 1 plus M 2 q dot it becomes q double dot minus this. So, minus M 1 g plus M 2 that is the equation, one thing might be worth mentioning ((Refer Time: 26:43)) that in the last problem that we solve the simple pendulum, where written the potential as minus m g l cos theta, taking this point as the datum.

Suppose, some of you say no, no I do not want to take that as the data, after all this fellow comes here that is the minimum position I will take that as the datum from measure it from there, yes you can do that. In that case this total length is here l and how much is the elevation l minus this do that no problem l minus l cos theta, but then what happens in that case just do it, it will be l minus l cos theta this whole thing m g, you would notice that this will have two components, one a constant component m g l minus m g l cos theta when you take the derivative this fixed term vanishes. And therefore, that does not appear, so it does not really matter where you take the datum is that clear, so that is up to you wherever you want you can take.

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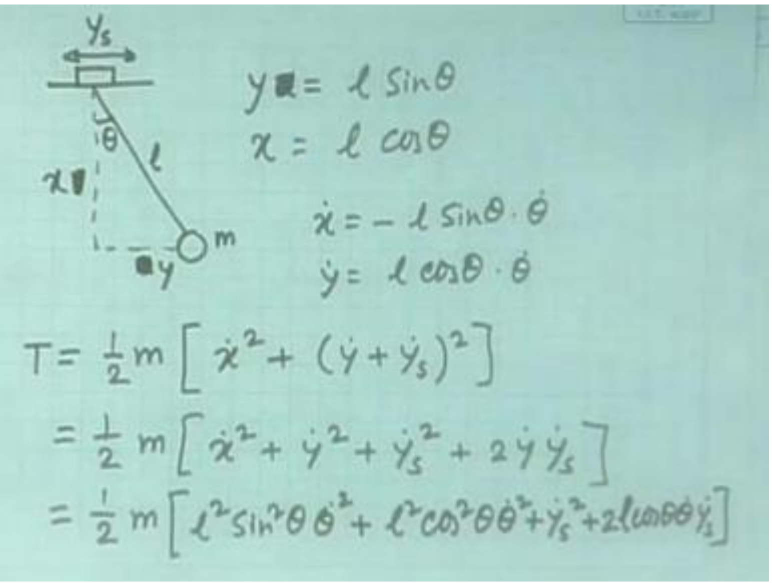


Now, let us go to a slightly more involved problem think of this, suppose you have a pendulum as before, but here the point of suspension is moving like this, if we actually simulate and see it is behavior such a thing, you would find that it has a very, very complicated behavior very, very complicated motion. So, let us try to write down the differential equation that can capture that motion, so this length is l here is your m, we would need the x coordinate and y coordinate, because this fellow is moving in the x coordinate.

So, here is the x and here is the y, now x is if this is theta x is l sin theta no let us redefine because things are happening in this direction, let us call it y and let us call this x, then y is equal to l sin theta and x is equal to l cos theta. Now, your additional thing is adding to the y, suppose that suspension's movement is called y s and this fellows' position is y, if you have this if this fellow is not moving, then you can say your x dot is minus l sin theta and y dot is l cos theta into theta dot.

So, these two are the velocities of this fellow and we have to add to that the velocity here, in order to obtain the actual velocity of the bob, it is if this were not moving whatever the velocity would be plus the velocity that is imposed on it, that will be the easy way of looking at it that is what we will do. So, ultimately when we have obtained this then you are the kinetic energy is half m v square.

So, we have to say half m and v square is x square plus y square x dot square plus y dot square, it would be then x dot there is nothing happening in the x dot direction x dot square plus y dot plus y s dot square, what you have done in this direction the movement the velocity was y dot added to that you have got some velocity here. So, this fellow will actually move with this velocity in the y direction and x square plus y square is the actual velocity square that is what we need.

So, let us break it up you have half m x dot square plus y dot square plus y s dot square plus twice y dot y s dot. Now, why we know what these guys are in terms of theta, because that is what we want to write it in, so we substitute m x dot is x dot square minus goes off l square sin square theta theta dot square, we have already done it plus this is l square cos square theta theta dot square plus y dot s square plus yes twice y this comes down twice y is this fellow y dot l cos theta theta dot times y s dot is it enough this screen, let us move it a bit then you will be able to see yes twice l cos theta theta dot y s dot right happy. What can we do from here, wait, wait, so it is done essentially these two can be put together, because sin square theta cos square theta it is one, so that can be simplified.

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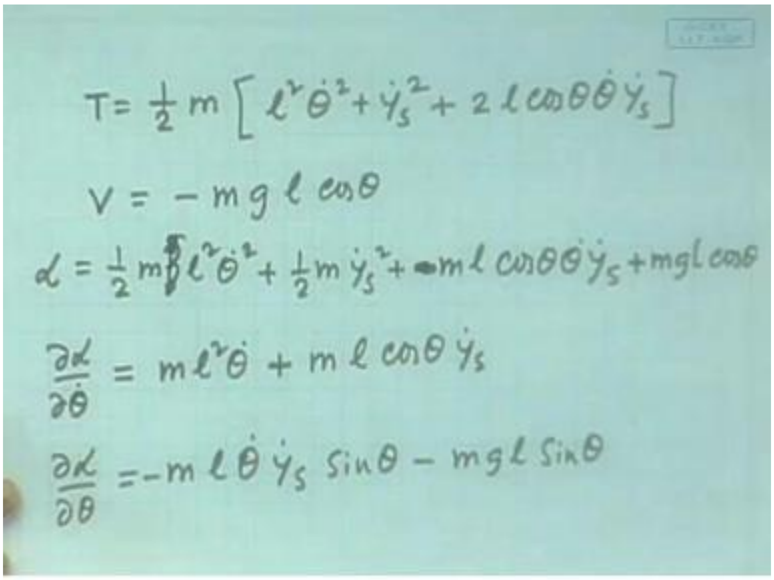


So, T is half m l square theta dot square plus these two terms they remain plus y s dot square plus twice l cos theta theta dot y s dot happy, look the procedure is pretty straight forward. What we have done, ((Refer Time: 34:48)) we have identified what the velocity would be and accordingly we have written in terms of the actual generalized coordinate, these are this theta and of course, y is also has to be one coordinate.

So, we have written the T and then we have to write V the potential energy and the potential energy is pretty simple, because the amount of potential energy is in no way different from for the simple pendulum, simple pendulum we had written minus m g l cos theta, so minus m g l cos theta. So, your Lagrangian term becomes this whole thing half m let me multiply let me get rid of this bracket, so half m l square theta dot square plus half m y dot s square plus no two and half of m l cos theta theta dot y dot s that does it T minus V, so plus m g l cos theta.

So, that is the Lagrangian, so see I am doing absolutely algorithmically I know that I have identified the coordinates and after that I am proceeding algorithmically. All these are simple algebra, no complication really even though this may look a bit longish, so you have with respect to with respect to what will you differentiate theta. So, you have theta dot and we will have to differentiate with respect to theta, now this becomes m l square theta dot, this goes off this remains, so this was what plus this plus this is not there.

So, you have if you differentiate it will be plus m l cos theta y dot s this goes off, now when we differentiate with respect to theta, this is this goes, this goes this remains yes m l theta dot y dot s and then cos theta, so you have...

Student: ((Refer Time: 37:58))

Sin theta minus yes no I am talking about this term here minus.

Student: ((Refer Time: 38:15))

Yes fine, now we have to differentiate these two and that is it, so can you write down now starting from here.

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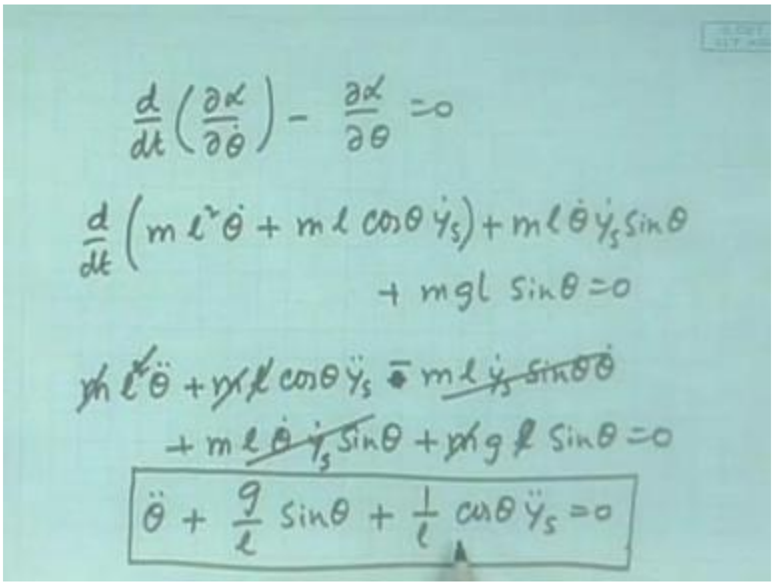

Now, we have to write down the d d t let me explicitly write it, so you have to say d d t of this term m l square theta dot I am doing this especially, because people make a mistake at this stage right. Now, you differentiate this, if you differentiate this what you get firstly, you get m l square theta double dot, now here are two quantities that are depending on time and therefore, you have to be careful in differentiating this term, you have to say plus m l cos theta y double dot s plus m l y dot s sin theta [FL] into theta dot.

And this is the crucial thing that people miss, so plus m l theta dot y dot sin theta plus m g l sin theta equal to 0 done that is what I am looking for what cancels.

Student: ((Refer Time: 40:59))

These two cancel off no yes, so these things three things remain, then we can cancel certain things. So, things become simpler theta double dot plus g by l sin theta these things were there in the simple pendulum that is why I wanted to write it this way plus 1 by l cos theta y double dot yes equal to 0, that becomes the equation. Can you now write down on your copy, if this y yes happens to be a sinusoidal oscillation, then what will this equation be.

Student: ((Refer Time: 42:14))

No that is the externally imposed motion, that is why it is not a independent degree of freedom of the system, that is why it is not necessary at this stage, her question is that would not this become another coordinate ((Refer Time: 42:34)) no it is not necessary, because it has no a internal degree of freedom it is moved by somebody y s is being imposed by somebody.

So, the system has no freedom in choosing that, that is why you do not need that as a coordinate that is imposed by somebody, even without solving the differential equation you know y s should be at any point of time that is why. So, you need a coordinate if in order to find out what that coordinate value will be at a specific point of time, you need to solve the differential equation, but in this case you do not need to…

Student: ((Refer Time: 43:18))

Which one this fellow.

Student: If it had a mass then kinetic energy you would have considered a term.

Yes her question then is if this fellow were a mass, then somebody would not be able to move it just like that, it would be application of force and this will then have it is own dynamics, in that case yes you need to…

Student: ((Refer Time: 43:49))

Yes in that case y s becomes an independent coordinate of this system an independent degree of freedom, but in this case the way we have defined the problem as yet it does not become, yes if this fellow had a mass you are applying a force and then the mass moves depending on the force and it is own dynamics, because this being pulled by this fellow also. So, in that case you do not know beforehand what the position of mass is at any point of time, you need to solve the differential equation for that and for that it becomes a independent coordinate.

So, in this case as I told you in case y s is a sinusoidal function, say y s is equal to say A cos omega t, can you see yes in that case what is the equation write down yourself, because most logical way of moving this fellow would be a sinusoidal oscillation. So, if that happens then what will be the equation, this will just become a omega square cos becomes sin with a negative sign here and that is it, only I wanted to you to know that this is a double derivative.

So, you will need to take a derivative twice that is it, but if it is a unknown quantity and something is imposed from outside, then it is simply this equation. Now, are you now comfortable with this idea, how the equations are derived, note one thing that you are getting second order differential equations along each position coordinate in a, so far we have considered problems with just one coordinate.

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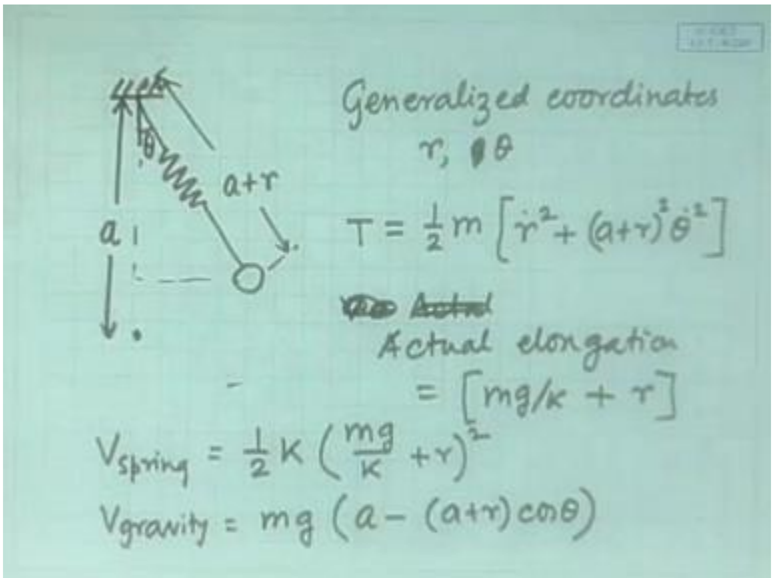


Let us now take the next step, let us take a simple next step to a system, where there are more than one position coordinate, say again let us take the pendulum, but let us make it a little more complicated. Suppose here it is, but instead of just being connected to a here is a spring and then you connect it [FL] what has happened, because of this spring, there is another degree of freedom in this direction.

Now, you really have to apply your brain a bit in order to identify the generalized coordinates [FL] here is your theta. Now, you would notice that there are many ways of defining the position coordinates theta is a natural choice alright, but how will you in the radial direction there is another degree of freedom I can see. But, how do you define that there are various ways of doing it, you might say that distance is bigger from here that is it, you might say that no let it hang freely here in this position, then it takes a particular length of this.

Remember in this range in this position it is not un stretched it is already stretched, but that let that be the datum and from there I will measure the difference, you could say that and they will actually make no difference. So, in order to do that let us do it that way, so what did we say, we said that when it freely hangs vertically, then your position is the datum position.

In that position, suppose it came here and this distance is say a and then this distance is the deviation from there. So, distance is a plus the coordinate that we are talking about r, we can do it this way that also uniquely defines it is position is it not plus theta, theta has to be there as one of the coordinates. So, r and theta, so then generalized coordinates becomes r and theta, now in terms of this r and theta what is kinetic energy and what is potential energy, that is what we will try to write.

Kinetic energy will be relatively simple to write, potential energy will require a bit of thought, the kinetic energy is half m v square. So, half m we will write first and then we will try to figure out what the v square is, now is notice that there is a component of the motion in the radial direction and there is a component of the motion in the circumferential direction, so if you can identify these two, then it will be simply this square plus that square.

So, in the radial direction the velocity is...

Student: r dot.

R dot yes, so it is r dot square plus.

Student: ((Refer Time: 50:04))

Yes a plus r theta dot whole square of this, so both this have the squares done, the potential energy in this case there are many sources of the potential energy there is a fun of this problem. One, it is contributed by the gravity, two it is contributed also by the spring and they have no difference we will simply add them up, because they are scalars that is the advantage fine. So, how do you go about it, notice let us start the argument from the position that it is un stretched, if it is un stretched it has some position, but from here it has already being elongated by application of a force.

Student: ((Refer Time: 51:06))

M g.

Student: ((Refer Time: 51:09))

So, how much has it already elongated.

Student: m g by K.

M g by K it has already elongated in this position by m g by k and I am now counting the difference from that position here and therefore, what is the actual elongation m g by K plus r. So, the actual elongation is let me start from here actual m g by K plus r that is the amount the actual energy, so the potential energy in the spring V spring is half K into actual elongation square.

So, you have, so that is the amount of energy in the spring, when it is hanging vertically in this position, then the gravitational energy is 0 and we are taking that as the datum and counting gravitational energy from there. So, how much has it gone up from there that position, this much is a plus r cos theta, so a minus a plus r cos theta is the amount by which it has gone up. So, V gravity is m g times this amount which is a minus a plus r cos theta is the line of argument.

The moment you have identified this two, the total potential energy is the addition of this two and the Lagrangian is t minus that and the rest is simply algorithmic, you do not have to apply your head just algebra. So, you obtain that differential equation starting from there, there is the rest of the problem is simply in a routine procedure, you do not have to really apply your head a lot for the rest of this problem, bring that in the next class.

Thank you very much, let us stop here.