Dynamics of Physical Systems
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Lecture - 32
Dynamics of non-linear systems – III

In the last class, we saw that in a non-linear system there can be a large number of possible behaviors, which are not possible in linear systems, so what have we seen, first we have seen.

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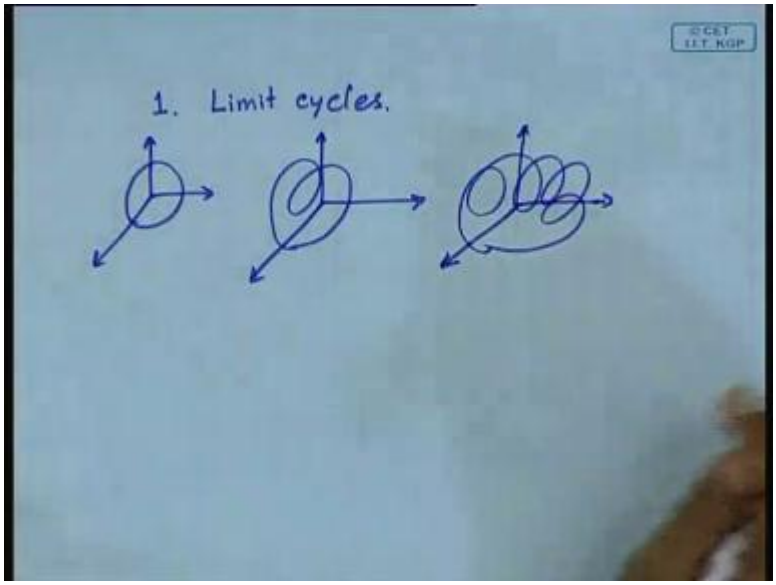

There can be limit cycles and in limit cycles, we have seen that there can be various types of limit cycles like they can be periodic orbit like this where I am drawing the axis here, they can be periodic orbit like these also if the system is three d, that could be a periodic orbit like that is also possible. And then we have seen that they can also be orbits that cannot be called periodic, the same state never repeats itself. So, in every terminate it refers as the new path and such system, we will learn are called chaotic and yesterday we saw one mechanism of that will happen, what was the mechanism.

In the system that it concentrated in the last class, there were two Eigen planes not parallel to each other they are intersecting with each other in one the rotation of; that means the corresponding equilibrium points are unstable and they have complex conjugate Eigen values with positive real parts. So, that this spiral outwards, but in one

the outwards spiral in behavior is clockwise the other the outwards spiral behavior is counter clockwise.

As a result as it goes out it gets thrown into this plane and as it goes out it gets thrown into this plane and as a result of this motion the whole thing remain bounded, so that was the mechanism that we learnt. There can be various other mechanism also of producing the phenomenon that we talked about what is the class of the phenomenon, one the orbit is bounded it does not go to infinity, two it has no periodicity and three there is sensitive dependence on initial quantization; that means, if you start from arbitrarily close initial conditions then after some time they will move apart.

What they it is physical implications are I will come to that little latter. But, first let us look at another system just to make our self comfortable because otherwise there is a possibly that we may we will late to think that happens only in that systems, so there are actually a very, very large number of physical systems in which happens. But, most physical systems as we have seen are yield somewhat complicated equations, here let us deal with somewhat simpler equations because it is also point that one is often late to believe that such things happen in very complicated systems.

No not true, such things actually happen in very simple systems and that is why it is more logical to consider a simple equations to show that even in such systems this kind of phenomena can happen. Let us take another set of equations called a Rossler equations.

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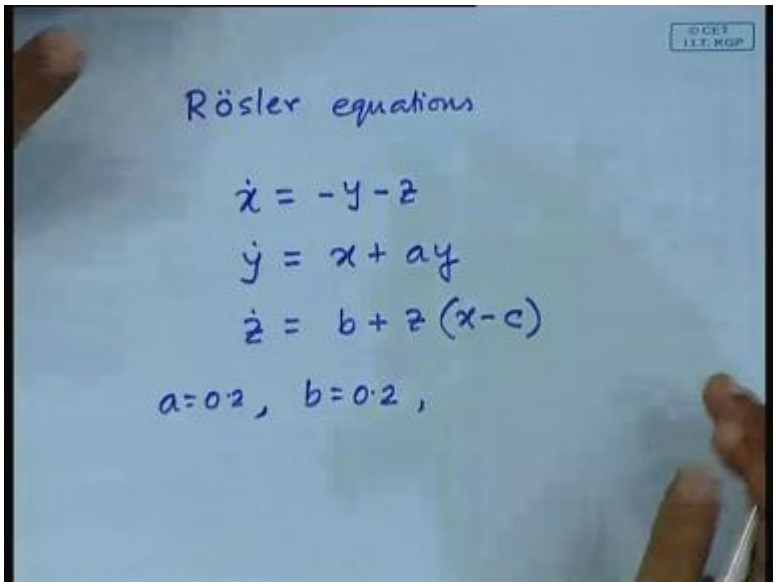

This is o with um love it is a german thing, with pronunciation Rossler, you make the lips as the if you are trying to speak o, but you will actually try to speak a, so whatever comes out from your mouth that is the oo, so Rossler Equation, equations are are x dot is equal to minus y minus z, y dot is equal to x plus ay and z dot is equal to b plus z x minus c, so these are the equations, three variables x y z and three parameters a b c. Out of that we will assume a is equal to 0.2, b is equal to 0.2 and we will vary c and we will see what happens. So, let us start with that let us start with this orbit.

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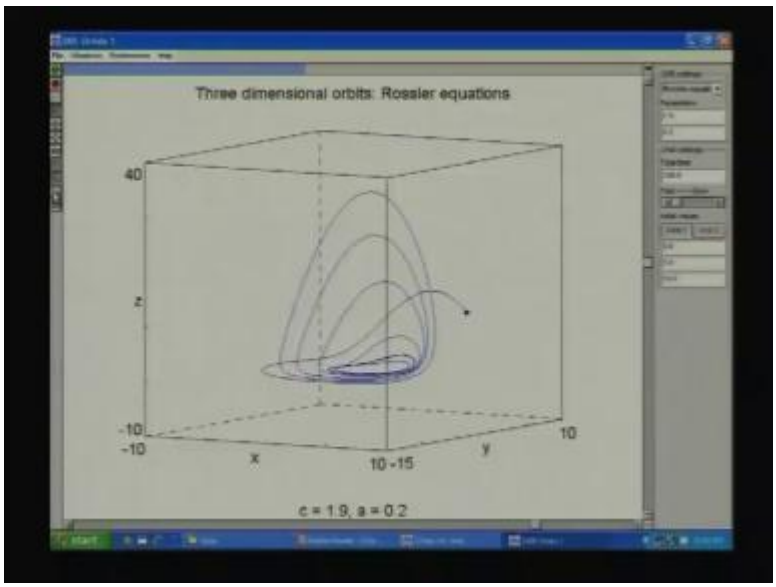

You see it is doing various things, but ultimately converging on to a periodic orbit, you can see that, so it has converge on to a periodic orbit. So, if I all these were initial transcend really, so that is not of our concern, we are concern with what is the steady state being and it is a periodic orbit. In this case you have parameter c was 1.9, let us increase it say 2.

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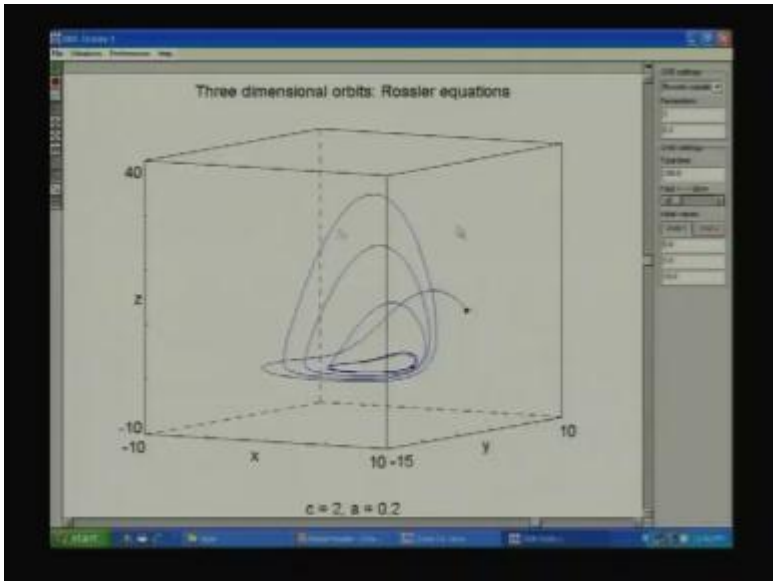

I will make little larger.

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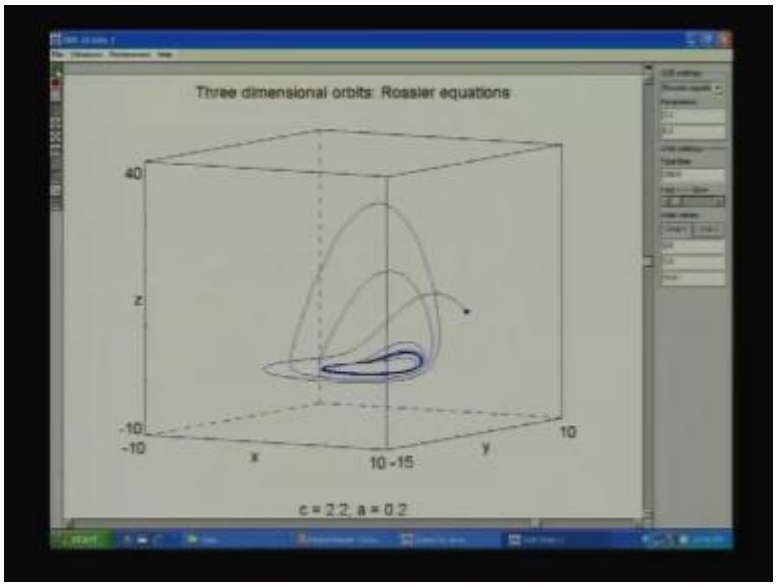

So, now, see what is happening, it is again converge on to an orbit which is a periodic orbit you can see that, so let us increase the parameter slightly say to 3.

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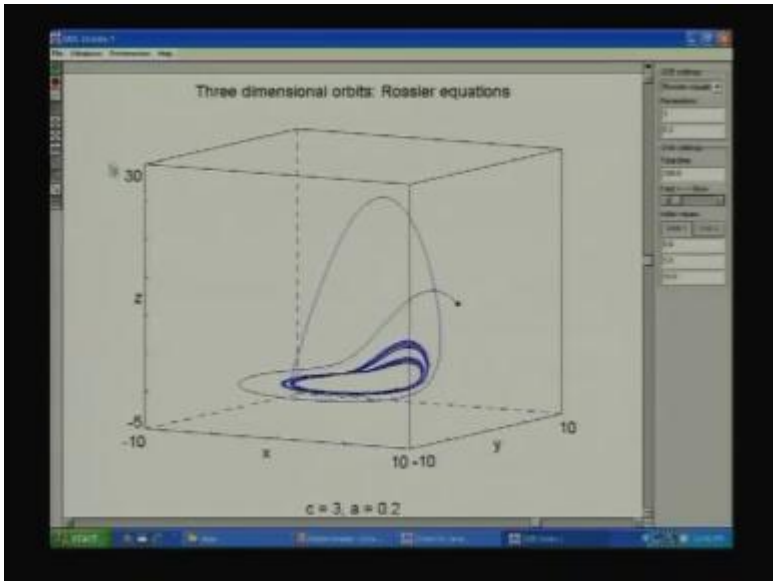

Then if you now run it, now you can see what is happening, it is actually traversing 2, I will make a slight less then it will be clearer, 2.8.

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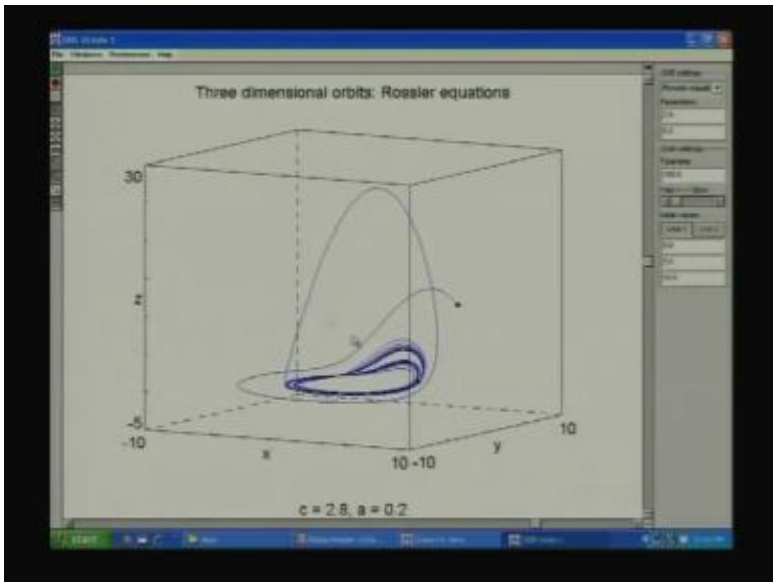

As it is doing, so let me now it will be easier to see, see what is happening. What we will call it a period 2 orbit, so it has two loops, clearly it has two loops, so it is slowly

converging on to that, that is why you see that as thick, but actually it is a period 2 orbit, so, if you now increase it say 3.2.

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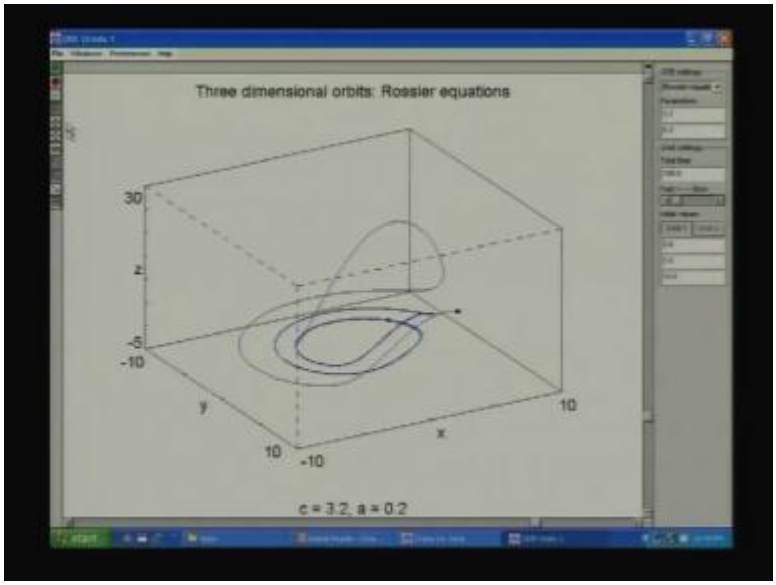

Still, a nice period to orbit, can you see that, I preferred to done it further then it will be easier to see. Now, I will make it 3.4.

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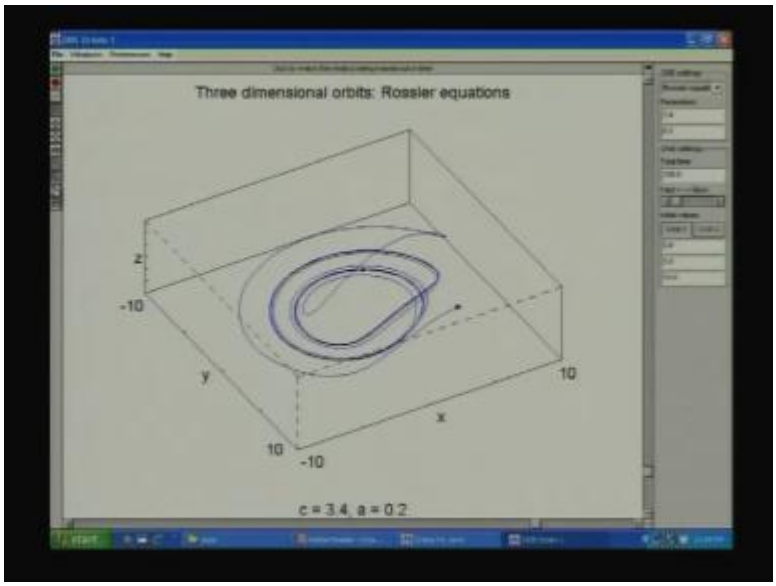

And slowly increasing the parameters in steps, it is still a period 2 orbit.

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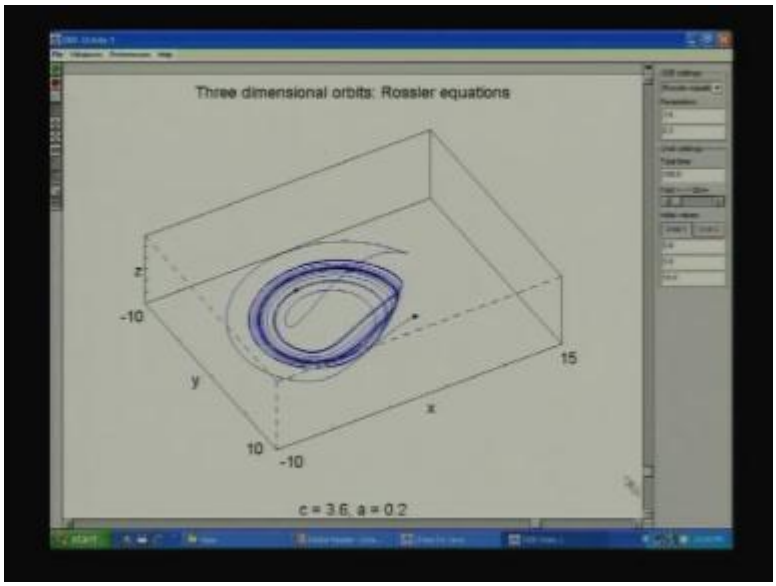

It is I do not know what, something strange is happening. Let us see it from again from that end where you start, your orbit is actually not stable that is coming close to it, but it is going away, can you see. Here you can easily see that these orbit is a, you can see the plane, you can see the plane and you can easily figure out, can we place initial quantization somewhere in the middle, this is minus 10, this is 15, so here you could use something like 0 and here also it will be 0 and here it will be like minus 5. Let us do that, 0 0 minus 5, let us start the initial quantization there and see what happens.

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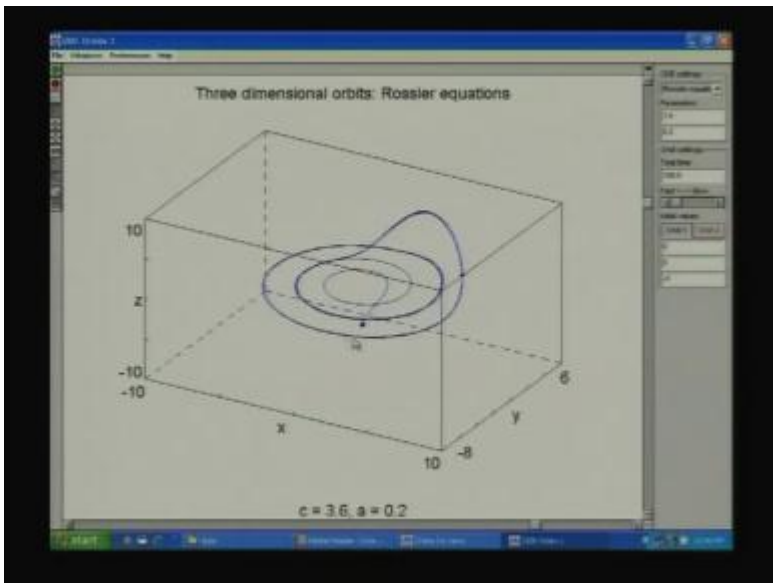

Can you see the plane, there is a plane passing through. So; obviously, that plane will have at the middle an equilibrium point, if you can calculate the equilibrium point from this set of equations that I given you, you can easily determine that equilibrium point. But, it is very clear from the orbit that it is definitely a complex conjugate Eigen value case with positive real point, which is going out, it has started and then it has gone out. But, then it did not really go out, something else happened as it is going out, it is again getting thrown in.

By that mechanism, it is producing this, what is known as period 4 orbit, you see, so far it is a period 2 orbit, now you see period 2 orbit has broken out, now period 4 orbit because this has now separated out. If I increase it further it will be clear, so let us increase it to larger values.

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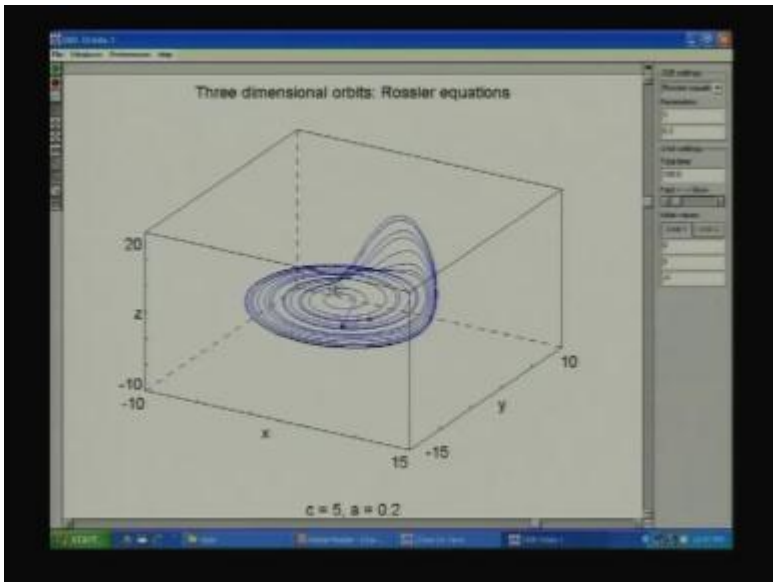

See you can easily see that it is spirally out, but it is failing to go out because something happening outside that is throwing up inside, that is another mechanism by which you can have a bounded orbit. Why will very close initial condition separate out because this fellow is anyway a spirally outward orbit, spiraling outward orbit means it will have a e to power sigma t plus sinusoidal term of omega t, that sigma t will have a positive value which means that for one initial condition it go in one way, the other initial condition it will also expand exponentially and as usual their separation will go out.

So, because of this expanding character, this will have that sensitive dependence on initial condition, I will clarify that further, but physically is that clear. So, let me, now his question is if I give the parameters are same, but if I now change the initial condition will the plane change, obviously not, because the plane is given by the set of equations. So, if you solve this set of equations it would not solve it, just find out the equilibrium point and find out the Jacobian around it and find out it is Eigen values.

Then, if it not depend on the initial condition, so you will always get the same plane same set of Eigen values, so that if any initial condition on that plane it will behave in a certain way, but supposing if you starting from initial quantization that is away from it, what will happen. It will be determined by the other Eigen direction and here that Eigen direction as that it is pulling it in, so pulling it in and that is what is happening you have the spirally outward orbit, but that outward see here is the plane that plane, normally if it is a linear system it is just a plane.

But, if it is a non-linear system the plane associated with complex conjugate Eigen value will bend and that what is happening, it is bending and then throwing the orbits in-wards again into the center, so again it is spiraling out that way it is keeping it bounded. So, these are sort of two possible mechanism, there are more, but I will not go into each and every mechanism, what I am tried to trying to point out is that this things are highly possible in a non-linear system.

Not only highly possible, actually this is a very prevalent in nature and as the story goes you see Newton's prime contribution was to show that all the bodies in this universe at in his time people believed that the invisible stars are the whole of the universe, now we have far better look, but nevertheless the point was that all heavenly objects, move by very deterministic loss and the deterministic loss are exactly quantifiable.

One says that any two bodies attract each other with a force g m 1 m 2 by r square and the second thing is that if you have a force then the body will move and how it will move. Force is equal to mass into acceleration, that is the differential equation, solve it you get the solution. Which means if you have an initial condition it will move, move depending on the this side of equation that you have.

With that it was possible to determine the trajectory of Mars, Venus things that were completely unthinkable earlier, it will earlier thought that these things are sort of move

on the own, but now things were quantifiable, determinable one could predict. So, that gave sort of a idea that everything in this world are deterministic, you can determine the values. There was a bit of problem, the problem was taken to account this solar system, how would you determine the trajectory of say Mars.

You would say the radius is cooled by the sun and therefore, there is g m 1 m 2 by r square acceleration attraction to it and the force will be exactly that and the mass into acceleration will be equal to the force. Which means exactly measurable, exactly determinable, you can, if you measure the initial condition, you can predict how it will move exactly which trajectory it will follow does it actually follow that. Now, there were a bit of complication because when we talk about this, we are only concentrating the effect of the sun, not on the other planets.

Yes there could be other planets into consideration, there are nine planets and therefore, you have to consider all the nine planets in their pool, with that idea people are slowly getting the idea that in order to talk about the whole of the solar system, we actually have a n body problem with n whole numbers. But, still nobody knew the solution of that kind of a problem because the question was essential is the solar system stable.

What will happen to the solar system later, it so happened that in 1887 or so the king of Sweden to celebrate his own sixtieth birthday, he announced a prize money was somebody who can solve this problem, N body problem Newtonian equation n whole number, you can assume n to be nine and then go ahead and proof that the system will be stable, many people tried it and failed, many people tried and gave up.

One fellow who did not give up was Poincare, Henry Poincare the famous mathematician, he reduced the problem he argued initially that forget about the nine body problem. If I start with the three body problem and if I can solve it then we can extrapulate the solution that logic into nine bodies, no problem. So, he took up the three body problem, there are three bodies say sun, jupiter the biggest planet and one other planet, then how would I write down the equations.

He proceeded a lot, he could not exactly solved it, but nevertheless whatever submitted that was mathematically, so very illuminating that he won the prize and whenever preparing the talk to accept the prize, he revisited the calculation and detected an error. And when he detected an error he wanted to recalculate and stuff like that, it is at that

point he realized that all the seeing by which he claim to shown that the solar system is stable are wrong.

So, there are there are issues that he had not realized before and now it is known that essentially he stumbled upon the fact that, that could be such orbit, chaotic orbits in the solar system and something that he did not understand and so he gave up. But, that was the first inkling that something could be a problematic, now what exactly is the problem notice it the problem is this.

Whenever I want to predict something, say I want to predict the motion of the mass 10 years later, what do I do, fine I can write down the differential equations, three body problem assume three body problem Sun, Jupiter and mars. The effect of the other planets are really negligible so neglect them, so you can write down the equation and then what will you do, you will measure the initial condition, you will look at it with the telescope and then say that now the position is this.

And then look for some time, so you can made at the change of the position from there calculated velocity, so this is the position, this is the velocity, we can do that and the masses are also known really. So, you can also calculate the momentum, so the position of the momentum initially are known, differential equation are also given, predict it, solve it, in those time there were difficulties, because you did not have computers. So, solving large differential equation means months of effort to obtain the exact solution.

But nevertheless forget about that, that was a technical problem they had, for us the matter is clear, you can solve it. But, then notice that whenever you are looking at a planet and trying to find out the initial condition it will always has some error, you cannot avoid it always be some error. Errors coming from the precision of your telescope, precision of your observation, so they will always be some error. So, imagine that the position of the momentum on the planet mass is plotted in this position momentum plane.

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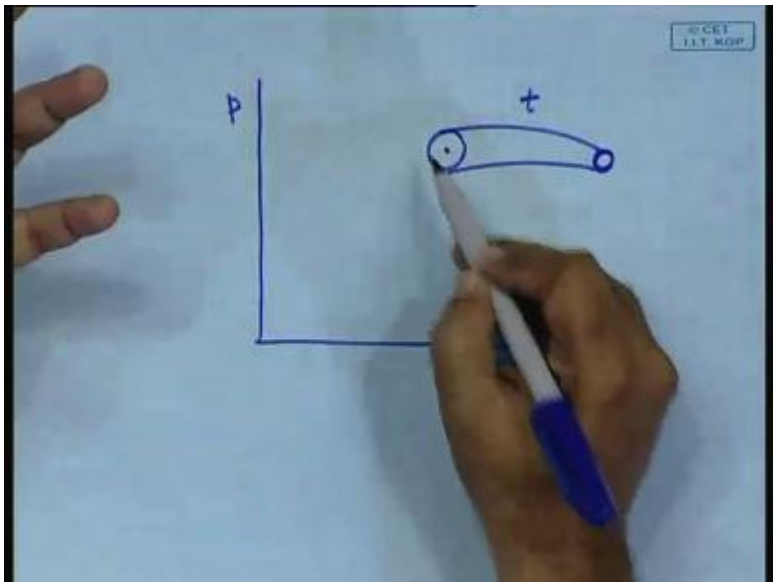

So, position x and momentum p, in that case say it was found to be here, can we say with confident that it is here, no because mathematically you will always say it is somewhere within this error ball. Because you know that my measurement is accurate to such an error, so there will some always be some error ball and then how will you predict, you will say notice that you will not say that let us predict, starting from this initial condition, you say that I have to do the prediction starting from this error ball.

Wherever this ball evolves; which means, all the points inside the ball evolves the final state could be anywhere in that, so say the ball evolves to something like this after sometime, so you say that after this much of time, you will also have this much of time, it is somewhere within this error ball. Now, if so happens that for most systems around you this resulting ball is smaller than the initial ball, the resulting ball; that means, after some time, if you predict, that means if you solve the differential equation starting from the all initial condition inside the ball.

Then you will get some object after some time that object in volume will be smaller than the initial that follows simply, because this volumes contract, if it is a dissipated system this volume within the state space contract, if it is a non-dissipated system they will be remains the same, but never less, the volume will be remain the same, but nevertheless you can do the prediction. But, now suppose you have a system that is that has a

equilibrium point which is unstable, say it has a saddle equilibrium point then what will happen. If it is a saddle equilibrium point say equilibrium point that is a saddle.

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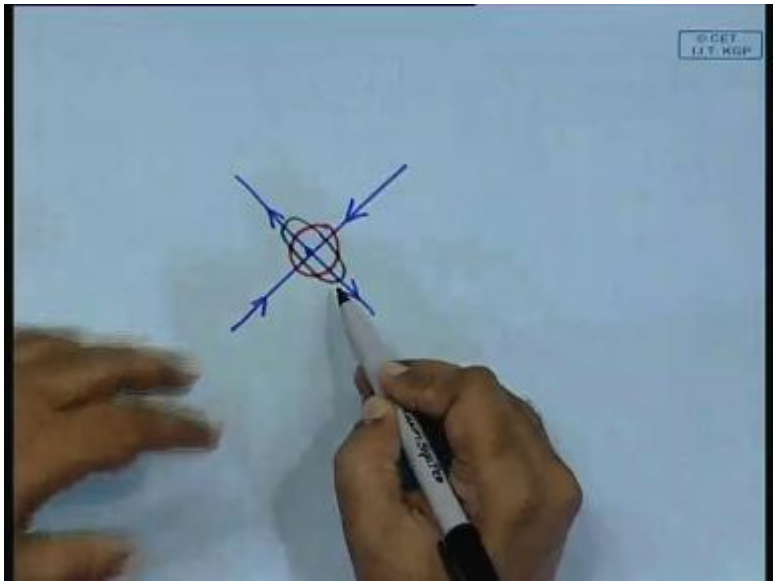

So, like this, now if to start with an error ball around it, how will that error ball evolve, yes this side it will be squeezed and this side it will be elongated and as a result of which you will find then after some time it will become like this, ellipse. If you start ideally from a circle it will become an ellipse, if it is a three dimensional system if you start from a sphere it will become an ellipsoid. But, let us consider the 2 D system first, now after sometime what will happen, it will further increase, further increase and finally, it will become a thin elongated thing.

And that is how it ultimately, in a linear system it will goes to infinity, in non-linear systems also if you have a saddle equilibrium point then the similar thing will happen that means it will it will expand to one side to infinitive. But, in a chaotic system that is not allowed because in that case it is going to infinity, you cannot alive it to go to infinity it will always to be bounded, how can you get that bounded then the only way you can get that bounded is if it is elongating and after some time, say I will draw a few.

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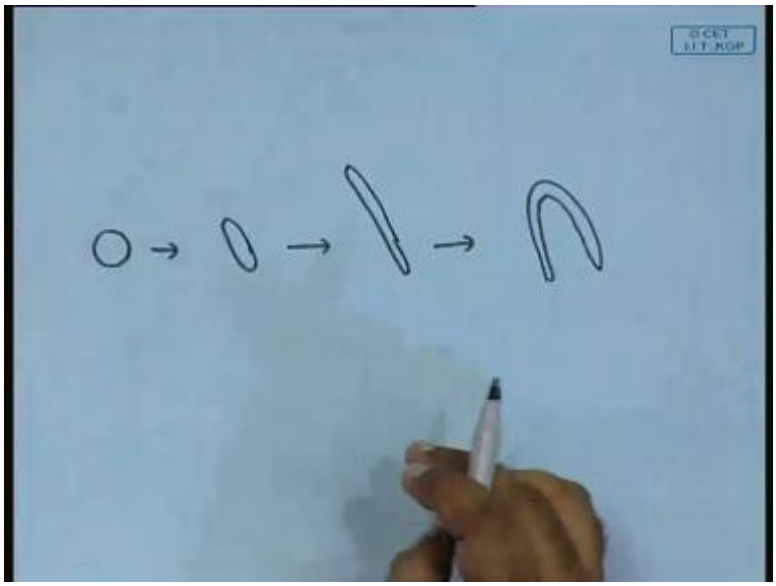

From here it has become elongated like this, from here it become further elongated like this, but that must be arrested, the only way it can be arrested is if it folds like this. So, this is the phenomenon that actually happens in a chaotic system, which means imagine that you have taken dough, have you seen mother making parota, so you take a dough then you roll it into a big thing and what does she do. She folds it and again then she presses it then again she folds it and that is why oil goes in those you know layers.

So, you have this layers structures, so in every iteration what is she doing, she is making it expand in one direction, fold in other direction, expand and fold, expand and fold, expand and fold and that is how the parota is made. Now, in a chaotic system something similar to that happens; which means, that takes place that is expansion it has to happen because you have a circle orbit for example. If you have a circle equilibrium point then the reason expanding in direction it expands, so it expands, but it expands direction folds again it holds the holding expansion folds and that way whole thing can be kept within limits.

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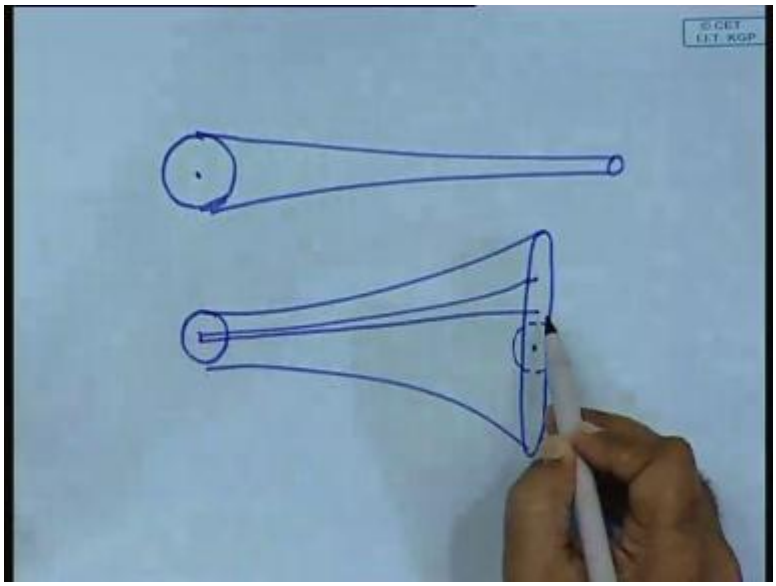

Notice another important thing, another important thing is suppose you have a system that is dissipative and you start from an initial estimate position like this, but an error ball like that and then you evolve it and suppose it becomes smaller, what does it mean in terms of the prediction. See it means that you can predict the future and the prediction is quite accurate, even if there was an error in initialized stage after the prediction the error, the possible errors come down smaller, that is the character of a stable dynamical system.

If you have a stable dynamical system, it contracts in all the direction and so prediction becomes very accurate, if on the contrary you have an equation like this, you have this, but then that expands in some directions and goes like this, that is what I found if you have a stable equilibrium point that would what have happen. Then what does it mean physically, physically it means that after this much of time, if you have estimated that it would be here and if you make a same error ball you would find that the future prediction has become unreliable because it can now be anywhere within this.

Future prediction has become unreliable, why that has happen because two very close initial conditions are actually moving away from each other. So, in case of chaotic system we have seen that that is what happens. They move away from each other that is meaning of the term sensitivity dependence on initial condition, but when that happens if you have very close initial condition moving away from each other then since you do not

know within some certain t, where the initial condition is actually here or here, your prediction become somewhat meaningless.

Now, it is known that yes the solar system is unstable it is known, but how can remained like this for a quite a long time because time scale is quite large in what way is it unstable. Suppose you made up the position of the window and you know that if you measure it there will be always some inaccuracy and suppose with that distance inaccuracy is only to the extent of one kilometer how will be possible. Having incurred an error of one kilometer in the distance of Pluto is that why pardonable offence, nobody will put somebody, some scientists gel for that.

But then it has been calculated that now with that error ball, if you allow Pluto and it is state to evolve then within something like one hundred thousand years the distance, the size of the error ball will become the size of one astronomical unit, you know what one astronomical unit is, it is the distance between sun and the earth. So, the error will become this big, uncertainty in the prediction will become this big, so we will not be able to predict with its which certainty it is position at all and the error prediction error will be this big, the distance between sun and the earth.

That is why it seem to be chaotical, the motion of the satellites of many of the major planets are also found to be in the same way chaotic, motion of the astronomic belt individual bodies in the astronomical belt are also found to be chaotical. So, that is one thing, that we live in a system which is easy itself unstable, still we have by the way when I say that one hundred thousand years it will become, so big you fell that one hundred and thousand years no, it is too big I do not have to bother about it.

One hundred thousand years is very small time in comparison to the time frame in the solar system. How old is the solar system you know four point six billion years, four point six into ten to the power nine years, one hundred thousand is a just a moment in comparison to the age of the solar system. The whole thing was stumbled upon when finally, people I mean pointed was very old after the he will forward about that thing. What I am telling you are all fished out, out of this papers very long, pretty much latter, but then the actual inclined problem came when people tried to study the way that.

And way the prediction is a business that scientists have been evolved in for a long time, people tried to predict the error and why does the way change because there are hitting

on the surface of the earth, there is cold temperature at the top, so there will be circulation. And the circulation follows partial differential equations, which can be determined down we can solve it, so there are always being the attempt to write down the partial differential equations solve them.

And then predict in which part there will be strong feeding up, in which part the pressure will go down, so that they will be some inclined way all that distance can be predicted. But, in order to do that what people do now is to you have whether monitoring station, where the humility, the pressure, the temperature everything is monitored and there is grid of sustains all over the globe and those things are relate to a certain place, where a super computer does the calculation and finally, we have the right to predict. Now, this man Edward Loran's in MIT was also in that business, he was trying to develop a model with which model of the way with which you can predict, so he considered a very simple system.

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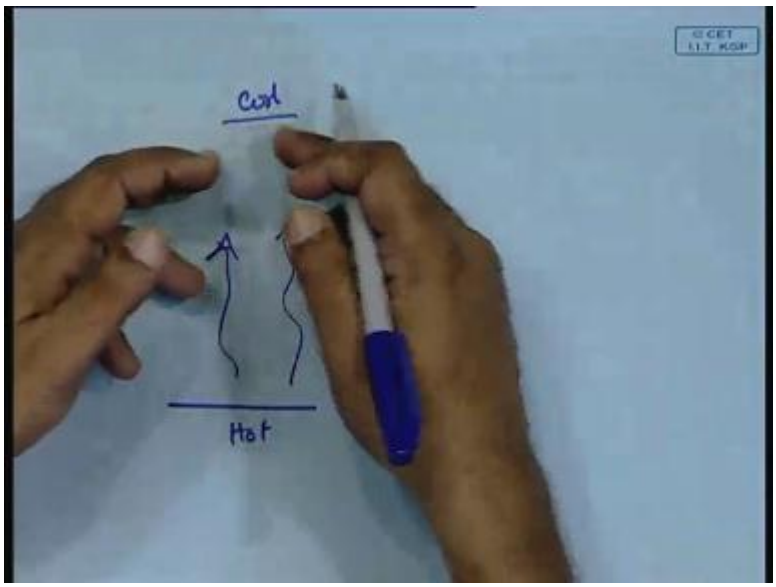

A ground being heated up, so that you have the air going up at top it is being cooled down, so this is hot and this cool, so what will happen. It will go up getting it will get cold and then it will come down, so if all the places get hot then how does it come down after all it has to come down.

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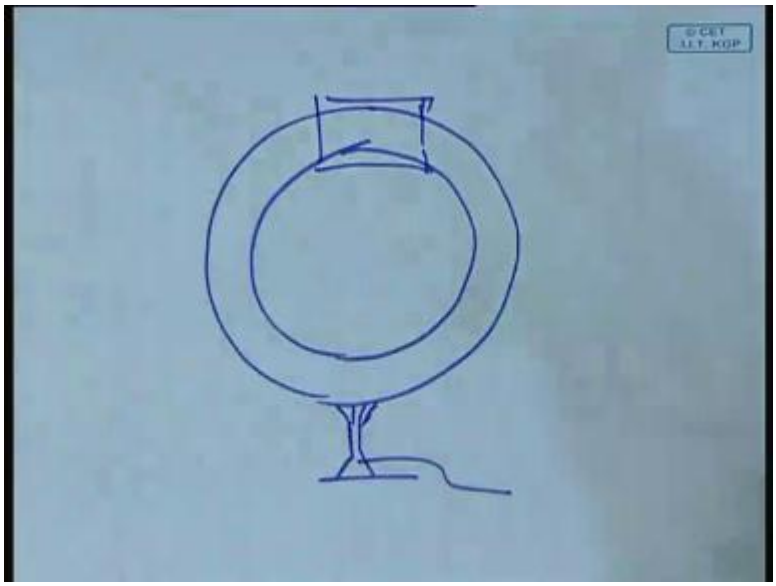

So, he devised a model in which he considered a loop emerging a physical loop made by glass and it is say heated by a Bunsen burner here and here it is a something to cool it down, so what will happen. This part will tend to go up, this part from here it will tend to go down, so it will establishes a circulation and you can easily write down the differential equations. And these differential equations initially were partial differential equations with some approximation you can obtain ordinary differential equation from there and that is got the Loran's equation square, which I tell in the last class.

And he, at the time there was the computers were very primitive 1963 very primitive, so that he would have to do calculation, obtain the result and at the end of the day whatever the results are he would have to enter that as the next, next day s initial condition and that will go on. So, after some time one day he did that entered the results, but at same time the computer went on calculating it, so he found that if we enters the initial condition by hand then it shows one prediction and if it continues the calculation by itself.

The computer does it by itself then it shows completely different thing then it realize that actually the amount of error incurred in his own hand entering of the data that means that is within certain precision that elimination of the last few digits or error in the last few digits is what is ultimately building up and finally, making the prediction impossible. So, that was the start of the whole game of known in the dynamics and ((Refer Time: 36:36))

So, you can see you can see various types of systems, some of the system that we have already obtained equations of, for example.

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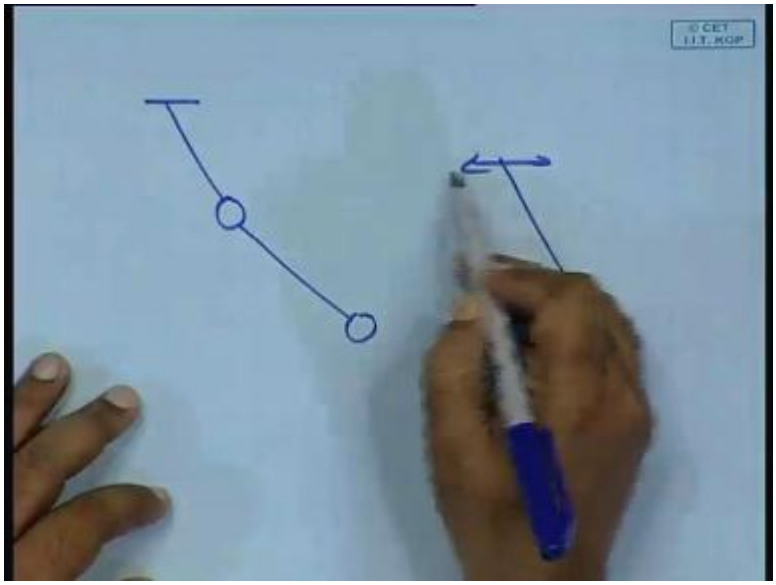

While we were doing the (( )) formulation, we have obtained the equation of the system, double pendulum you can take that double pendulum and simulate it, you will find that start from any arbitrary initial condition and it will go into a chaotic motion, you will see that. You have also done a pendulum with oscillating support, you have done that, simulate that with the all the tools in your hand, you have differential equations, you have the Runge-Kutta method, simulation code written up, simulate that you will find that you will get the same kind of behavior.

Take any value of mass and the length whatever, but yes ignore friction in this case why because if you have friction after some time it will come to vertical downward position there will be no interesting diamonds. So, in this case at least ignore friction in this case you may consider friction because there is a force symbol, so you have all these different types of systems which ultimately lead to a chaotic behavior.

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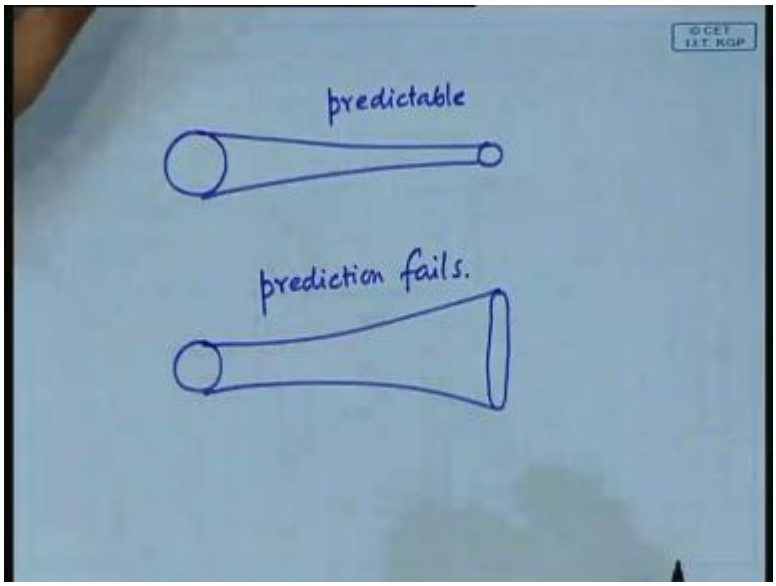

Now, an important thing, if you have a system like this, where you are yielding a smaller, final error ball starting from a larger initial error ball then it is predictable. If you start from a smaller error ball and it goes and beats a bigger one it is where prediction fails. Now, if all systems in the world were like this, can you visualize such a world, what will happens, what will happen is that, it is a vision that Newton started and forced Newtonian scientists to some level.

What was the idea of Newton, he says that any bodies fuse the state can be predicted from the initial condition if I write down the differential equations; which means, that the future of anything is uniquely determined by its past. If the future of anything can be uniquely determine by past I can also say that what I am saying now; that means, my vocal cords are vibrating in a particular way to utter those words, my tongue is moving in a particular way to express my thoughts, the change inside my brain are moving in a particular way to embody that thought

All these were you know contained in the past in some way, so that normal prediction with the initial condition differential equation will ultimately yield this lecture will be delivered, so somewhat observed. The reason is that you have smile is that yes that is outside, so there must be something in nature that prevents from that happening, in-fact that heart deterministic is called; that means, everything can be predicted was nicely written by Laplace, who is the definitely you have heard.

Laplace was a French mathematician, he was at the center of the French revolution also he was a statesman he was everything, mathematician. In his book he wrote that if there is some intelligent being who has the access to all the initial conditions of all the bodies in the universe; that means, you know the initial state, initial position, initial momentum of all the bodies. Then that intelligent being would be able to predict infinitely into the future what happens in this world.

Not only that it will be also able to called predict ret rode it into the past, what happen in this world, now that vision has been proved wrong, why because the reason that you will laugh. This is quite absent that all my tongue, my brain, my hand, my vocal cord and all these are moving in such a way, why because something happened in million years, initial condition the million years ago, determined that, no that is not possible. There must be something, something in nature where there is a possibility of doing something, which is not exactly predictable even in theory.

You can see that if everything were like this predictable, in that case that is true, now it so happens that we cannot predict into the future, we cannot predict into the future not about everything, we can predict the position of the mars, we can predict the position of the Jupiter. But, then not into million years, why because that is also a unstable system and things around us we cannot predict even today, I cannot say where he will be tomorrow, can I, I cannot even tomorrow after going out of this class where you will go, I do not know and nobody can predict.

Where he will his molecules be, it is not possible to predict, even though I know his initial condition this position momentum smiling I know. So, it means that in nature there are things where the prediction does fail and there are things where the prediction does not fail. Both types are there and because this types are also there you have physical inverse as you know, that is why the organs of such non-linearity are actually essential to the action of the world that we know physically. We have learnt about a few types of behaviors in non-linear system.

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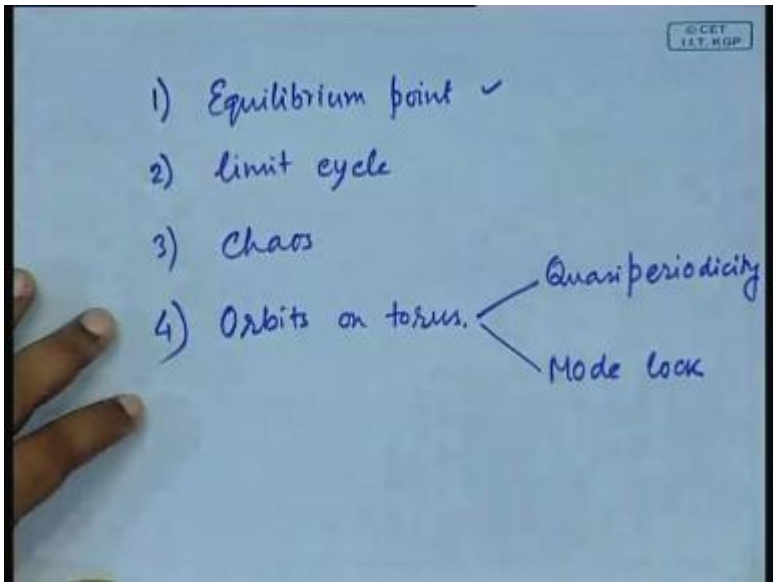

One, there it could be a equilibrium point be stable behaviors, It could be a stable equilibrium point, two limit cycle, limit cycle could be high periodic period to n, It could be chaos. There is one fourth thing, which I need to talk about before we leave the subject that is where what I have to say.

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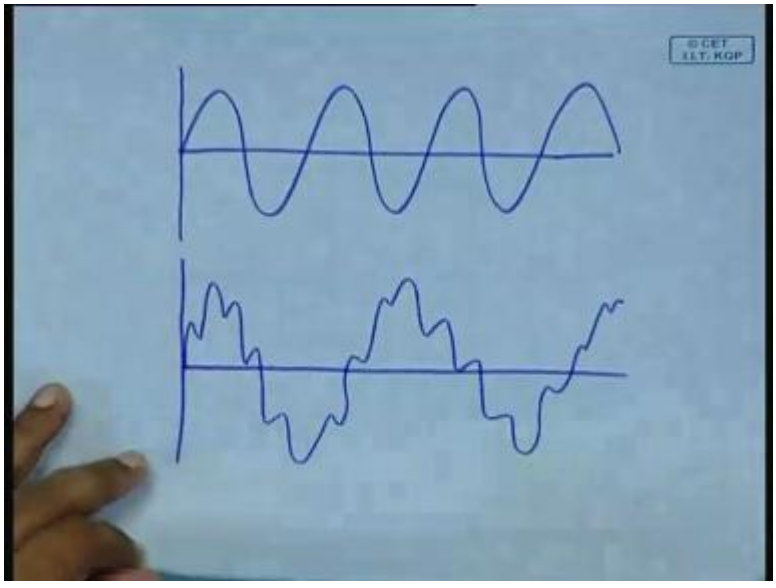

In any oscillation, we can always identify a frequency of oscillation, now if there are two frequencies then what will the wave form be, say one frequency is this, another frequency is say some few times that frequency it will become like this, you can see

another frequency characteristics. If you take the Fourier transform, we will see the two piece, one correspond to this frequency and other correspond to this frequency, now such systems also can happen only in non-linear systems such behavior, you cannot have such behavior ever in linear system, that is typical non-linearity, but there is little more to it. As you have seen, we are trying to visualize everything as motion in the state space not in the time domain, so this is a wave form in the time domain. But, in more clearer geometric intuit idea will appear, if we tried to figure out how will this orbit look in this state space.

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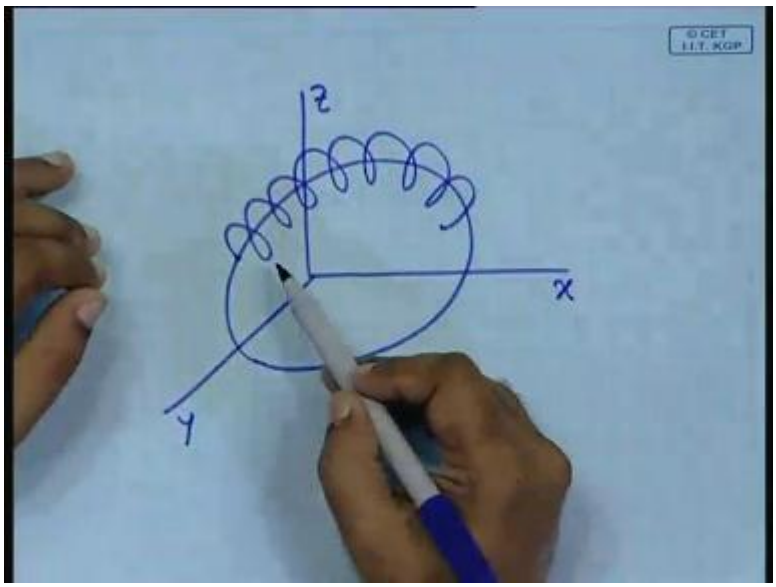

Now, in the state space an orbit like this will be a close loop, so it will be a close loop, it will have a certain frequency, that frequency will determine if you start from here after how long you come back to that same point. Now, there is another frequency, how will it be manifested, It will be it will manifested you can see how It will be manifested by a motion like this, ultimately you can see that the systems behavior in the state space. So, it is x y and z in the state space will be on the surface of a duress, you can see that. It will ultimately be combined through its surface of a duress.

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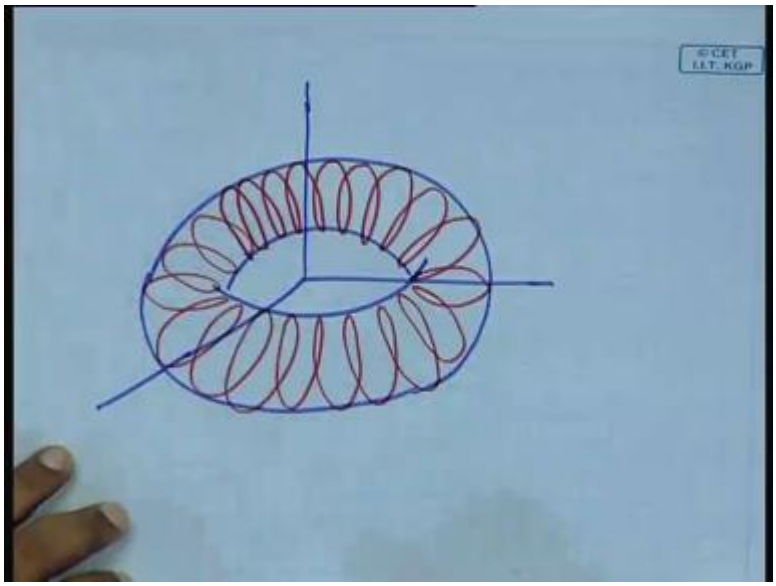

Let me draw out very clearly, say there is a torus like this and the orbit is actually going around the torus that is how it can have two frequencies. One frequency corresponding to the rotation in the bigger circle, another frequency corresponding to the rotation in the smaller circle, smaller circle and the bigger circle, there are two frequencies, f 1 and f 2. Now, these two frequencies could be commensurate and could also be incommensurate what does commensurate means one that you can define a number that is integer multiple of both, that is a concept of commensurate, you can define a number, for example.

Say 2 and 5, 5 is not a integer multiple of 2, but nevertheless 10 is a integer multiple of both, so you can define a number like that. If that is, so then what will happen suppose, you the frequency here is 2 and the frequency here is 5, 2 hertz, 5 hertz. What will happen by the time you give 5 goes 5 rounds around that, this will have gone many rounds, but after 5 rounds of this it will come to the same state; which means, such an orbit will be periodic

So, even if that is an orbit on a torus, it will be a periodic orbit, it will be a periodic orbit apart if this two frequency are incommensurate; that means, the time taken to go around the bigger circle and the time taken to go around the smaller circle are not know there ratio is a irrational number, then it will never come back to itself., then it will forever go running around the torus, but it will never come to the same state, you see that. So, you have the a periodicity condition satisfied, It is still a periodic.

But no in this case there is no sensitive dependence on initial condition because there is no moving away from each other, If you start two states very close to each other they will forever remain close to each other that distance does not increase does not decrease they will remain on the same. Such behavior is called quasi periodicity, which one where the two frequencies are incommensurate. The ratio is a irrational number then you have what is known as quasi periodicity.

If the ratio is a rational number then you have ultimately a periodic orbit, but then there is a difference between that periodic orbit and the other types of limit cycles that were hard- off, so far, why because they are two frequencies. If you draw the Fourier transform we will see two sharp frequencies only there ratio is a rational number, so that is different from what you have in the normal periodic orbits. In order to signify that there is a special name given to such periodic orbits, that is mode locked periodic orbit.

Why mode locked, because there are two frequencies they are locked into a state, where there rationally related you can easily imagine that such a system is created by a physical system in which there are parameters and you can change the parameters the way we were changing so far. We were changing the parameters and as you are changing then it will the two frequencies will change, so if say the frequency ratios will change, Strangely you will find that for a large parameter range the frequency ratio gets locked to a certain range.

And that is why it is called mode locked periodic orbit, do you know any very common example of mode locked periodic orbit, very common example, earth and the moon. How does the moon rotate around the earth, how many times, what is the speed of rotation, what around twenty-nine days and moon will also rotating around its own axis in how long exactly the same, exactly the same why, could be something different. No, it is not something different and that is exactly why you always see one face of the moon.

The reason is that they rotate at exactly the same frequency, the speed at which the moon rotates around the earth, it rotates around its own axis at exactly the same speed and that is why you always see the one face of the moon. That is an example of a mode locked periodic orbits, here the locking ratio frequency ratio is one to one and there are other satellites in which the ratio is 1 to 2, 1 to 3; there are certain like the in the solar system like that one.

So, you have this kind of mode locked periodic orbits, the mechanism by which the mode locking will cause will be a bit difficult for us to deal with, now we will try to do later if there is time. But, now that more or less completes the orbits on torus could be one quasi periodicity and mode lock, so these are the possibilities. So, in general if these are the new possibilities that has opened up the moment you started to consider non-linear system, in the linear system equilibrium point was there, but nothing else all these possibilities are typical and specific to non-linear systems.

Now, for us engineers there is another matter of concern, As I told you whenever we engineers is do something design something for example, we design a oscillator we like that to be stable and there are situations in which we want something to work in chaos, where ever for example, we need random number generator yes physically we do, cell phones for example they need, so all the time in your pocket there is a chaotic system running it has to be.

So, we want those things to run like that and to run stably like that, it has to be after all stable behavior, chaos should also be stable in the sense that the chaotic aware must be re level and wherever you want a quasi periodic behavior, mode lock behavior those things are also should be there. But, then if you have, if you have a linear system it is stability is trivial, how do you know understand the stability, we calculate the Eigen values, if the Eigen values have negative real part it is stable, if there are positive real part they are unstable, that is what you have learnt.

But that idea cannot be applicable to all these systems that is applicable only to this, so how do we then visualize the concept of stability of a limit cycle, concept of a chaotic orbit, concept of stability of orbit on a torus, yes that is what we will deal with in the next class.

Thank you very much.