Dynamics of Physical Systems
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Lecture - 30
Dynamics of Nonlinear Systems – I

We have been talking about the local linear neighborhood of equilibrium points. That means, we had earlier seen that most of the dynamical system that we have modeled, they yielded non-linear differential equations. And then we said that we will first look at the equilibrium points.

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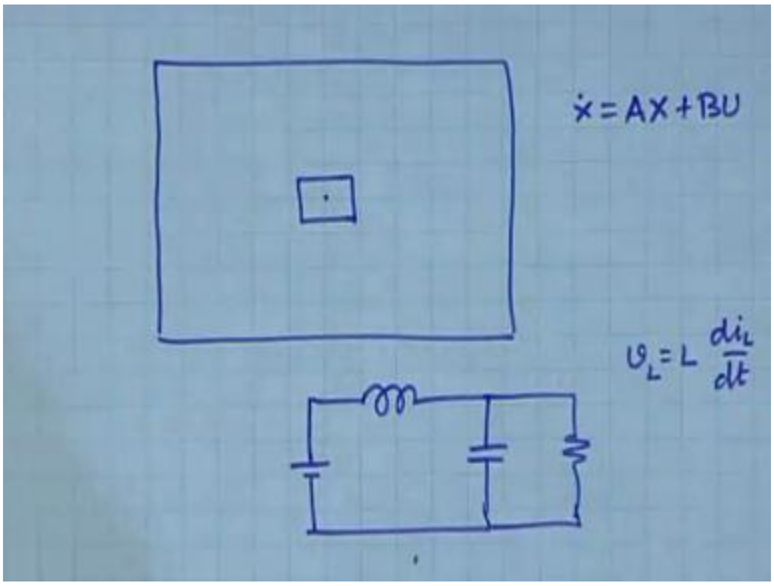

So, if this is the whole state space we said that, first we will locate the equilibrium point show them here. And then we will take a local picture around that equilibrium point by taking the Jacobian matrix, so you get a linear equation around that. And then we had in the last few classes understood what happens in this local linear neighbourhood. Today we will take a global loop and try to understand what happens elsewhere that means, the character of the vector field that can happen elsewhere.

But one thing is certain that in many text books you will find that, this local linear neighbourhood is all that is dealt with that means, normally there are many text books especially in complex theory, that deal with equations of this form, x dot is equal to A X plus B U and that is all. Now, it is important for us to realize, that this is always a local approximation. And there, had been problems you have dealt with where the equations obtained, were linear in nature were not there are cases.

For example take this circuit, you have got this battery, you have got this inductance, you have got capacitance and say the resistance. Now, if you obtain the differential equation it will be a linear ((Refer Time: 02:40)) differential equation why, because there also a process of linearization has gone in no system in nature are linear can be, can really be linear. In this case we had a non-linear set of differential equations and then we locally linearized it by taking Jacobian, but in this case we have already linearized the characters of these elements.

We said that the inductance is a linear element why, because then we said that the v L is equal to L d i L d t. So, if you apply, if you allow the current to change the voltage induced will be proportional to that and the proportionality constant is L. Now, if you allow this change to happen in a certain range then this is fine, if this range is very large, so that the inductance is ((Refer Time: 03:44)) into saturation obviously, this is wrong. Even in air cored inductors there would be issues like, the magnetic lines of force will be going through the air and linking various things and all that cannot be absolutely linear all the time.

So, it is linear only in a certain range of these values v L and i L, in other words this is also, this equation is also representing a local linear approximation in the neighbourhood of the position v L and I L 0 correct. Similarly, the case with the capacitance, so capacitance is also ideally considered a linear element with the assumption that, current is c d v d t. So, if you allow the current to change within the certain range then the voltage induced will be proportional to that, else other things will come in. You can easily imagine if you take a small capacitance and apply a kilo volt on it obviously, the changes will not be linear it will burn.

So, all these things are linear approximation, so whenever you obtain a set of linear equation you have understand that I have already done the linearization somewhere. And in most cases you actually obtain non-linear equations and then you do the linearization.

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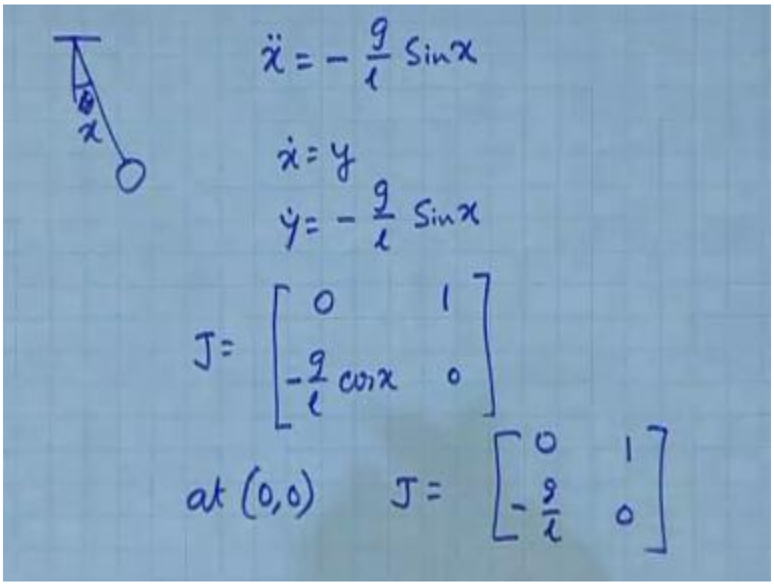

Take the this simplest possible example, in the in the case of the simple pendulum in the case of the simple pendulum what was the equation, say if this is theta the equation was, x double dot is equal to minus g by l sin x. So, this is x minus g by l sin x, so if you now write down in first order form you would say, x dot is equal to y and y dot is equal to minus g by l sin x fine, so this is a non-linear differential equation, because the sin term is there.

What is the next step, to locally linearize it by taking the Jacobian, so the Jacobian matrix is 0 1 minus g by l cos x and 0. So, if you take cos x, if you consider the equilibrium point what is the equilibrium point here 0, 0 is the equilibrium point at 0 0 if you substitute 0 at here, then this term it becomes 1, so at 0 and 0 this becomes 0 1 minus g by l and 0.

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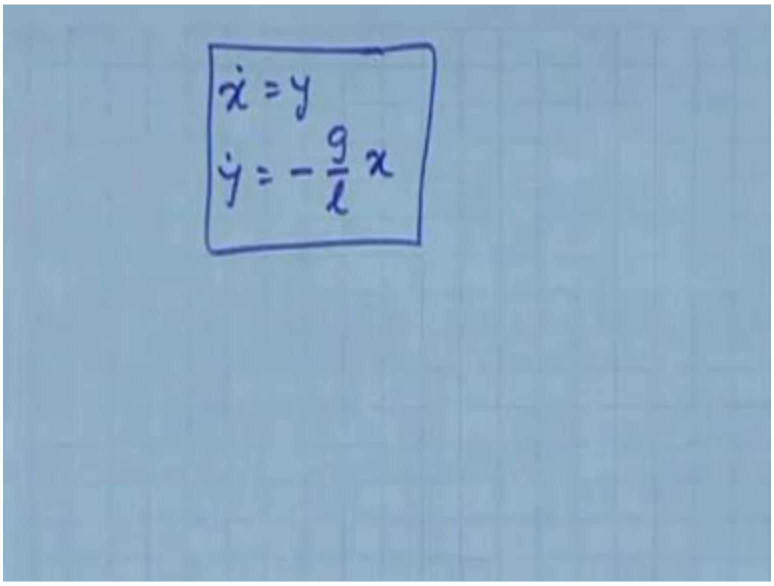

So, the resultant equation then is the local linear equation would be, x dot is equal to y and y dot is equal to minus g by l x, and this is the equation probably, you have learnt in school fine, so this is the equation you have learnt in school. So, this is a linear set of equation and that is a result of the local linearization from this equation. So, the point that, I am trying to drive home is that you never come across linear systems in nature all the systems, that are that actually occur in nature are non-linear.

But, for our own purpose we locally linearize it and then make the statement that if in case of, the pendulum if the deviation from the vertical position is small then this equation is a reasonably good representation of reality. And that is true for all systems that we talk about, so if you realize that, then the after we have understood what can happen in the in the neighbourhood of the equilibrium point, then next step is obviously, to take a bigger local at it, what happens elsewhere.

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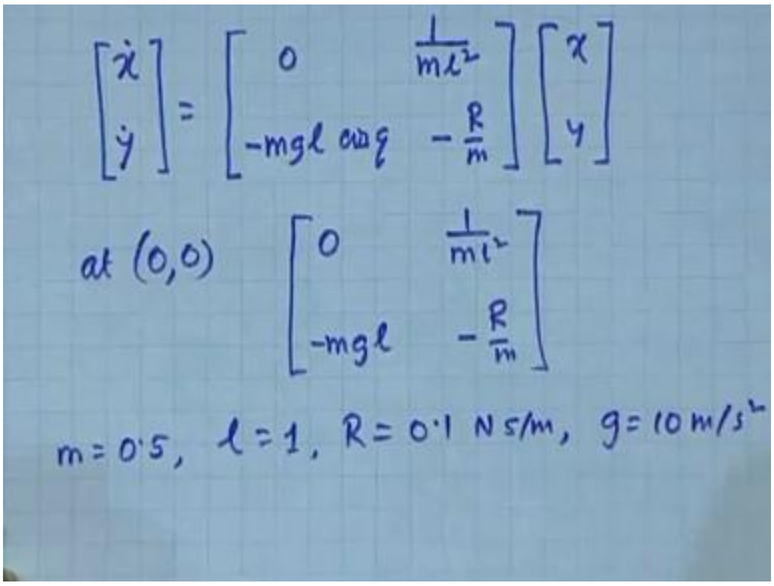

Now, what happen elsewhere means, we our process, so far was that there was an equilibrium point and we had taken a loop around the equilibrium point. Now, what can happen elsewhere, if you ask this question one thing is sure that in case of a linear system, the behaviour everywhere in the state space is the same. So, if the system equation is a linear, you have got one set of equations and then one set of Eigen values one set of Eigen vectors and that is what determines the behaviour over the whole.

But, if it is a system is linear non-linear then obviously, there is no reason to believe that the behaviour here would be the same as behaviour here, they could be different. Let us take an example, of again the simple pendulum whose equation we have derived just recall you have got the page. I will just write down the equation, because we have already derived that. It was x dot y dot it was 0, 1 by m l square, no this is already linearized, no I will not write this, I will write the original one.

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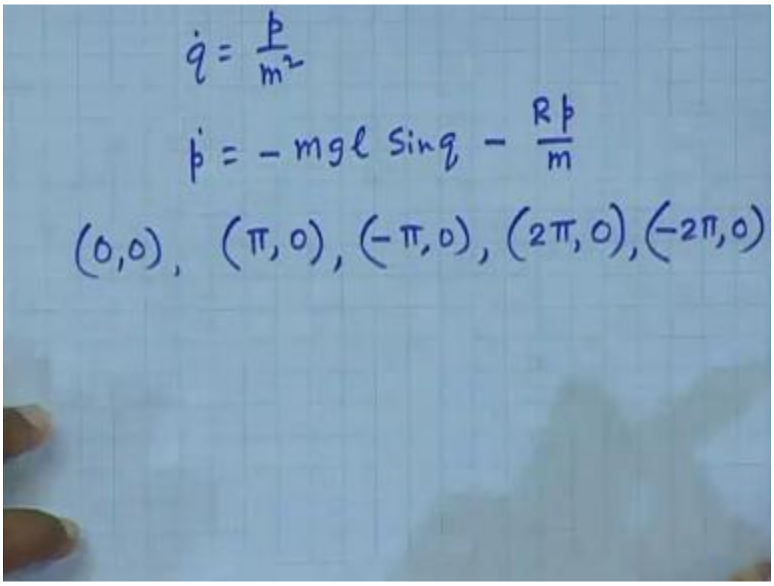

Q dot is equal to p by m square, and p dot is equal to minus m g l sin q minus R p by m, that is where we were just see we had derived this equation earlier, so we will start from there. In this case what are the equilibrium points 0 0 is an equilibrium point why, because p is 0 q dot is 0 and then this goes 0, and then sin q with q 0 is also 0, so p dot is 0. So, that is definitely an equilibrium point, so (0,0) is an equilibrium point anything else.

Obviously, pi 0 is also an equilibrium point twice, pi 0 is also an equilibrium point, so pi 0, minus pi 0, twice pi 0, minus twice pi 0 all these are equilibrium points and there are an infinite number of equilibrium points. So, the next question is, we had done the local linearization and had obtained all that around this equilibrium point, is the behaviour around this equilibrium point the same let us check that, so what we have to done. That is what I was starting to write ((Refer Time: 11:33)), the Jacobian matrix would be, 0 1 by m l square, minus m g l cos q and minus R by m x y fine.

So, now if you substitute ((Refer Time: 12:04)) (0, 0), if you substitute (0,0) you have this turning into 1, so you have at (0,0) you have the Jacobian matrix 0 1 by m l square this is minus m g l and this is minus R by m fine. And we had taken at that time, the values and accordingly we had tried to obtain what are the values, we had taken m is equal to 0.5, l is equal to 1, R is equal to 0.1 Newton second per meter and g is equal to t 10, we have done this problem earlier fine, do you remember that, good.

So, I will not do this for this particular case, what were our conclusions at that time, if you substitute these values here, what did you have, you had,

Student: ((Refer Time: 13:29))

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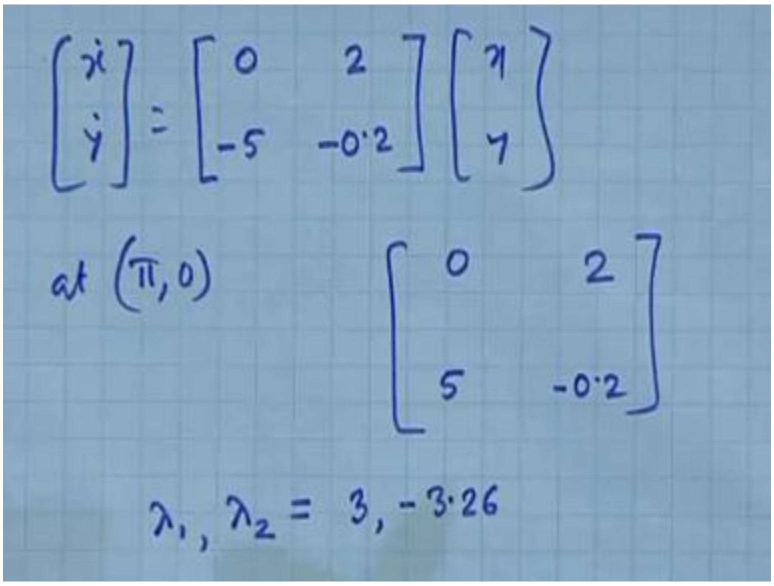

No, you had the Jacobian matrix as, 0 2 minus 5 minus 0.2 happy, and we had calculated the Eigen values we found that they are complex conjugate with negative real part. So, we had computed that it will be a incoming spiraling orbit.

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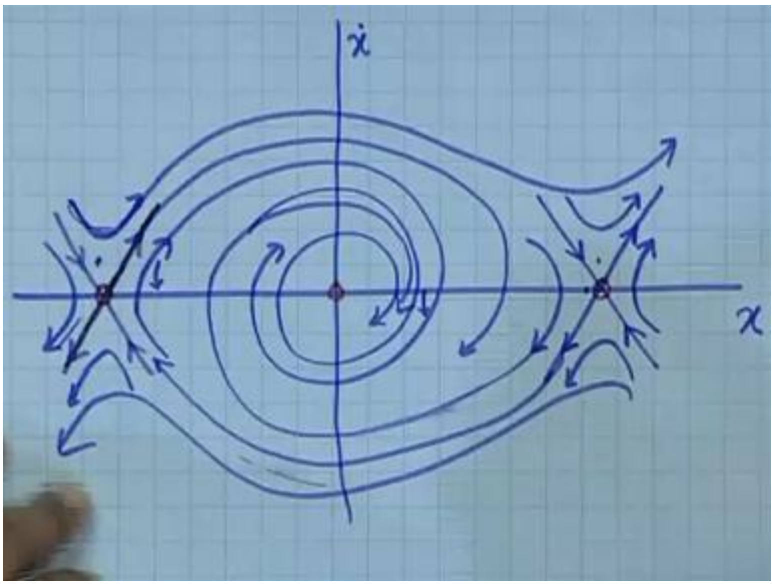

So, which means, I will start drawing it here, we had an equilibrium point at, 0 0 and the behaviour around that was a spiraling orbit clockwise spiral or counter clockwise spiral, clockwise how did you decide, ((Refer Time: 14:34)) it is rather simple, because here we would say x dot y dot is equal to this x y fine. Take a position that is along the x direction, say a small distance away say 1 ((Refer Time: 14:51)) and y is 0, so you have x dot is equal to something 0, y dot is minus 5 1 and this is 0.

So, you have minus something which means y dot is minus something. So, you have a clockwise spiral that is how we decided, so around that you have a clockwise spiral something like this, that goes into the equilibrium point we have decided, now let us do it for the next equilibrium point pi 0. Substitute the values here ((Refer Time: 15:42)), this was the local linearization you substitute pi 0 pi cos pi would give 0 here, no minus, so this will become plus right and the other things, so substitute the values what you get yes, ((Refer Time: 16:15)) so at pi 0 you have 0 2 this will become plus 5 minus 0.2 fine.

Now, obtain the Eigen values, should be possible to do it quiet quickly do it, what you get, what are the values.

Student: ((Refer Time: 17:10))

Yes, yes take the calculator just calculate it and tell me, soyou have understood that it is a saddle.

Student: ((Refer Time: 17:29))

Minus 3.26 is that right, so the Eigen values are such good, now naturally we have to obtain the Eigen vectors along these directions. So, what are the Eigen vectors well what,

Student: ((Refer Time: 18:05))

What Abishek is that right.

Student: ((Refer Time: 18:12))

What values?

Student: ((Refer Time: 18:16))

3.06, it is not very away…

Student: ((Refer Time: 18:24))

So, this is this is fine, so I mean 3 may not be exactly rounded 3, I would be happy if the numbers are, so chosen that this gives round number, so that you do not have to press calculator much., but nevertheless lets go ahead with this. So, you have this as the Eigen values and what are the Eigen vectors,

Student: ((Refer Time: 18:50))

Give me round about values, I do not want to calculate exactly that is, not necessary in this case, ((Refer Time: 18:59)) so you have say the equilibrium point, I was marking with red here and there, is another equilibrium point minus pi. Here, you have two Eigen directions probably they will be inclined more or less at 45 degree see check, because the Eigen values are almost the same with plus and minus values. What are you getting Teja,

Student: ((Refer Time: 19:40))

2 3, 2 3 means 2 3 like this.

Student: ((Refer Time: 19:52))

So, that is the outgoing, so this is the outgoing one and the other incoming one, will have something like this, you can easily calculate the exact slope, so this is the incoming one.

Student: ((Refer Time: 20:11))

Like this, what?

Student: ((Refer Time: 20:16))

2 and minus 3 more or less like this fine, So, you have these as the Eigen directions at this point, so you can easily see that the behaviour around this is quite different from the behaviour around that. So, at different parts of the state space, we have completely different behaviour that is one of the important aspects of non-linear system you have to keep in mind.

That there is no there may not be a global behavior, so whenever you have something around a equilibrium point, that does not mean elsewhere it will be the same you can easily see from this simple example, that here it is a saddle. Similarly, from symmetry, I can also say that this will also be a saddle why, that is simple ((Refer Time: 21:18)) because, this was the expression of the Jacobian matrix, so if you substitute plus pi and get something minus pi will give it will same thing.

So, that is why I can easily see that this will also be, sorry blue one and in this case what will be the outgoing and incoming, this, so around this around that we know the behaviour will be like this fine and inside the behaviour is like that. And there will be another equilibrium point at twice pi, another equilibrium point at minus twice pi, so the whole structure will repeat, so let me draw just one. Now, you see do you understand each of these cases, what is this case this equilibrium point.

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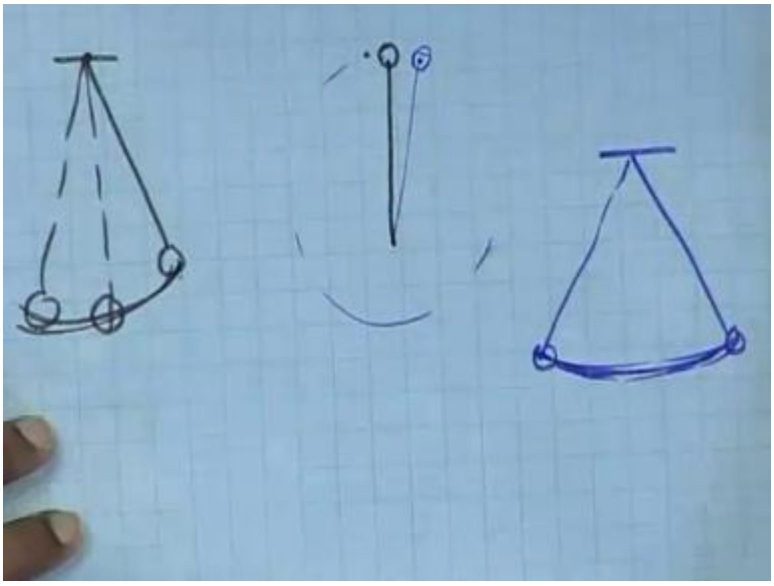

When it is hanging vertically downwards, it is oscillating like this and then the oscillation slowly dies down, because there is a friction and that is what is happening here what about this. Yes this is when it is vertically above vertically, above means it is an equilibrium point in the sense, that if you release it mathematically exactly there it will remain there.

But, physically it will not, because slight perturbation will make it move, but depending on the perturbation it will move, so that is representing that saddle equilibrium point it is a saddle. Now, notice from here can I construct logically the character of the vector field elsewhere notice, there are some clues to it.

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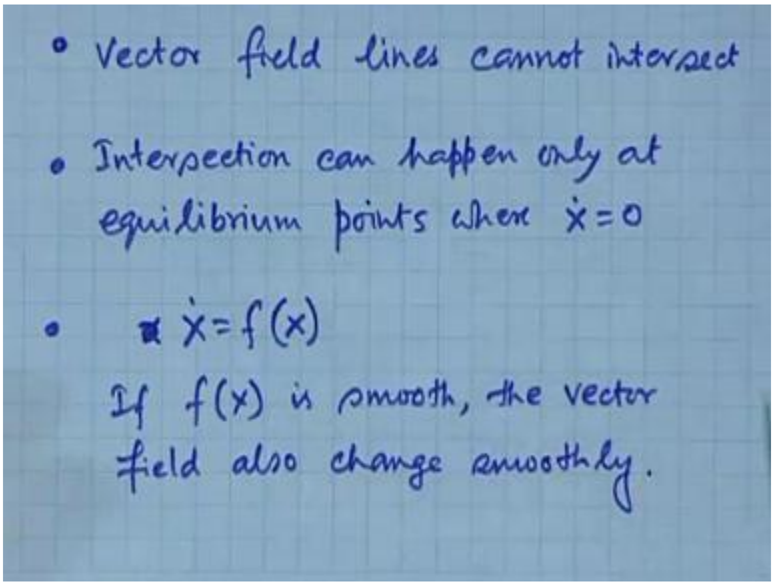

One that vector fields cannot intersect, so vector field lines cannot intersect we will use these as the rules, second it is not absolute true, that they cannot intersect they do. For example, these are vector field lines they have intersected, but only at an equilibrium point where the vectors itself is 0, so X dot is 0, the other thing is a bit settled one. You have, started with equations like this sorry, X dot is equal to some function of x, if this right hand side is a smooth function smooth function means, differentiable function then X dot will also change smoothly, which means, that you cannot have a sudden turn or twist in the vectors, say a vector is here the next point it cannot be like this, they will all be smoothly changing provided this is a smooth function. So, if f x is smooth the vector fields also change smoothly, as I told you these characters are more or less similar to the character of magnetic field lines ((Refer Time: 26:10)).

So, on that basis can we construct how will the behavior, elsewhere in the state space be, notice that starting from here what will what will happen, it will have to go on this spiral fine starting from here it will also have to go on to this spiral. If you start from here along this line then also you will have to go into this spiral, start from here it will also have to go into this spiral, so ultimately it will go into the spiral start from here, well it may or may not because it may go like this.

But, if the dissipation or the friction element is large, it will, it may also go into and get into the spiral, what is happening, what is happening is that what is this equilibrium point. This is the vertically upward position this vertically upward position, what is the meaning of this part of version, what is the meaning of this part of version, what is the meaning of that part of version let us try to understand, this is your x axis and this is your x dot axis, x is the angle, x dot is the rate of change of the angle.

So, at this point it is vertically up give a slight perturbation this may means what, the starting point is slightly like this, what will happen ((Refer Time: 27:55)) now, what is the meaning of a point somewhere here or say here, it is exactly vertically, because the x coordinate theta is pi, so x coordinate theta is pi. But, x dot is not 0, ((Refer Time: 28:30)) which means it has a starting, so you are giving a push in that direction what will happen, if the friction is small it will go on rotating for some time going on rotating means, continuously the angle is changing positively it is not oscillating the angle is not oscillating.

So, it will go like this, so starting from here there is a possibility that it will go like this, but if the dissipation is strong, then ((Refer Time: 29:01)) if the initial perturbation is small then it will not be able to come back to this position, may be stopping here. And then going on like this and that will be represented by going into this clear, the whole is the picture clear, let me do in a somewhat neater fashion, so that you can understand it well.

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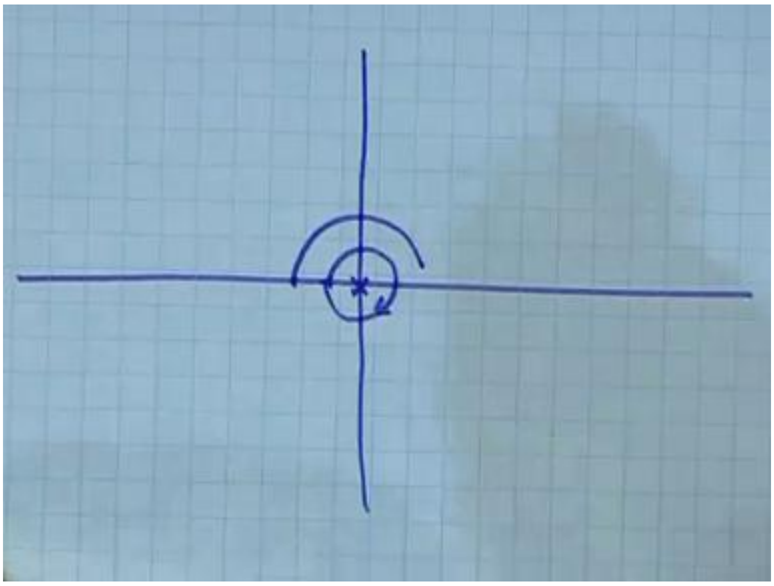

Here you have an equilibrium point, around which the behaviour is circle, no not not circle sorry, incoming spiral.

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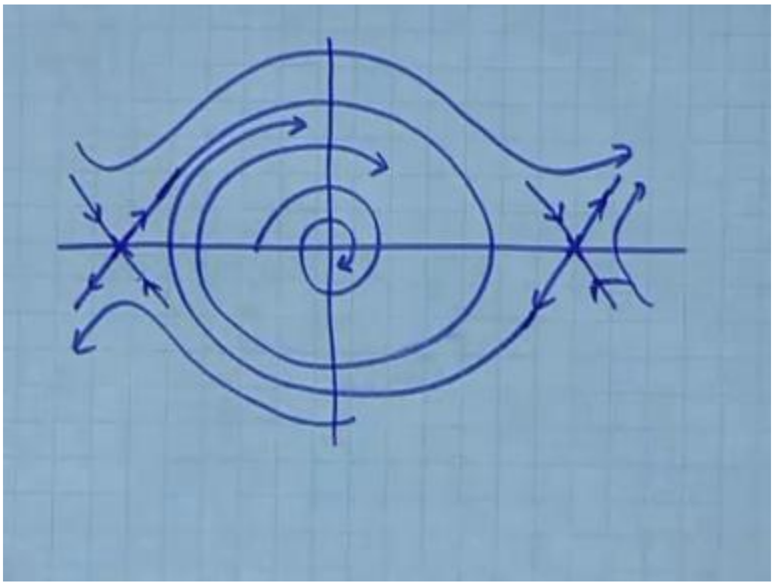

Here is an equilibrium point, here is an equilibrium point plus pi 0 minus pi 0 and here there are two, now, from here it will be like this, from here it will be like that elsewhere, it will be elsewhere it will be, so these are outgoing these are incoming yes this is outgoing this is incoming. Is the whole picture consistent nowhere, where these guiding principles broken fine, so simply by working out the local linear behaviour around the equilibrium points, we could figure out what will happen elsewhere seen that. What will happen if the friction was 0, look at the equation if the friction is 0 this term goes to 0, ((Refer Time: 30:57)) in that case m can be ignored we have seen that.

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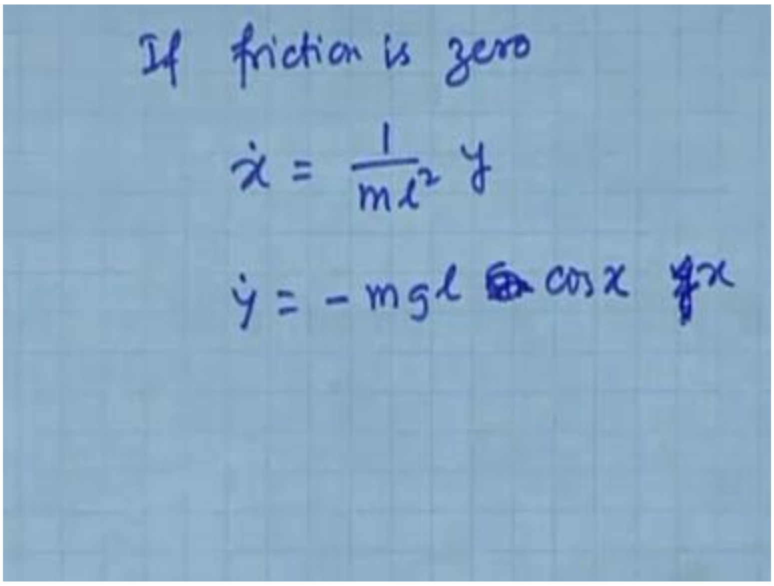

So, if friction is 0 then the equations are, x dot is equal to 1 by m l square, y and y dot is equal to minus m g l cos cos x y, no, no, no, no, this is written wrongly, here, I have written q and here x that is not the right thing to do. Yeah cos x, so in this case is that clear, we have already seen that in this case, we can further simplify it by cancelling 1 l and m 1 l m.

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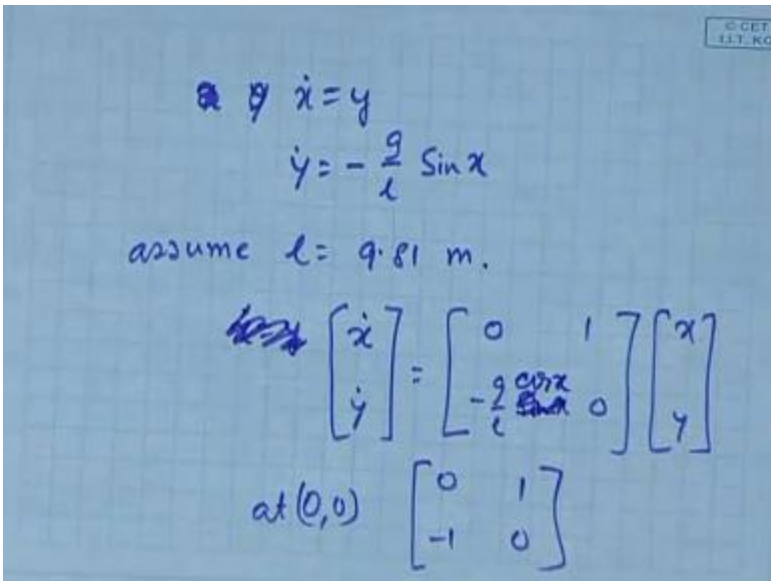

And so ultimately that can be simply written as X dot y sorry, x dot is equal to y, and y dot is equal to minus g by l sin x, that is a simple way of writing it assume, because in this case there is no role to be played by the mass, so we do not have to take a mass. Assume l is 9.81 meters the result is this is 1 fine, if this is 1 the whole calculation can be done without ever pressing a calculator, so you have x dot is equal to y, no I will write it as, the local linear expression x dot y dot 0 1 minus g by l sin x 0 x y.

Now, we see at 0 0 this fellow is 0, this term is 1, so the at 0 0, you have 0 1 minus 1 0, this is cos, so at 0 you will you have this 1, so this is the expression. What are the Eigen values, this you can do at least without pressing a calculator, what are the Eigen values,

Student: ((Refer Time: 34:27))

Plus minus j right.

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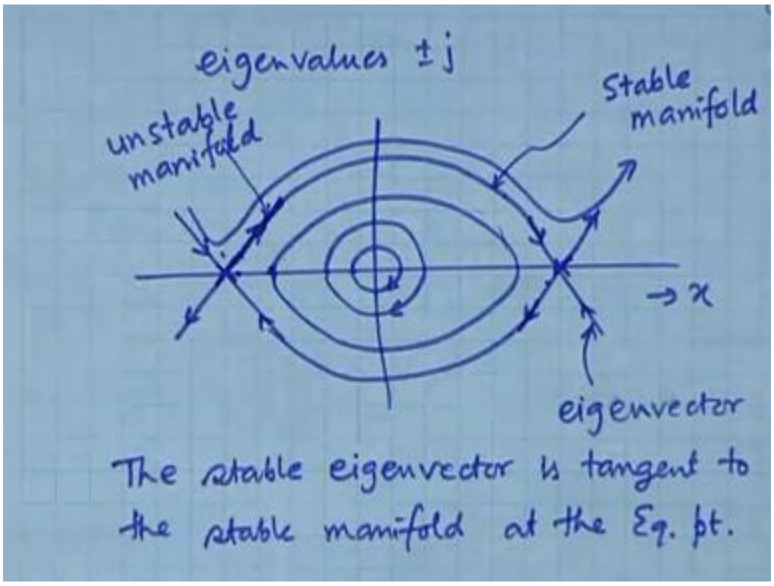

So, the Eigen values are plus minus j, which means this will be perfect circles here and here, there will still be those two equilibrium points minus pi 0 and plus pi 0, at that those two points what are the, ((Refer Time: 35:04)) this will be plus 1, if this is plus 1 what are the Eigen values, it will plus 1 and minus 1 clear. And the Eigen vectors would be exactly 45 degrees check that, this will be outgoing and this will be incoming, this will be outgoing, this will be incoming and what will be the behaviour like.

Now, notice that here it is perfect circle start from anywhere, here it will still be a closed curve, but not perfect circle why because, now the effect of the sinusoid will be coming in. It will be perfect circle only in the very close neighbourhood of that equilibrium point because we had approximated, sin x by x that is how we got it, so elsewhere it will be still closed loop, but not perfect circles. And this is, clear from the actual say actual system, ((Refer Time: 36:14)) because here it is this and that it is just oscillating.

And since, there is no dissipation whatever kinetic energy it loses by coming here, or potential energy it loses exactly the same potential energy, it gains by coming there. So, these two heights are exactly equal, so it will be perpetually oscillating without any decay, which means that it will have to come back to the same state, that is why from our knowledge of the system, we infer that this will be perfectly a closed loop. So, anywhere here it will be a closed loop, what happens if we start from here, if you start exactly on the Eigen vector then now what is that Eigen vector,

Student: ((Refer Time: 37:06))

No, no that is one all right that is associated with 1 Eigen value all right, but what is that physically what does it mean, it means that, let us consider this situation that as simple. Here, it is minus pi 0 and here it is slightly less than that ((Refer Time: 37:26)) which means, a position something like this what will happen then it will go like this, it will go like that that will perpetually oscillate like that.

So, which means that it will go in a loop, start from here which means ((Refer Time: 37:42)), it is vertically placed and then you give a initial push, what will happen if there is no friction, it must go like this fine. So, you can see that there are two different types of behavior, one type of behaviour is the oscillatory type of behavior, here the other type of behaviour is a specific type, where the x continuously goes on increasing.

Y has a oscillation x dot has an oscillation, but x has continuous increase, ((Refer Time: 38:17)) that means this fellow will go on rotating, that is when the angle goes on monotonically changing. What does it mean, these two behaviours entirely different behaviours that happen for different initial conditions, so if you ask me what divides them, what is the condition in which, slightly this way it will go this way slightly that will go that way, what is the condition.

That is a very unique situation, because now you have to understand something more, what were these, these are the Eigen vectors, the Eigen vectors had what property, that if you place an initial condition on that Eigen vector, it will forever remain on that Eigen vector that was the idea. Now, that idea was obtained from the linear system description and then we obtained that these the lines with that property would be straight lines, the straight lines were the Eigen vectors.

But, elsewhere it is linear as you go away from it is a non-linear system, so there is no reason to believe that the line with that property will still remain a straight line, so you can visualize that the line actually gets extended, but that does not remain a straight line. But, you can still conceptually figure out that, if you take an initial condition on this line I can still draw a line, on which it will forever remain.

That line will no longer remain straight line it will bend, now what happens is that it bends and then joins this one, this one bends and then joins this one and that is what separates the two types of behaviour. Can you see it makes an island, it makes an island inside it is a oscillatory behavior, outside it is a you know rotating behavior, an oscillatory behaviour or rotating behaviour in the state space is the oscillatory behaviour in the actual system, a straight increase in one direction means, a rotating behavior in the actual pendulum.

So notice here, these where actually the Eigen vectors, but then this fellow has started as the Eigen vector, but then it has bent and gone elsewhere with the property that it is, if an initial condition is placed on this it will slowly converge on this equilibrium point it is stable. So, these have different names in non-linear system, it is called stable manifold and then this is the unstable manifold.

So, what is the property of the stable manifold, it is that line if you place an initial condition on that line, it will convert onto the equilibrium point in forward time. And what is the character of the unstable manifold, if you place an initial condition on that it will convert onto the equilibrium point in backward timing. So, that is the unstable manifold in forward time it goes away from it clear, so then we have the Eigen vector as nothing but, the tangent to the stable manifold at the equilibrium point.

This Eigen vector is the tangent to the unstable manifold at the equilibrium point, so the stable Eigen vector is tangent to at the equilibrium point fine good. Now, this may give you an impression that simply, by looking at the local linear neighborhoods that is, what we did we obtained the local linear description drew initially the behaviour around that, equilibrium point and then we sort of ((Refer Time: 43:04)) logically, built up the vector field elsewhere, it might give you an idea that this could be always done. So, the behaviour of a non-linear system is rather simple, because you can always work on the local linear things and then build up the rest. That idea is wrong, let us re illustrate that with the help of one example.

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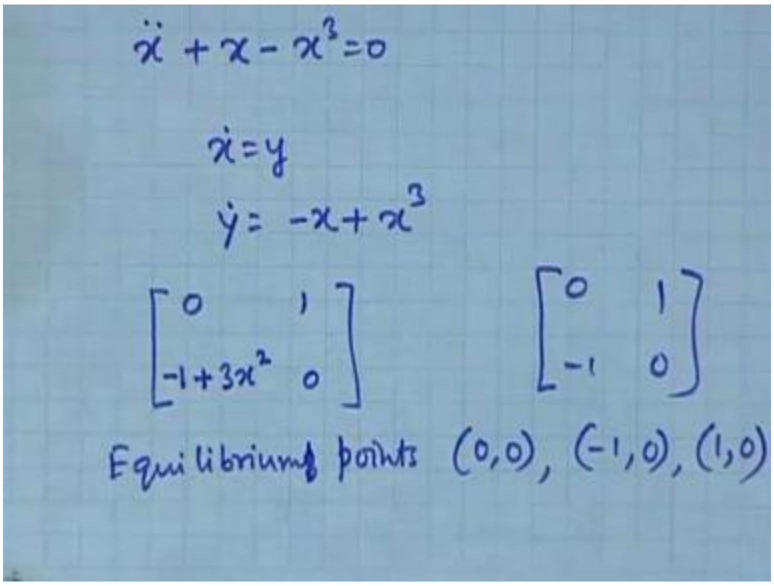

X double dot plus x minus x cube is equal to 0, suppose your system description is this then how we will obtain the equilibrium points and go ahead. First, we will write down the first order equation, we will say x dot is equal to y and then y dot is equal to minus x plus, so your Jacobian matrix is 0 1, this is minus 1 plus 3 x square and this is 0, what are the equilibrium points.

Student: ((Refer Time: 44:29))

What,

Student: ((Refer Time: 44:31))

Yes, there are three equilibrium points, because this is a cubic, so equilibrium points are 0 0 minus 1 0 and 1 0, so substitute 0 0 here, you get 0 1 minus 1 0. And this tells you that the, if the behaviour will be we have just done it will be circle yes, and if you put minus 1 here you get minus 1 or plus 1 does not matter, because it is square. So, you have minus 1 plus 3, you have plus 2 here, 0 1 2 0, what is the behaviour of that again a saddle.

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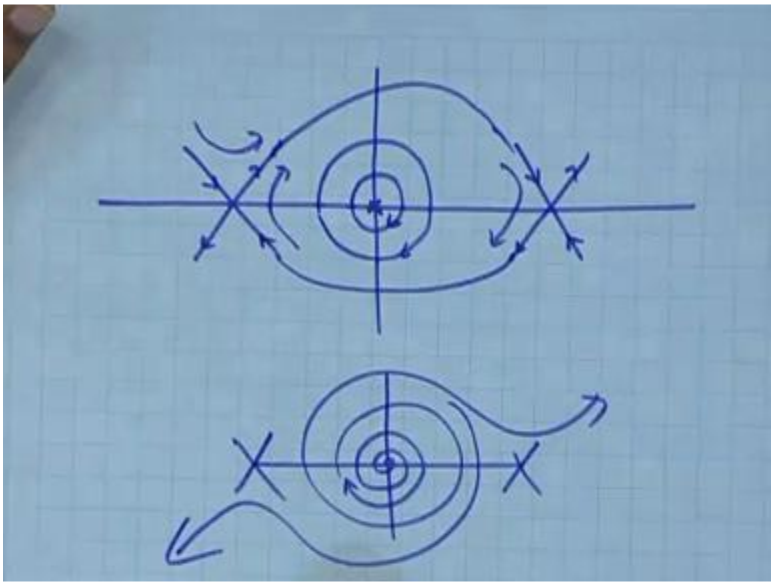

So the way, we have just done, if you do the same way we will conclude it that here it is a circle behavior, and here these are saddle behaviours. So, there will not be infinite number of equilibrium points. But, nevertheless around this it will be more or less the same the behaviour will be like this and then like this like that and so on, so forth. Now, notice an important issue, how did you conclude that this point was a circle, you had concluded on the basis of this, ((Refer Time: 46:32)) we had substituted here obtained 0 1 minus 1 0 here and it is Eigen values are perfectly imaginary.

But, that is true only when this x is exactly 0 notice, if this x is 0.001 then this will not become minus 1 if it is not then the Eigen values, will not become purely imaginary it will have some real part it will become complex conjugate. So, apart from exactly the equilibrium point at no other point, you can say that the behaviour is exactly a circle behaviour got the point it is an important point.

It is an important point, in the sense that this will tell you that the behaviour at the middle cannot be a circle in case of, the pendulum how did we conclude. We had taken request to the actual system description if there is no friction then it will be oscillating in the same way, that is why we concluded that it will come back to the same state, but here we cannot, so if we on the basis of the Eigen values conclude that this will be a circle that will be error.

So, give some number here, small number and see what is the real part negative or positive, give a small number here say, 0.1 it will be a positive Eigen value positive real part take that, which means actually here it will be an outgoing spiral get the point, so there is one way at least we have understood that in non-linear system, we have to take the linear local linear description with a pinch of salt.

Sometimes we have to be cautious when, because when the Eigen values are exactly placed on the imaginary plane imaginary axis have you got it why, because the imaginary axis Eigen values been placed on the imaginary axis is something that happens with infinitely small probability. There will always be some part, some small part in the real and depending on that it will be either a converging spiral or a diverging spiral. And it become exactly imaginary is a very, very unlikely situation.

So, whenever in a physical situation, you get a plus minus j you have to be cautious, is it real, or has it happened because, we have locally linearized it. Now, what actually happens here, if you start from a from a initial condition very, very close to that equilibrium point it will diverge all right, but diverge with a very, very slow rate. But, as it goes away that rate will increase, because these term will start playing its role and after sometime it will ((Refer Time: 49:46)) that it is really a diverging spiral.

So, the behaviour is actually like that, so it will diverge like this, so start from here it will diverge like that fine understood. So, that is one way in, which we cannot say that simply by working around the equilibrium points, we can conclude about the whole behavior, now let us take one important step ahead.

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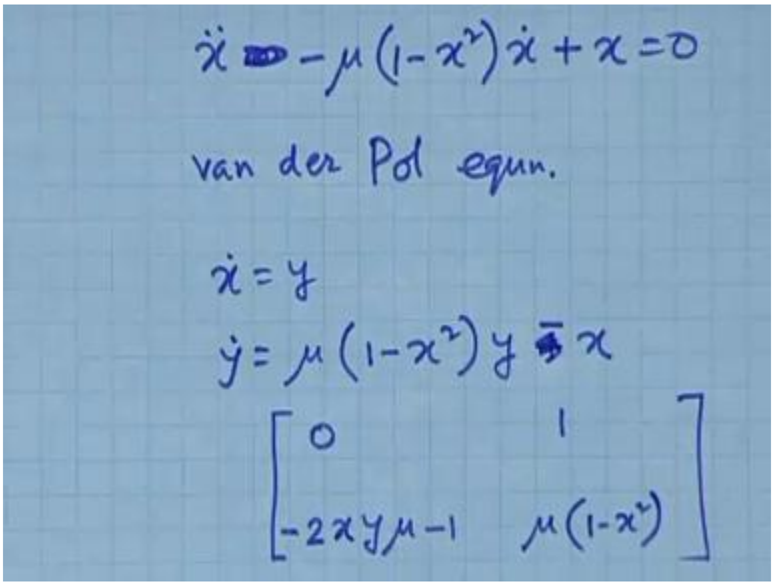

You have say a equation something like this, x double dot is equal to or minus mu 1 minus x square x dot plus x is equal to 0, this equation is actually called van der Pol equation. So, in this case how will you proceed again proceed in the same way, so we will say x dot is equal to y and y dot is equal to mu 1 minus x square x dot is y plus x fine. So, your Jacobian matrix is if you…

Student: Minus x.

Minus x, you are right, so you have 0 1 the first term is minus twice x y mu this is there minus 1 and mu 1 minus x square fine. So, that is the Jacobian matrix are you comfortable with my directly writing with it, because this is your f 2, so this term will be the derivative of f 2 with respect to x, so that is what I have done good. So, now what are the equilibrium points, only 0 0 there is no other only 1 equilibrium point fine. So, there is only one equilibrium point and you substitute, 0 0 here, what have you 0 0 you substitute.

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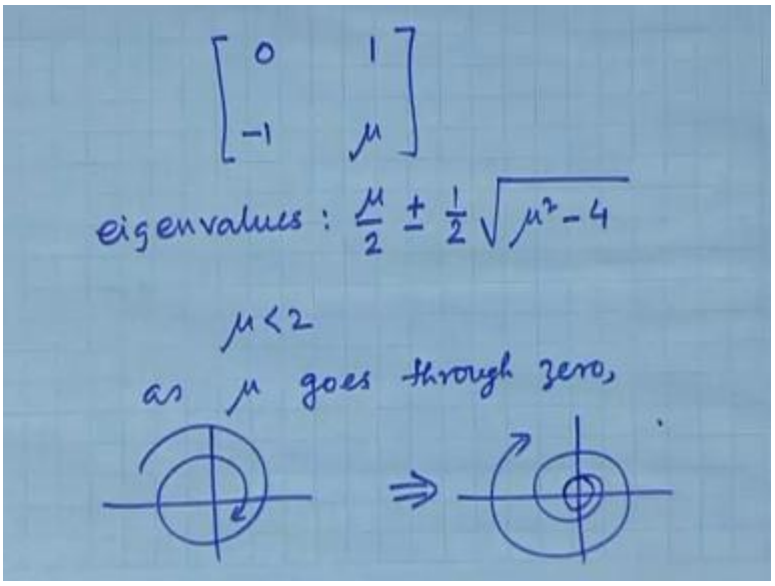

You have the Jacobian matrix as, 0 1 minus 1 and mu right, this term goes to 0 minus 1 this goes to 0, so mu fine, what are it is equilibrium, it is Eigen values, what are it is Eigen values, do it fast this is rather simple. So, you have mu by 2 plus minus, this immediately tells us that, so long as mu is less than 2, it will be complex conjugate. So, let us consider mu less than 2, so that we are dealing with complex conjugate Eigen values now, this tells us that as you mu from a negative value to a positive value, mu can be some parameter, then it becomes from a incoming spiral to an outgoing spiral.

So, you notice that as mu goes through 0 it will be from this to this fine, so the system losses stability at mu is equal to 0, it was initially stable and then it lost stability. Now, based on your linear intuition, what would you say that it has lost stability this system will collapse the state will go to infinity right, but then notice again what have we done.

We have locally linearized it and then on that basis we have taken the conclusion, how can we say that that will apply to say this point. You cannot say, because that is true only in the in the in the close neighbourhood of that fine, so as you go away from it, we can no longer say with confidence that whatever, we concluded on the basis outgoing spiral that we concluded on the basis of the local linear approximation will be valid.

So, the behaviour was actually incoming spiral, but then on the basis of the local linear approximation we have concluded that in the neighbourhood it becomes outgoing spiral. But elsewhere away from it we cannot say that fine, so what will happen?

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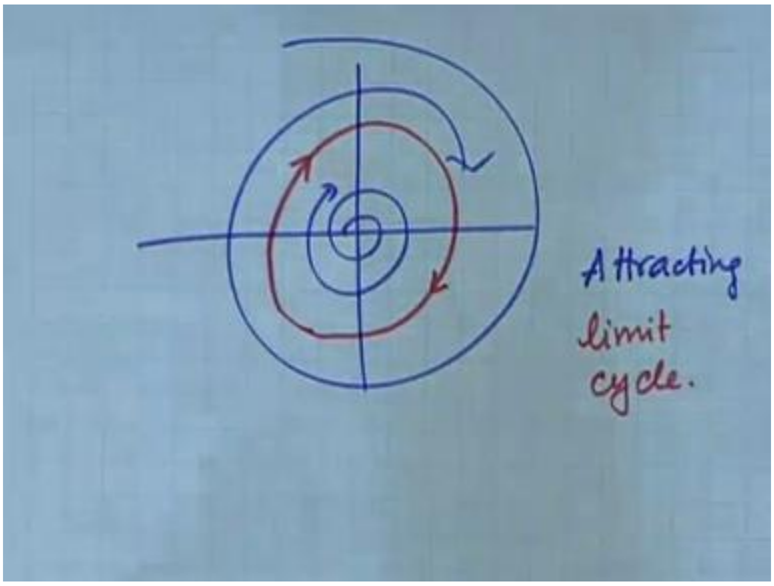

What will happen is something like this very interesting, that here it will be outgoing spiral all right, but outside it will still remain a incoming spiral, so the result is you can easily see that there will be some intermediate closed loop on which, it has to converge right. If you place an initial condition inside it will be guided by the local linear approximation and it will be followed it will follow a outgoing spiral.

Outside it will still remain an incoming spiral and so wherever you place the initial condition it will ultimately converge onto that orbit, which is a a closed loop, it is stable in the sense that, if you part of the initial condition it will still come back to that closed loop. This is a very special situation that can happen only in a non-linear system, it is called a limit cycle it is called a limit cycle.

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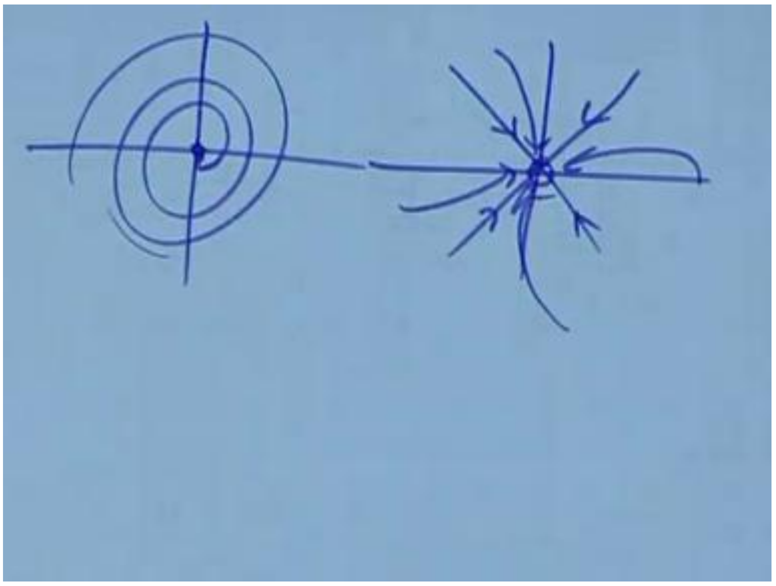

Now, there is another name for it, for example in a linear system we have seen that there is a situation something like this, so any initial condition anywhere in the state space in a linear system is then attracted to that equilibrium point. A situation something like this, where you have got all the incoming Eigen vectors, real Eigen values then everything goes to that, then also this equilibrium point is a stable equilibrium point and, but notice the property that starting from any initial condition is attracted to that.

That is why, these equilibrium points are called attractors, these are called attractors, these attractors in a linear system can only be point attractors points, equilibrium point can either be attractive or repelling. So, they these are point attractors, while in a non-linear system, you can have a whole closed loop as an attractor, so this is an attracting limit cycle, we will continue with that in the next class.

Thank you.