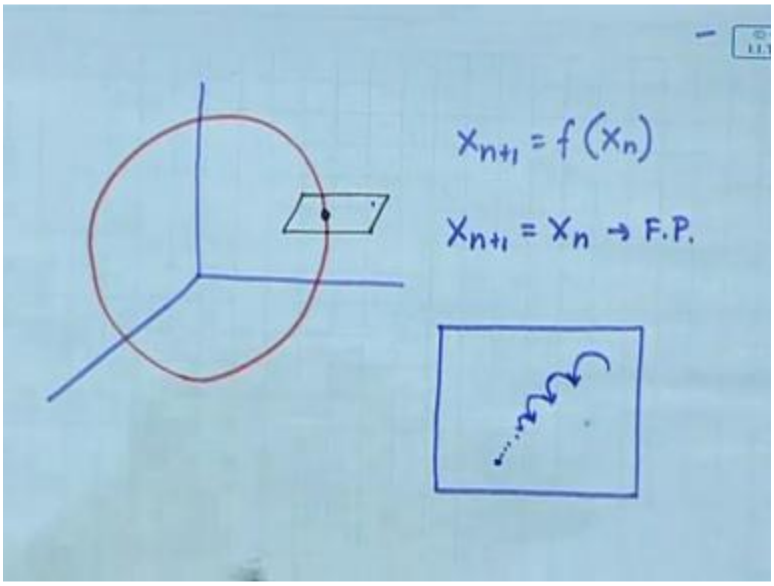
Dynamics of Physical Systems
Prof. S. Banerjee
Department of Electrical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 34
Discrete-time Dynamical Systems – II

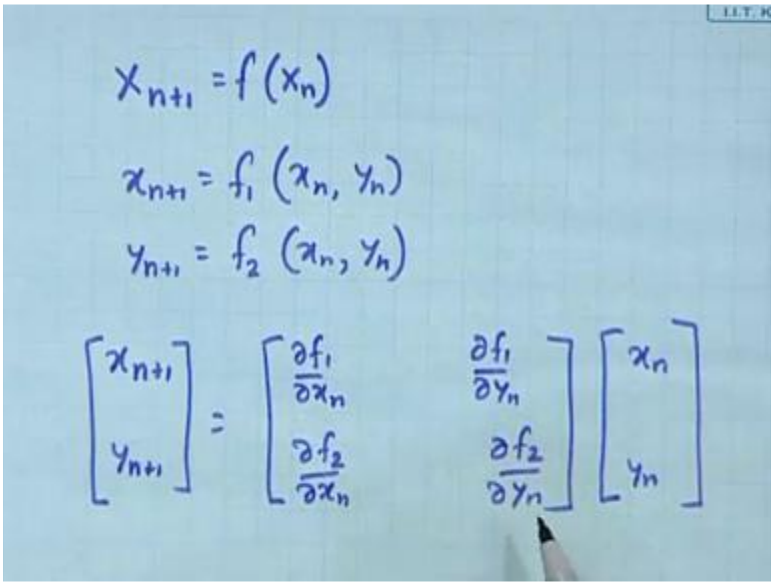
(Refer Slide Time: 00:55)



In the last class, we have seen that if you have a periodic orbit something like this. A limit circle then in order to study its stability, what we will do is we will obtain a Poincare section and it will be seen as just one point, so the on the Poincare or the section it will be defined as a map, X n plus 1 is equal to function of X n and this specific point has the property that X n plus 1 is equal to X n, this particular point and that is a fixed point. Then our ultimate question was how to study the stability of this orbit.

Now, it is not difficult to see that if the orbit is stable then if you start from an initial condition away from it then in successive rotations or iterations on this Poincare plane, it will come closer and closer to that specific fixed point. So, you will see a sequence of points, a point mapping to a point, mapping to a point, mapping to a point, but they are coming progressively closer to each other ultimately, it will converge on to that fixed point. So, whether or not this is true we are trying to test.

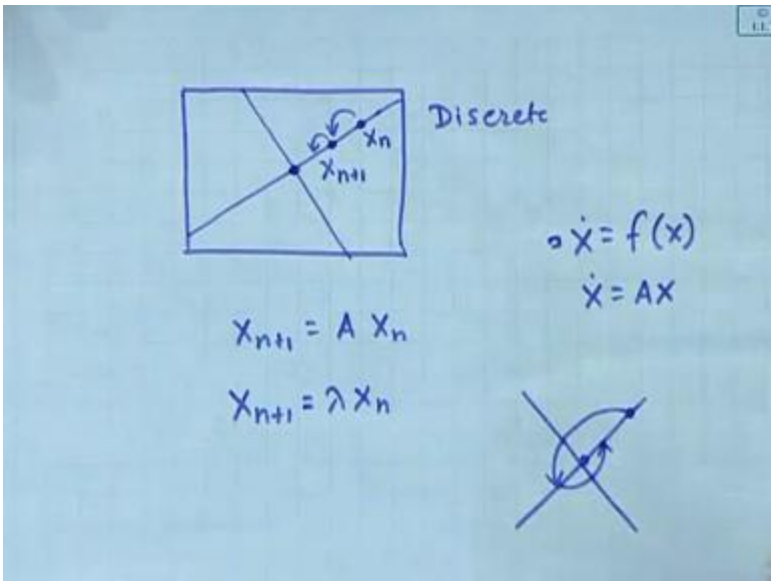
(Refer Slide Time: 02:45)



Now, if you have an equation something like this normally, this function will be a non-linear function some non-linear function. In that case if I say that here is my fixed point and I want to study the local linear property, what will I do. Suppose this is a two dimensional system as I have depicted there in that case it will be x n plus 1 is equal to some function f 1 of x n and y n and y n plus 1 is another function of x n y n, here these are the vectors, now we have written in individual scalar terms.

In that case your local linear behavior will be something like this, x n plus 1 y n plus 1 is something times x n y n. Now, what is that something it is again the Jacobian matrix which is. Having obtained that you would have numbers here, because whatever will appear in these parts are functions. And then if you substitute the position of the fixed point you get numbers here and those numbers again in the matrix will yield Eigen values. Now, what would those Eigen values mean, what they would mean is for example, here I have.

(Refer Slide Time: 04:58)



I am drawing a discrete state space, so this is discrete remember; which means, points jump from one point to the other or map from one point to the other and say this is the point, this is a fixed point. So, if this matrix yields real Eigen values then there would be real Eigen vectors also, these will yield, so this is X n plus 1 is equal to A say that matrix times X n these are vectors, these are X n plus 1 and X n are vectors.

Now, in that case what would the implication of the Eigen values be, notice again we would say that in this space there would be two directions; such that, if you take at initial condition on one of these directions Eigen directions then in subsequent iterations it will remain on that, that is the core definition of the Eigen vector. So, you are taking an initial condition X n on that Eigen vector, in that case your X n plus 1 will be the Eigen value times that.

See, what does it do a matrix operated on a vector is giving a vector, earlier when we were dealing with equations of this form X dot is equal to f X continuous time dynamical system then also this was true, X dot is equal to A X. And then a matrix operating on a vector was giving a vector, but this vector was the velocity vector X dot. In this case what does it give A operated on X gives what not the velocity vector, but exactly where it lands in the next iteration.

So, that is one essential difference between a continuous time dynamical system given by this and a discrete time dynamical system given by that. So, A operated on X, X is the vector gives the vector where it lands, actually lands not how it moves, so if the initial condition is here say in the next iterate it jumps like this then this distance divided by that distance will be the Eigen value or this is the initial distance. So, what will we say if you have this in the Eigen direction then X n plus 1 is equal to lambda X n, lambda is a number.

If that number is say 0.6 or so, then what will happen if you have X n here times 0.6 will be X n plus 1, so this is X n plus 1 and this is X n. So, you know already how to obtain the Eigen values of a matrix, so I am not repeating all that this is rather trivial, but here you have a conceptually slightly different situation where this scalar number lambda operated on the vector is giving me the next iteration here. So, when will I say that this fixed point is stable when the next iterate lands closer to it.

And if you operate again A on X n plus 1 you will get X n plus 2, which will be further closer and in further iterations it moves closer and closer to the fixed point and ultimately converges into it, so what is the condition for that to happen. Now, notice the lambda is simply multiplied with the initial vector to give me the final vector, what should the condition on the lambda be, so that it lands closer it should be less than 1. Notice here an important difference with continuous time dynamical system.

We had decided that the condition for stability is that the Eigen value should be negative for continuous time dynamical system, here the Eigen value should be less than 1, what happens if it is negative. It goes to the other side, say if this is a fixed point and here is the initial condition and the Eigen value is negative what happens, this initial vector multiplied by a negative number will come somewhere here.

In the next iteration it will come somewhere there and so on and so forth. It will toggle, it will flip between the two sides and ultimately it will converge on to that, so still here the condition is what the magnitude should be less than 1 that is it, so I do not care whether it is negative or positive, the condition really is that the magnitude of the Eigen value should be less than 1.

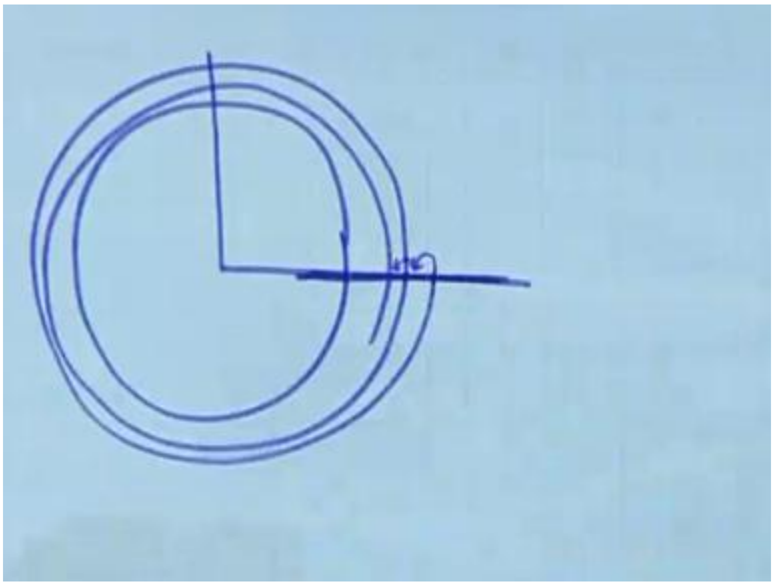
So, now notice the whole thing that we have done, we have started with a continuous time orbit like this, we have observed it in each piercing, as a result we have obtained this map. If we have obtained this map then we would simply locally linearize around this point by obtain the Jacobian, this is the Jacobian matrix. If you have obtained this then we will write down the Jacobian matrix we will obtain then Eigen values of it and if the Eigen values are less than 1 in magnitude we would conclude that the system will be stable, which system this orbit, so we had initially landed with a problem that in non-linear systems we have not only equilibrium points, but also periodic orbits like this and those orbit are also important in nature as well as in engineering. So, we are interested their stability, so we decided that instead of we cannot handle the stability or we cannot probe the stability, simply by locally linearizing in the continuous time space state space, why because it is not a fixed point, it is not a equilibrium point. None of these points would on this orbit are equilibrium points we cannot do that.

So, we took a Poincare section then on that we obtained the map, so what is the character of the fixed point. The fixed point is exactly representation of the periodic orbit because its falls exactly there, so the fixed point is exactly the periodic orbit then in order to study the stability all we need to do is to obtain the Jacobian and find out its magnitude, magnitude of the Eigen values.

Student: ((Refer Time: 13:01))

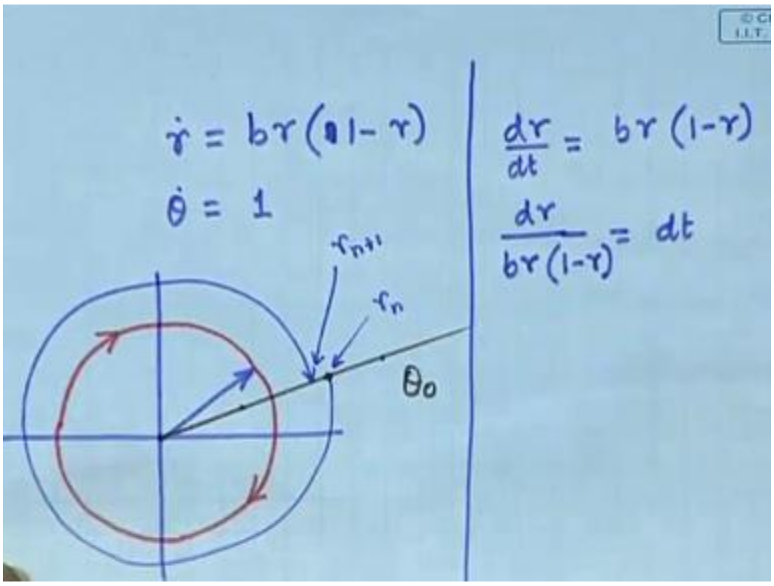
His question is that in obtaining this we are differentiating a discrete function with respect to a discrete variable, how is it possible. I will deal with that question, in order to deal with that question, let us try to understand it in one dimension why. Because in one dimension there are a few advantages let me clarify that. In one dimension the equivalent of the Jacobian matrix is nothing but the derivative, so whether or not we can take the derivative less that is what you have to understand. In that case we have to take a larger system. A system where say, there is an orbit something like this.

(Refer Slide Time: 14:05)



In two d space then how can you take place a Poincare section then just a line will be the Poincare section, just a line will be the Poincare section. If this is your limit cycle in that then this will be, it will go like this and you can say that might this line is my Poincare section, so that this point match to this point, match to this point and. So, on and. So, forth clear. Now, if we start from that premise then it will be somewhat easy to proceed because then we can possibly do the whole job with just one dimension. Let us start with a problem, where you have this kind of a orbit in two d, consider the set of differential equations.

(Refer Slide Time: 15:20)



r dot is equal to b r 1 minus r and theta dot is equal to 1, start with this set of differential equations, so theta and r immediately talk about the radius vector and the angle. So, it says that theta dot is equal to 1 means there is a radius vector which is turning and theta dot is the same; which means, it is turning at uniform speed and r dot this radius vector is changing and the rate of change in this direction is given by this. If you put r dot is 0 then under what condition is that possible r is 0 and r is 1.

r is 0 means right here, so there is an equilibrium point right here, but there is also something very important at r is equal to 1 what is it. If r is 1; that means, this radius vector is 1 then r dot is 0 which means it does not expand or contract. Moreover if r is between 0 and 1 then what is r dot between 0 and 1, this is positive number, therefore r dot is positive which means it should be going out, r dot is positive, if r is greater than 1 then what negative which means it will come back.

Which immediately means that it harbors a limit circle at r is equal to 0, start from the initial condition inside it will go outwards and converge on to it and start from an initial condition outside it will again come inward from outside and we will converge on to it. So, we have got a typical example, of a dynamical system it is a toy dynamical system all right, but nevertheless it has the property that we wanted to study and it turns at a constant rate of theta dot is equal to 1 radians per second.

Now, how do we then discretize the system, we have to place a Poincare section and that Poincare could be placed just any where there is no distance between this point, this point, so it could be placed at any arbitrary angle say this angle. Let it be theta dot, theta naught, so in that case what we will ask, on this Poincare section if we start from an initial condition somewhere here where does it land next. If I start from initial condition somewhere here where does it land next.

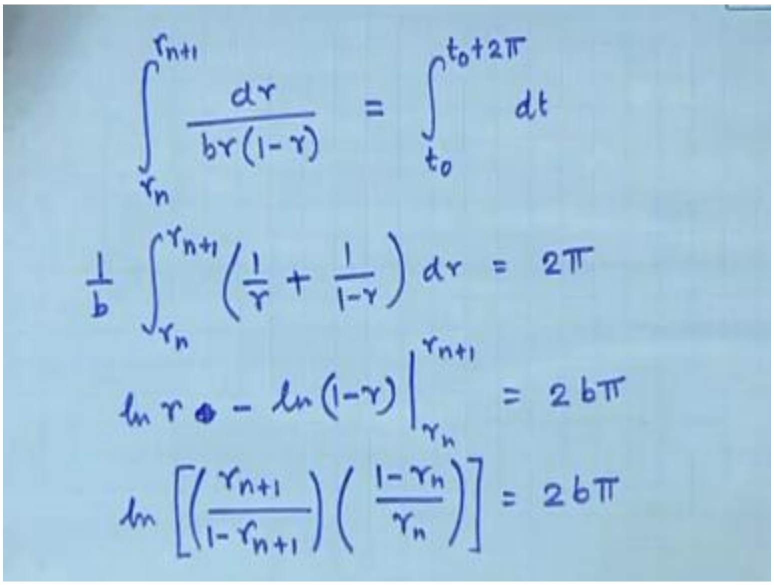
So, we will essentially be talking about an orbit like this, it started from here and it landed it here, we have already decided that the r dot is negative if it is outside, so it will land it here. So, what are we trying to find out, we are trying to find out this r initial is mapping to the r final, so this point is r n, n th point say and this point is r n plus 1. So, our objective now is to find out r n plus 1 in terms of r n that we will do.

Now, to your question it is not really though we are talking about discrete jumps, but actually you could frame this question. Suppose, I now move r n by a slight amount, slightly different amount yes it will come back to some other point again slightly different amount, so this r n is really continuous thing and r n plus 1 is also a continuous thing. Start from any place after having constructed the map we should be able to answer the question starting from any point, where does it land next.

Let us let us go about doing this then it will be clearer, so how do we obtain r n plus 1 in terms of r n. Since these two equations are not coupled they are separate theta does not appear in the first equation we can solve the problem simply by integrating this, so that is what we will do. So, we have here d r d t is equal to b r 1 minus r, let us separate the variables, so you have d r by b r 1 minus r is equal to d t.

Now, let us integrate it, so can I move to the next page, but we will have to come back to this picture, so just keep this picture in mind and let us do the algebra in the next page because it may become little bigger. So, we had this equation and then this will have to be integrated from r n to r n plus 1 and how much is the t is actually a complete rotation, a complete rotation is plus pi, so the limits of the integration would be this is…

(Refer Slide Time: 21:39)

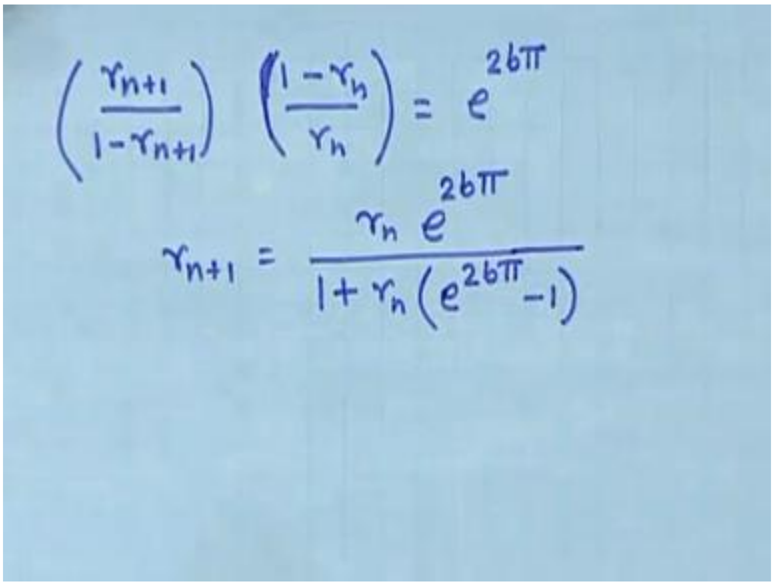


r n to r n plus 1, what are we integrating d r divided by b r 1 minus r is equal to t is integrated over some t naught initially, some time here to t naught plus twice pi d t, do this integration you know how to do this integration better than me, b can be taken out, b common. So, you have integral to r n to r n plus 1, we need to separate them out, it would be 1 by r minus 1 by 1 minus r, this whole thing d r is equal to we can directly integrate it will be 2 pi.

Student: ((Refer Time: 22:53))

So, then we can do the integration, let us take the b to this side it will be l n r minus l n 1 minus r this will be from r n to r n plus 1, b has come to this side, so twice b pi, then this will be l n of I want to write r n plus 1 in one side and r n in the other side. So, it will be r n plus 1 by just check this steps, minus r n plus 1 this is one term and the other term is 1 minus r n by r n is equal to twice b pi. So, now we can since, we have taken the l n out we can get this out and this would become exponential twice b pi, then we can extract it out.

(Refer Slide Time: 24:50)



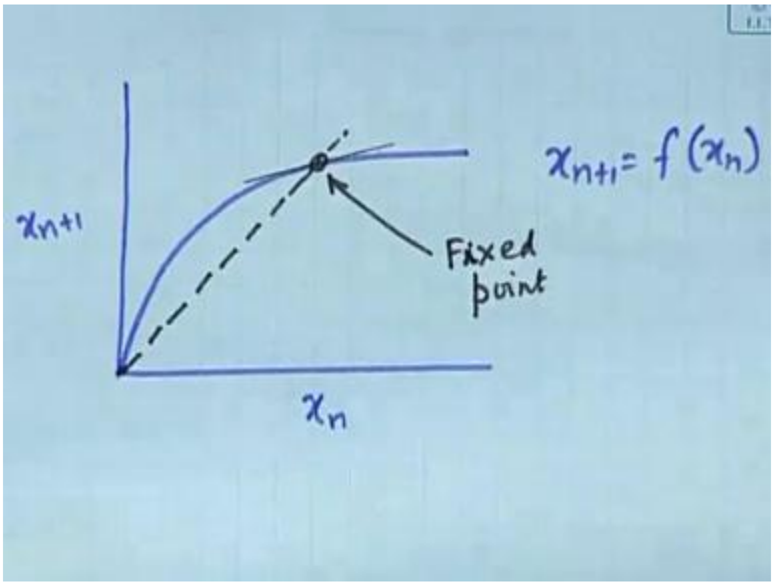
So it will be r n plus 1 divided by 1 minus r n plus 1 times 1 minus r n by r n is equal to e to the power twice b pi. So, from here we can easily extract r n plus 1 in terms of r n, just algebra from here, so if you obtain it you will get r n plus 1 is equal to r n e to the power twice b pi divided by 1 plus r n e to the power twice b pi minus 1. Now, notice his question r n could be just anything, r n is at the n th iteration if the distance is this much, where will it land next that is the question we are asking.

So, this much is it could be anything, we could ask this question related to anything there is no reason to say that it is one inch and not one point one inch, so if this question can be answered with respect to one inch distance, it could also be answered with respect to two inch distance, it could also be answered with respect to 1.005 inch distance. And therefore, r n is a continuous variable, r n plus 1 is also a continuous variable, so even though we are talking about discrete jumps, ultimately we have obtained a function and it is a continuous function.

Student: ((Refer Time: 26:45))

No, the time scale is one radians per second, see theta dot that is why we did that, this is a special case that is why normally you will have to take that into account. Now, so in general therefore, we will get for a two dimensional thing we will get a one dimensional map like this. So, if it is a one dimensional map the advantage is that we can draw a graph of the map; what does it means, it means that we can draw say we have obtained.

(Refer Slide Time: 27:43)



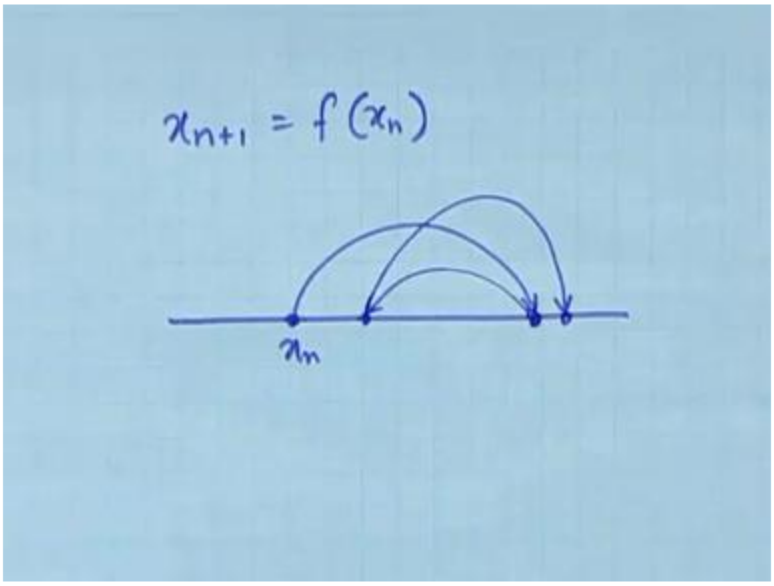
x n plus 1 is equal to some function of x n this is 1 b, so I am not writing it as vector, so here I will write x n and here I will write x n plus 1, this function can be plotted as a graph, this cannot be possible in two dimension or higher dimension, but in one dimension its possible. For example, this particular function what will be the character, if r n is 0 then what is r n plus 1, 0, now differentiate r n plus 1 with respect to r n. We can at r n equal to 0 what is the slope if you do it that way, you will find that it has a shape something like this.

So, you do not draw a graph, it is possible to draw a graph and then you will see that much can be obtained simply by looking at the graph, much can be inferred simply by looking at the graph, like what. Like for example, if I ask you what is the fixed point of this system, it can be simply obtained by drawing the 45 degree line and here it intersects, where it intersects x n is equal to x n plus 1, so that is the point which is the fixed point.

So, the fixed point is obtained simply as the intersection of the graph of the map with the 45 degree line. The equivalent of your Jacobian matrix is nothing, but the slope the derivative yes you can take a derivative, now you see at every point you can take a derivative and what is the meaning of locally linearizing at the fixed point. It is just this line having a specific slope, a straight line which is tangent to the graph of the map at the fixed point.

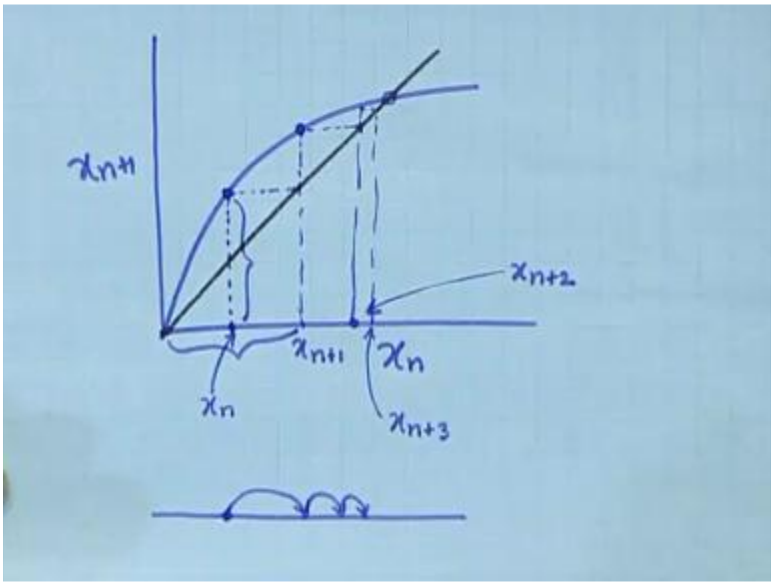
Now, then much more can be inferred, if you look at this graph because if the initial condition is close to that fixed point then it will more or less follow this particular straight line. So, let us try to understand what happens, if you have this straight line, before that let us try to understand something more.

(Refer Slide Time: 30:49)



When we have x n plus 1 is a function of x n, what actually x n is a point on the real line any point on the real line, what does f x n d, it takes it to another point on the real line. It is like, suppose it is a real line and here is my x n, and then this function is saying that in the next iteration it will jump there, in the next iteration it will jump here, in the next iteration it will jump there. So, on and So, forth, that is what it is saying. Now, that can also be obtained graphically, how suppose your graph of the map is as we have drawn.

(Refer Slide Time: 31:37)

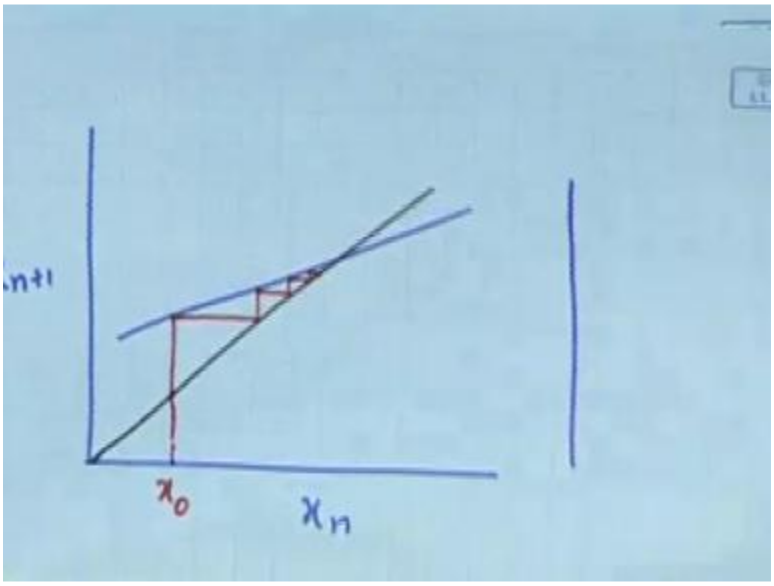


For that example system like this x n and x n plus 1 and this is the 45 degree line and here is our fixed point then suppose we have started from an initial condition somewhere here. What will be the next value of x n plus 1, it is just this, if you go up is this value. If you now want to iterate what will you do, x n plus 1 will have to give rise to x n plus 2 which means this value, this value, you will have to take along the x axis and then find out where it goes next.

A simple way of doing it is to come to the 45 degree line and come down here, so this is your x n plus 1. So, here we went up to the graph of the map and then we wanted to bring this one, this one here that is what we have done, this part we have brought here and what will be x n plus 2 simply from here go up to the graph of the map here, so this is x n plus 2, where will be x n plus 3. Simply come to this and here, this will be x n plus 2 and then go up to the graph this will be, if you go to the 45 degree line it come down here this will be x n plus 3, here is your x n.

So, what has happened on the real line we started with x n it jumped to x n plus 1, it jumped to x n plus 2, it jumped to x n plus 3 and so on and so forth. Do you see that it is the distance is progressively going down and it is converging to something, converging to what the fixed point, why and under what condition would that happen, let us try to consider that.

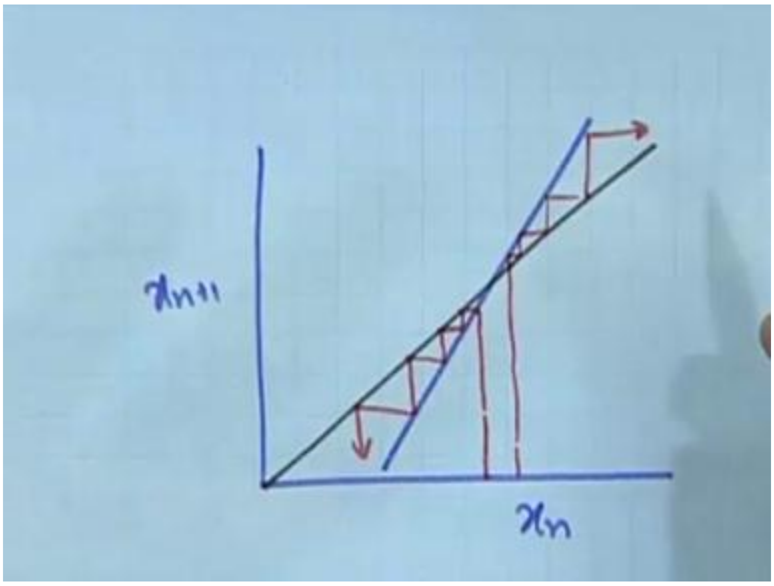
(Refer Slide Time: 34:30)



Now, in order to consider that, this is the 45 degree line and let us say the graph of the map is like this, I am taking the only the tangent to it, so that we are essentially blowing up what happens in this neighborhood. Now, suppose you start from this point, I will go in red, start from this point how will you go this my x naught say starting point, then I will go up go up this way, go this way, go this way, can you see that it is converging, it is progressively converging, why did that happen.

That happen because, what should I say because the slope of this line is less than unity therefore, this line is always bigger than this line, so it will always go like this and converge, just contrast it to the other case. Suppose, you have I will draw in another page.

(Refer Slide Time: 35:51)



And suppose the graph of the map is like this with the slope bigger than 1 and suppose you start from a initial condition say here, say I will start close to the fixed point, it will go like this, next iterate would be here to obtain the next iterate you will have go to the 45 degree line again come back to down to here, again come down here is going outwards. Start from a point very close to the other side going outwards, so you can see that we are doing it completely graphically.

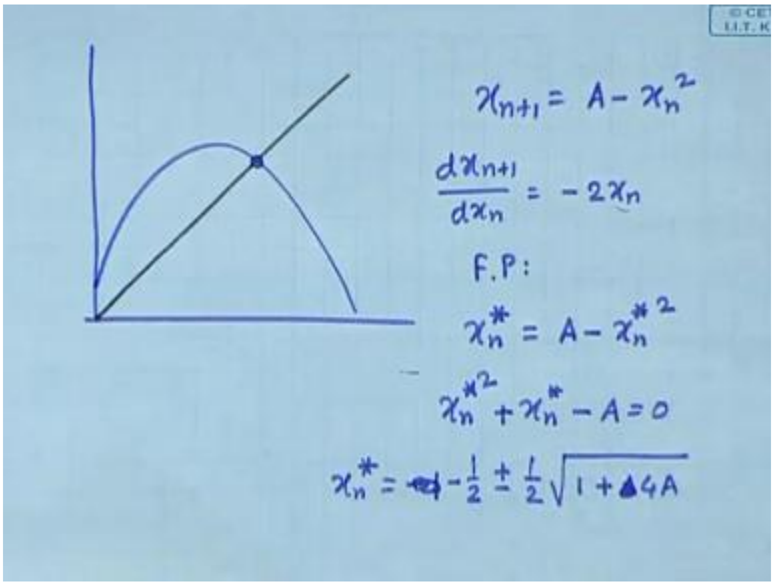
We can easily do that by pressing a calculator all right, but nevertheless we are trying to develop an understanding, a graphical understanding and we can see that the moment the slope of the graph is greater than 1 it goes further iteration go outwards, go away from it and so the system is unstable. While in this case, if the slope is less than 1 it is stable that is what we were decided really, when we talked about the Eigen values we decided that the slope, in that case we talked about the Eigen values.

The Eigen values becoming less than 1 will guarantee the stability, here this Eigen vales are equivalent to the slope and this slope becoming less than 1 we will guarantee stability, the slope where, the slope at this point. So, again let us take stock of the situation, we said that a continuous time orbit we had difficulty in starting the stability, we said that we will place a Poincare section, obtain the map then obtain the local linearization by the taking the Jacobian, if it is a 1 d map the Jacobian is equivalent to taking the derivative.

Fixed point is obtainable by the intersection with the 45 degree line and then the stability of that whole system is given by the slope at that particular point. So, we have essentially solved the problem of stability of the continuous time orbit that was a limit cycle, we have understood how to tackle the question of stability of the limit cycle, but now there is something more to it than meets the eye at this stage.

The point is that in a linear system, if you find that by obtaining the Eigen values around an equilibrium point, you find that the Eigen values have positive real part you know that it is it become unstable, unstable means initial conditions starting from any initial condition it will run to infinity, this system is unstable the system will collapse. One peculiarity of a non-linear system is that even if a system can become locally unstable, there is no reason to believe that it would be globally unstable. So, if one behavior becomes unstable, another behavior can become stable. Let us illustrate that.

(Refer Slide Time: 39:57)

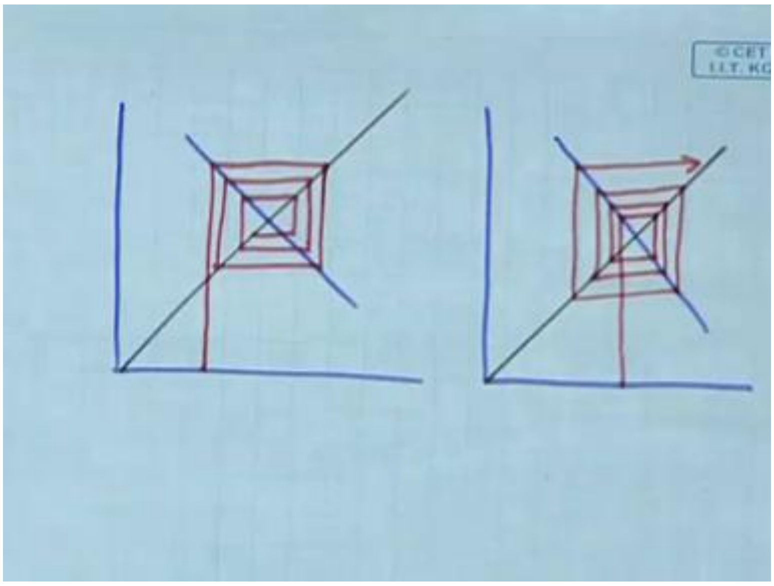


Suppose we take a map x n plus 1 is equal to A minus x n square a very simple, non-linearity just a square. How will the graph of the map look parabola, parabola like this, so say this is the parabola, how will it be yes you can start drawing like it, so where is the equilibrium point, where is the fixed point here and what is the slope of this. Notice we can do that by hand, how what is the slope of this d x n plus 1 d x n is equal to minus twice x n. What is the slope, in order to know the slope exactly we have to substitute x n.

That means, you have to substitute the condition for the fixed point, what is the condition for the fixed point x n plus 1 is equal to x n, so the fixed point is given by x n I will call it x n star is equal to A minus x n star square. So, this gives a quadratic x n star square minus x n star no plus minus A equal to 0, so, x n star is equal to minus half plus minus half root over 1 plus 4 A, so it depends on the value of A, the position of the equilibrium point depends on the value of A.

Now, suppose we take a certain value of A then we will get this position exactly and substitute that here, we will get the slope exactly. Now, if I ask you at what value of A does it exactly become minus 1 can you say you can easily say because in that case it will have to be half and in order to get half here, what you have for A you can easily calculate got it. So, this is the parameter at a specific value of the parameter the slope becomes exactly minus 1. And we know that when it become minus 1 it will become unstable, exactly what happens when it becomes minus 1, let us check out.

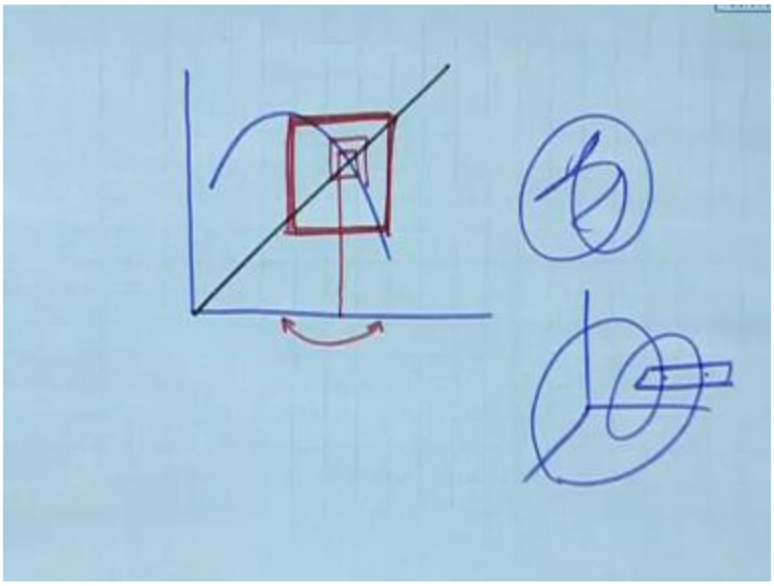
(Refer Slide Time: 43:44)



If you have the slope of the graph like this exactly, no slightly less than minus 1, I am drawing then how will the iterations go, iterations go like this then for to the 45 degree line to here. So, if the slope is less than minus 1 then it will go like this, have you understood this graphical iteration process, we are first going to the graph of the map then the 45 degree line, again to the graph of the map, 45 degree line and so on and so forth.

If the slope is greater than minus 1 like this then how will it go, it will go like this, it will go out, so the limiting condition is exactly minus 1, so long as it has a slope less than minus 1 then it will be stable, if it becomes a slope greater than minus 1 it will become unstable. So, in this case at a specific value of A it becomes unstable below that it will be stable, so long as it magnitude is less than minus 1 it will be stable or after that it will become unstable. Now, when it becomes unstable, at this point the slope has become greater than minus 1. So, let me draw it separately.

(Refer Slide Time: 45:49)



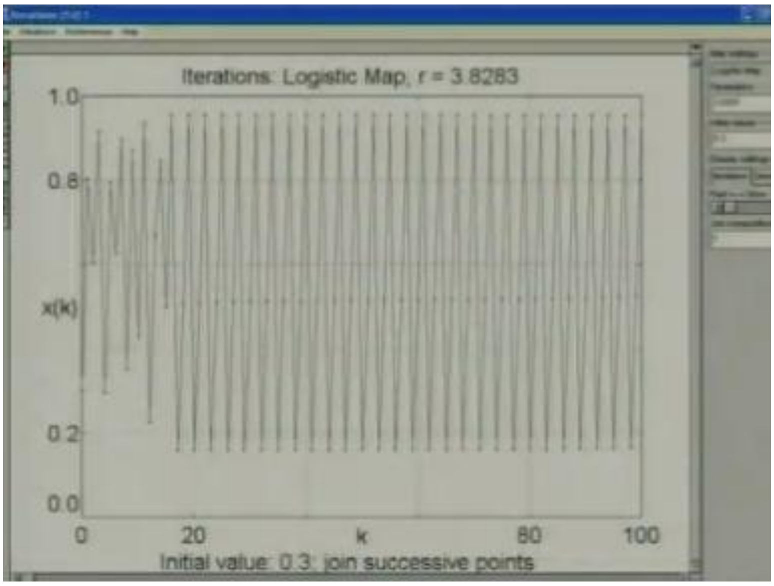
The slope has become greater than minus 1, so take any initial condition close to it, it will diverge out, but will it be able to diverge out independently no because the slope is reducing here. So, after sometime it will reach a state where, it will become like this, what does it mean, it means that it is become what a period two orbit, because it is toggling between two values, it is going from here to here again back here.

What is meaning in continuous time dynamical system because, ultimately what are we doing, we are we are looking at the Poincare plane and what is happening there, there we find that it is become a toggling between two points. What is happening in the continuous time dynamical system it is actually, now an orbit like this a period two orbit, so that if you have two there are two intersections, so this is how even though one fixed point becomes unstable the system does not become globally unstable, it is only locally unstable, but globally another behavior develops a period two behavior.

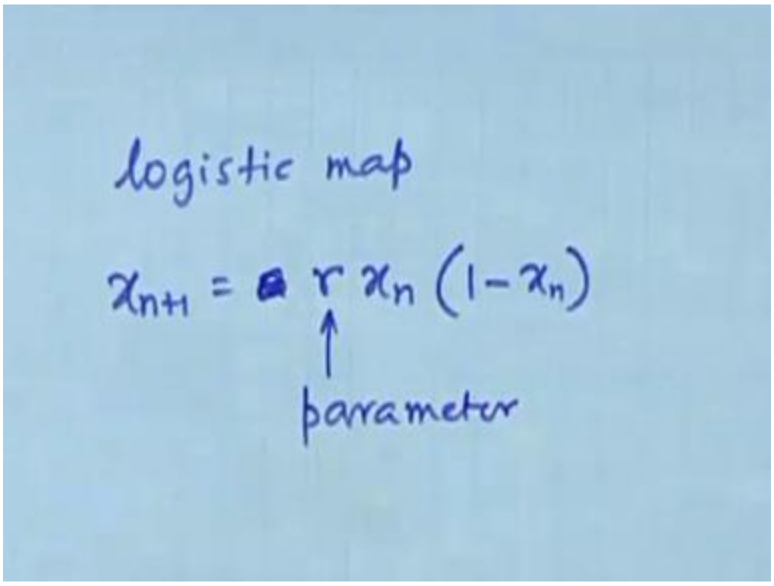
So, similarly you can figure out that in a non-linear system if some place, some part of the space becomes unstable that does not mean it is globally unstable, you have a large number of possibilities then the whole state space will be divided into parts, in some parts it is unstable that does not mean it elsewhere unstable. In this part if it is unstable means a whole orbit can become stable. If this whole orbit can become unstable still a period two orbit may became become stable.

A period two becomes unstable still a period four orbit may became become stable, which means that stability needs to be understood in a different context in a non-linear system. In the linear system the stability is often understood as whatever we infer at a local level is also true at global level, while in a non-linear system whatever is true at a local level is to at a local level only, you cannot extrapolate that idea to a global level. So, with that understanding, let me illustrate a few things on the computer before we stop today.

(Refer Slide Time: 48:48)

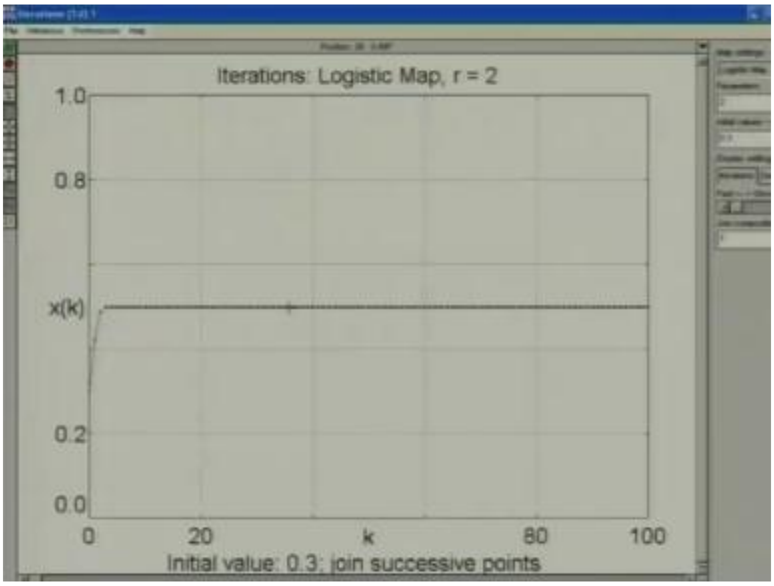


(Refer Slide Time: 48:57)



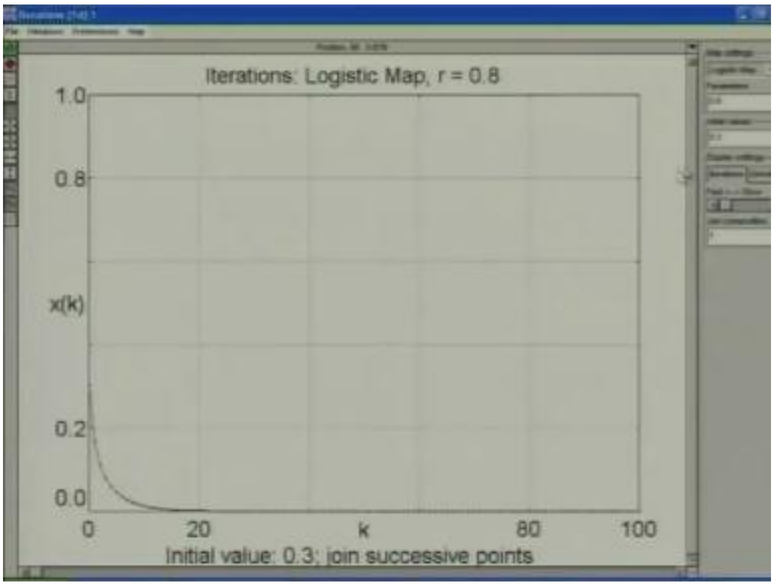
For example, here is the map that I just no not that, here is the map called a logistic map, which is x n plus 1 is equal to a or normally it is written as r, r x n 1 minus x n, r is a parameter, so in order to study this behavior, its behavior what will you do, you will start from any initial condition and go on iterating it. So, start from x naught this will give you x 1, put that back here it will give you x 2, put that back here it will give you x 3 and. So, on and. So, forth you get a sequence of numbers. Now, if this were say.

(Refer Slide Time: 49:50)



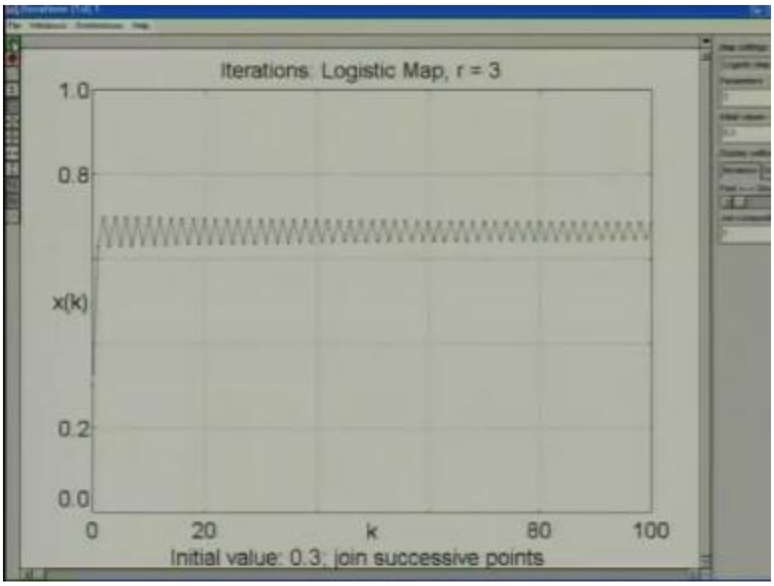
Do you see that all of them ultimately starting from some initial condition converge on to a number and then is steady there flat, start from a smaller number, smaller value the parameter say 0.8.

(Refer Slide Time: 50:14)



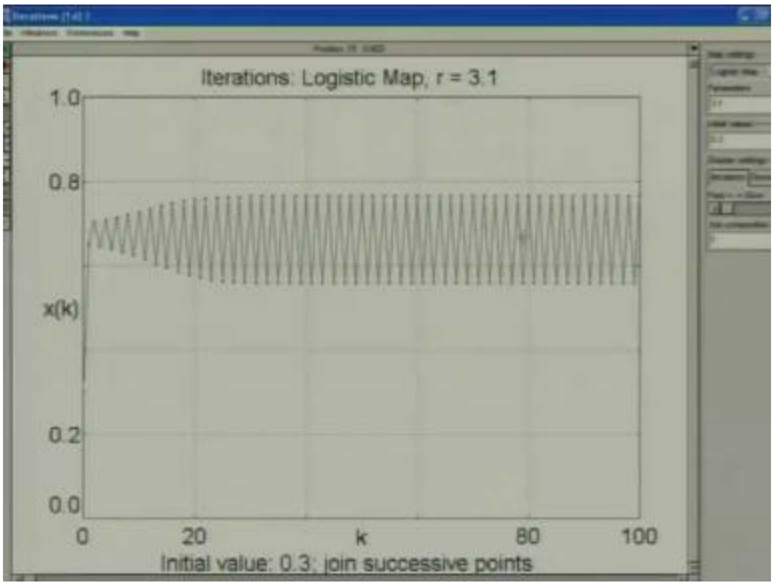
It goes to 0, start from any initial condition you get a sequence of number that goes to 0 everything dies down, but if you start from say the parameter r is equal to 2 then it goes to a definite value, but now make r 3.

(Refer Slide Time: 50:38)



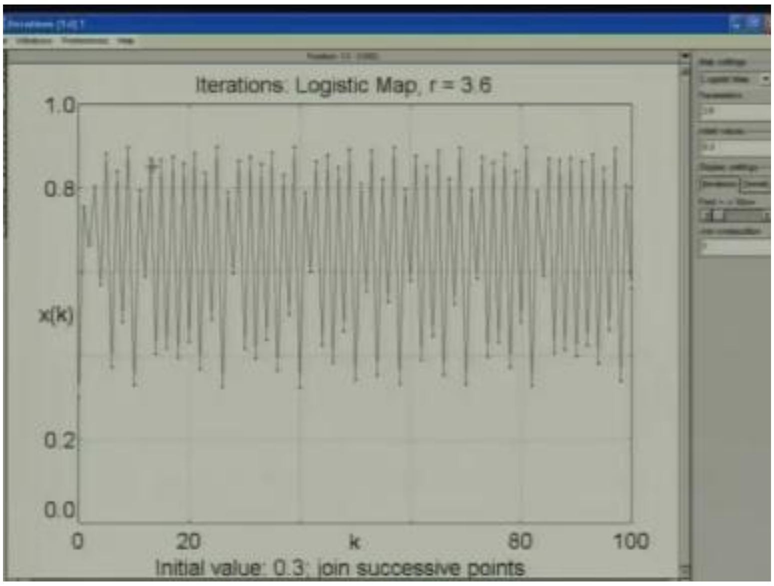
See, it toggles between two numbers, there is a initial transient that goes for a long time, let me make it slightly bigger it will be better.

(Refer Slide Time: 50:54)

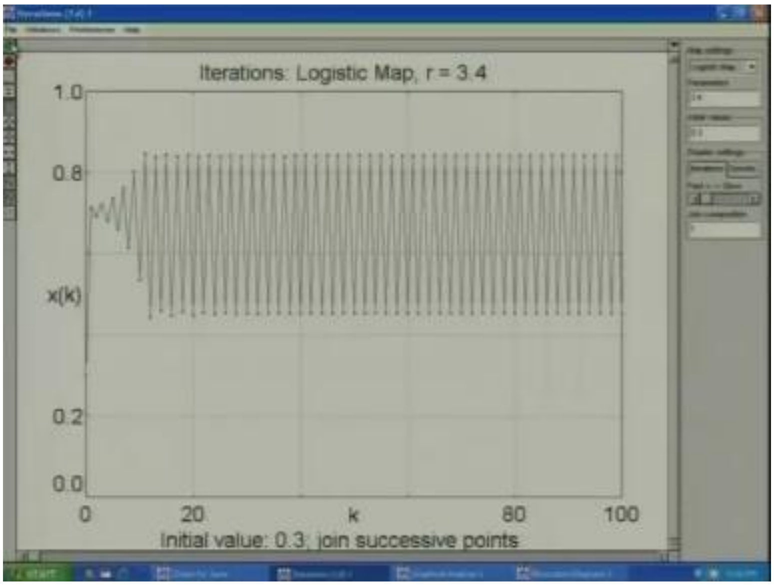


See there was some initial transient, but ultimately it converged on to two states repeating each other; which means, it is a period two orbit. In the continuous time dynamical system it is a period two orbit like this, but here it is seen as the state toggling between two numbers. Let me now increase the value.

(Refer Slide Time: 51:24)

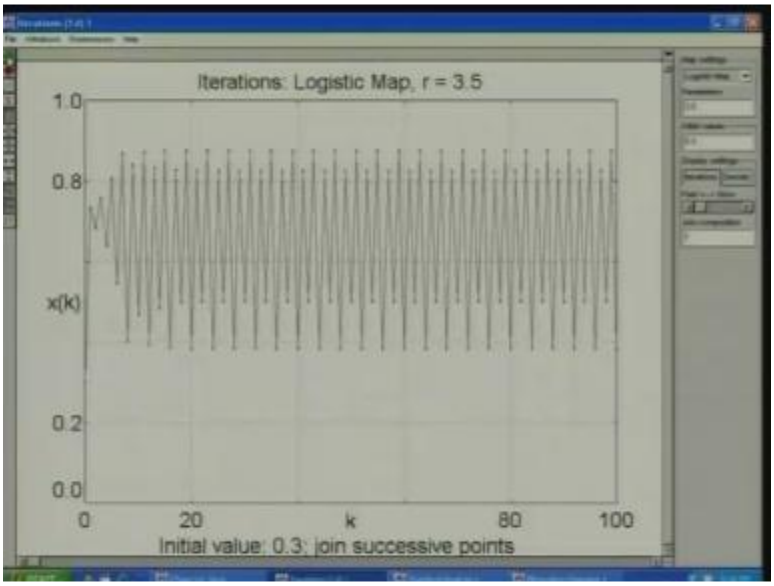


(Refer Slide Time: 51:31)



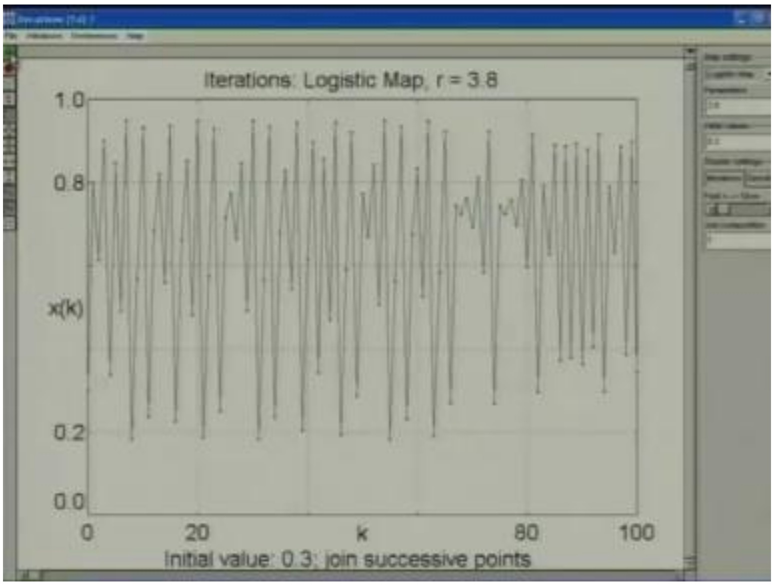
It is still a period two orbit. I have now put 3.4 and now I will put 3.5.

(Refer Slide Time: 51:39)



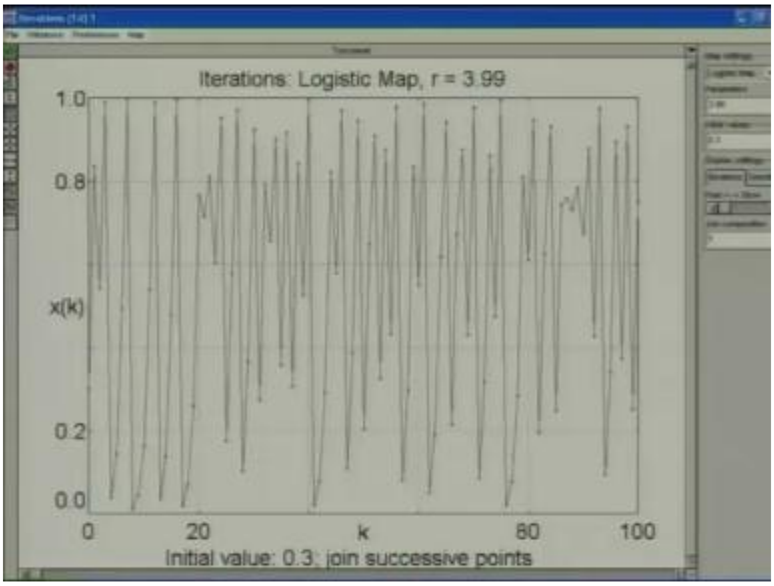
It is a period four orbit, so I am changing the parameter it is going from dying down to 0 to period 1 orbit, period 2 orbit, period 4 orbit and then if I further increase it.

(Refer Slide Time: 52:04)



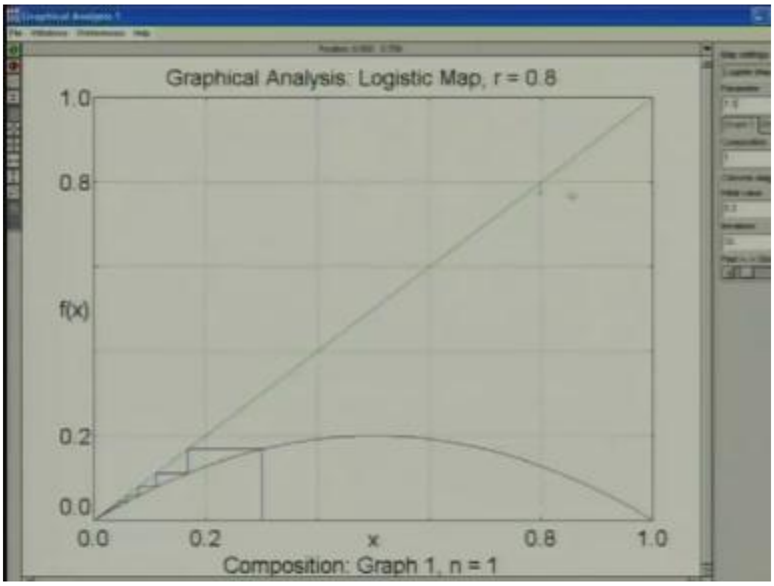
It is completely, what is it chaotic because no state is repeating itself, no state is repeating itself.

(Refer Slide Time: 52:20)



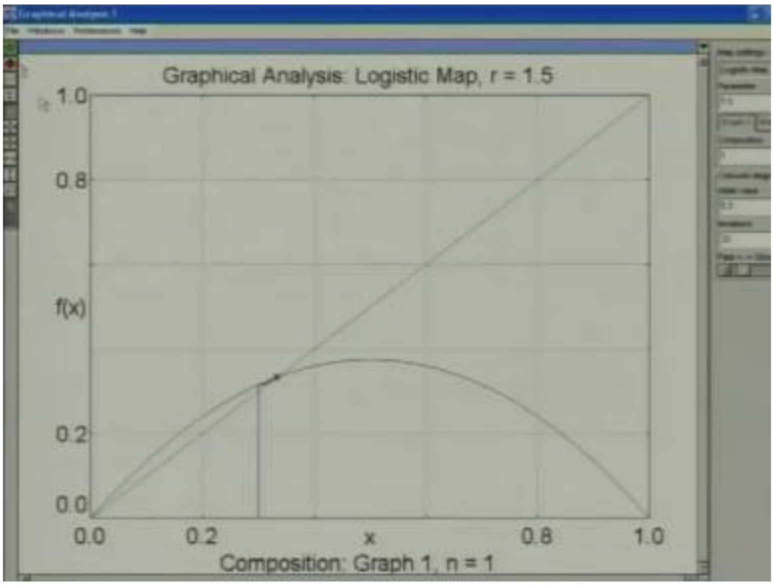
Yes no state is repeating itself. Now, let us take a look at it from the point of view of graphical analysis, so let me start with, we started with 0.8.

(Refer Slide Time: 52:41)



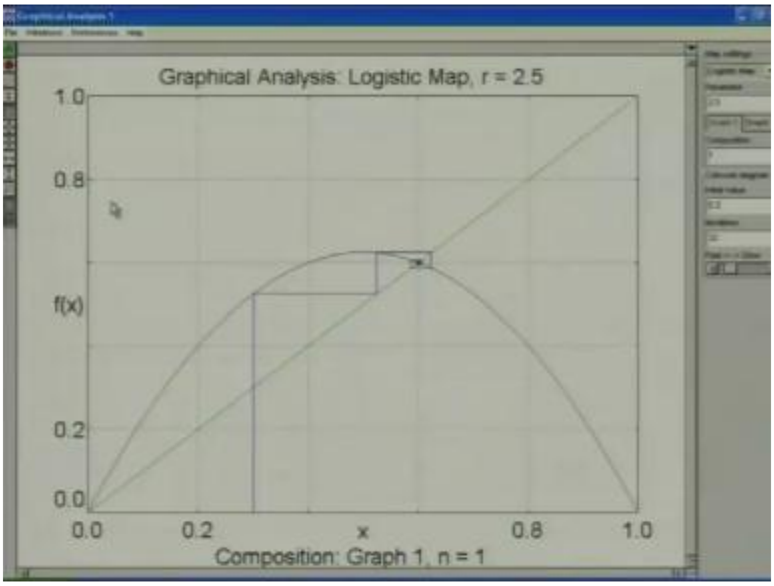
See then this is the graph of the map and since the slope here, this is the fixed point the slope here is less than 1, therefore starting from any initial condition it goes to how can you see how it goes. Let us change it to 1.5.

(Refer Slide Time: 53:01)



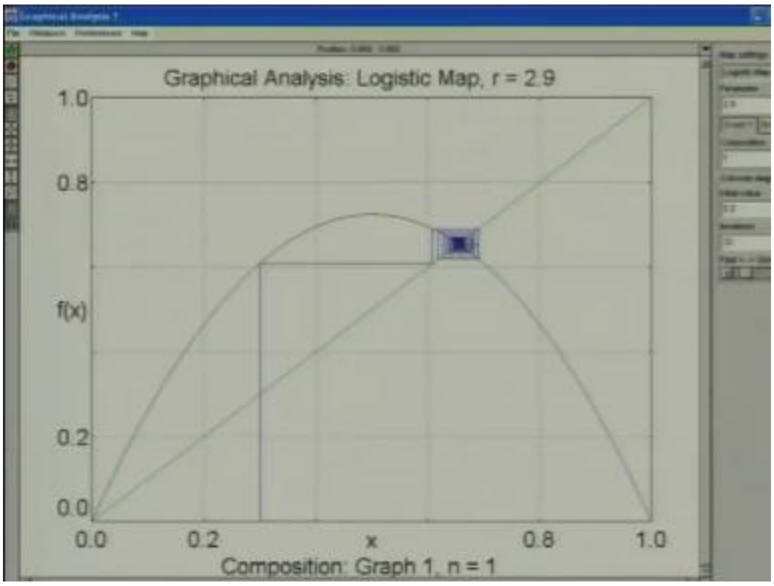
See now this is a fixed point, this is another fixed point here the slope is greater than 1 and here the slope is less than 1, and so starting from n initial condition it will converges on to that, now let us make it 2.5.

(Refer Slide Time: 53:23)



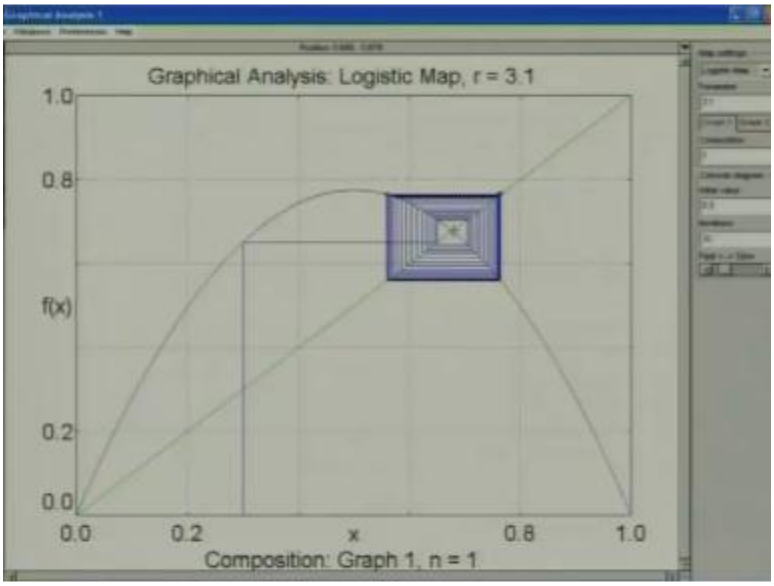
The slope has increased, but it goes like this and finally converges, but here at this point the slope is negative, so it converges from both the sides.

(Refer Slide Time: 53:44)



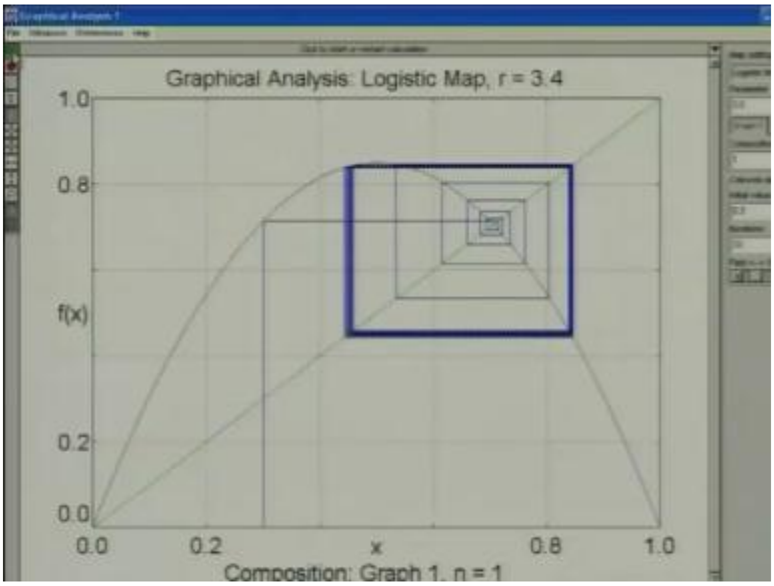
Let us increase it 2.9 may be can you see how it is converging, it is converging from there. Now, let us make it above 3.

(Refer Slide Time: 53:57)



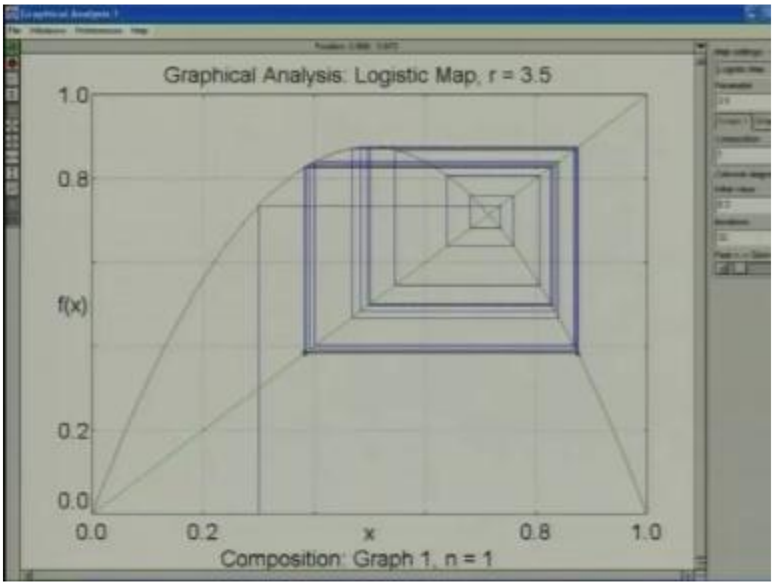
See it goes there and then after that it goes out because this point has become unstable. As it has become unstable it goes out, but after that this orbit a period two orbit has become stable that is what you saw earlier the same map, that is what we saw earlier when it was a toggling between two values, all that can be inferred graphically. Now, let us increase it to 3.5 or 3.4.

(Refer Slide Time: 54:31)



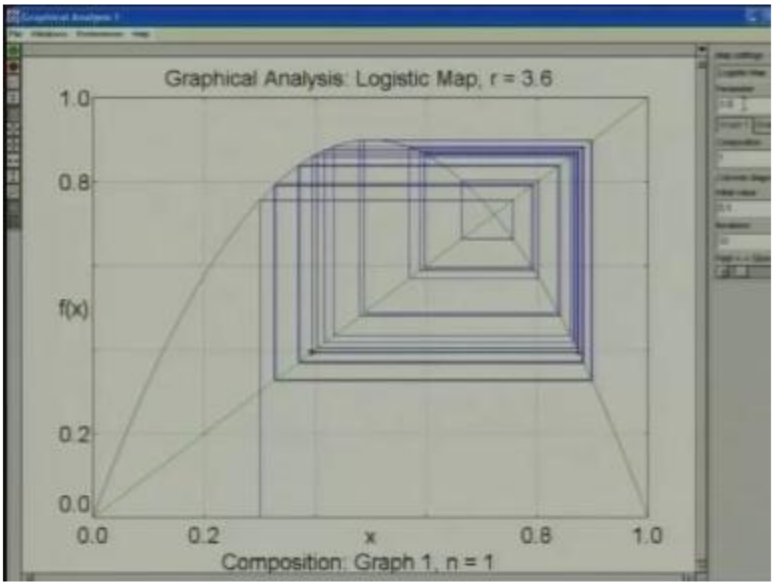
Still period 2.

(Refer Slide Time: 54:38)



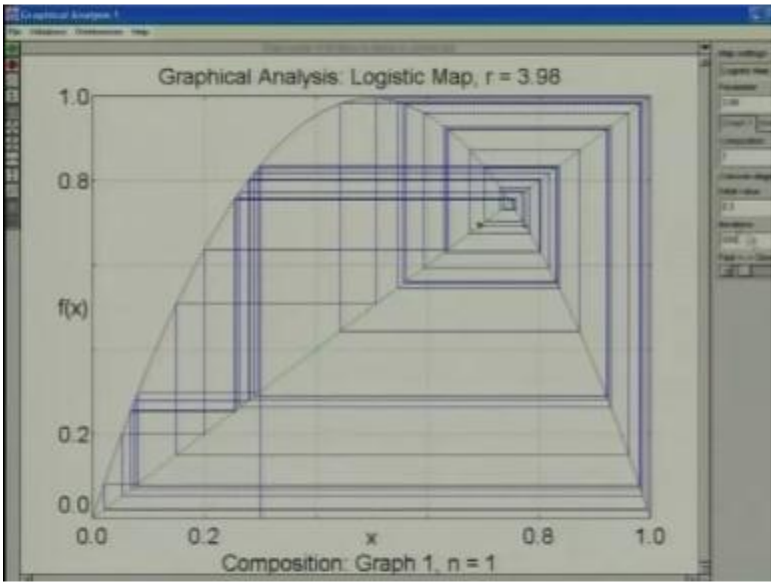
Can you see now it is period 4, let me do that again.

(Refer Slide Time: 54:48)



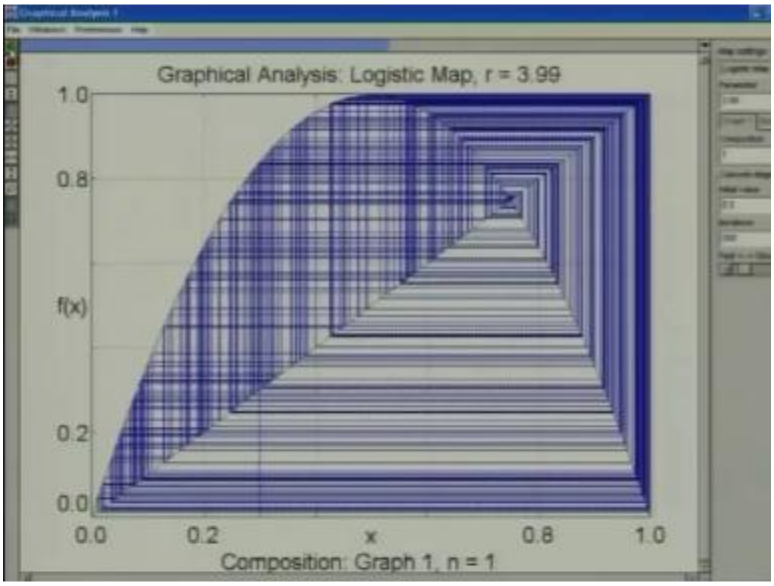
So, it is now going on with that let me, increase it then will be clearer, see now same points are repeating. It was homing on to that that is why this part is find a little thicker, but actually it has to be homed on to a period four orbit, but now if I increase it to a large value.

(Refer Slide Time: 55:27)



It is completely chaotic, but now I will have to draw large number of points otherwise it will not be clear.

(Refer Slide Time: 55:40)



Keep on seeing what is going on, no point is repeating itself and that is what is chaos. So, we can see that even with this very simple map, which you can do any algebra with simply by hand, still you can find this kind of completely chaotic behavior appearing in it, that is enough for today. Let us call it a day.