Dynamics of Physical System
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Lecture - 3
Derivation of the Lagrangian Equation

Start today by briefly recapitulating where we were in the last class, we have seen that even though the propositions of Newton. Where they laid the basic frame work for deriving differential equations, there are a some practical difficulties and we identified that difficulties.

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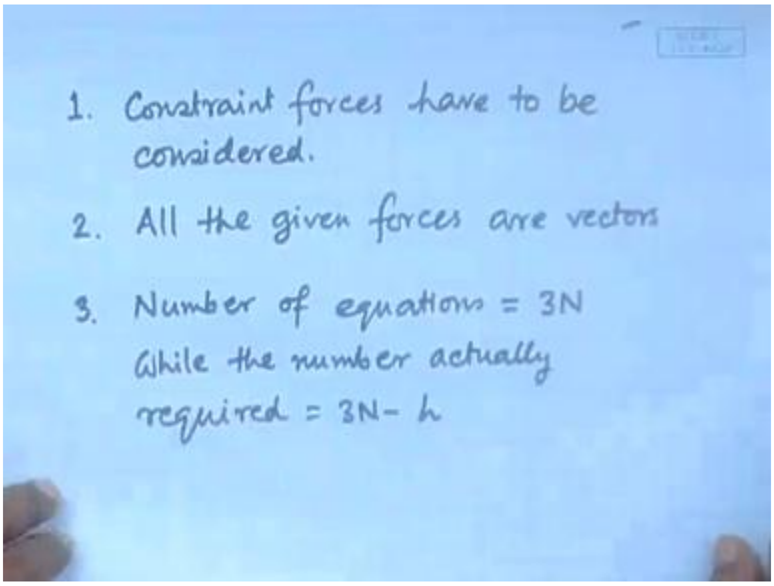

Let us enumerate them one by one, the difficult number 1 was that in a given system mainly constraint system there will be constraint forces, just to recapitulate what the constraints, where there were two types of constraints, holonomic constraints and non holonomic constraints. And the holonomic constraints were those, which can be expressed as a algebraic equation, why the non holonomic constraints are those, which can be expressed as in equation.

So, that was what we did and then we are said that in order for the body to follow the constraint or constraint surface, the constraint surface has to apply some kind of a force on it. So, there will always be some force acting on the body the constraint force and in order to write the Newtonian equation for each body, you will have to take in account all the force acting on it and therefore, the constraint forces will have to be considered.

So, number 1 was the constraint forces have to be considered, so that was one difficulty, because the constraint forces are not always easily quantifiable. Number 2, difficulty was that in a inter connected system various body will interact with each other with forces and these forces are in general produced by springs. If two bodies are connected by springs, this body will be applying a force on the other body by means of this spring, if two bodies move against each other by means of some kind of frictional element, the one body will apply a force on the other body.

So, for each body this forces have to be considered and all this forces are vectors, so that was another difficulty that we the formulation becomes very complicated. So, all the given forces are vectors and we would like to simplify the matter by writing down the differential equations in terms of things that are scalars, one would be comfortable with that. The third problem was, that we have seen that in case of a system with holonomic constraint for example, the pendulum, it is completely stupid to write three differential equations, because it does not have it does not move in all those directions.

So, in general if there are n total number of configuration coordinates no not n thrice n, if there are n bodies there will be three times of n number of configuration coordinates. But it is somewhat unintelligent to write all the thrice n equations, rather we should be able to write thrice n minus h number of equation, where h is the number of holonomic constraints. So, the number of equations while the number actually required, where h is the number of holonomic constraints. So, these where the three major difficulties with the Newtonian formalism, that we need to overcome.

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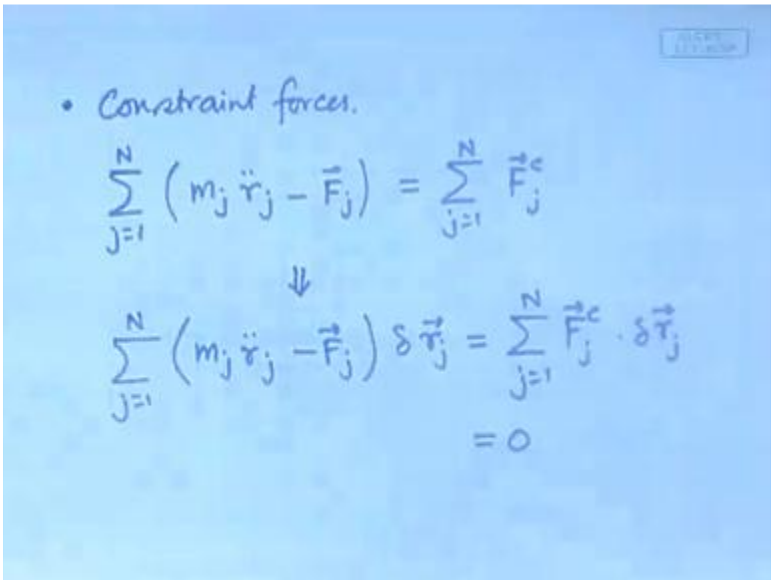

So, let us go one by one, first let us consider the situation of we consider the situation of the constraint forces one, how to and in countering this problem we argued that even though the constraint forces are there in general they do not work. Because, the actual motion is orthogonal to the constraint force, there are situations somewhat rare situations, most situation the constraint forces do not do any work.

But, where the holonomic constraint is what we termed as rheonomic, where the holonomic constraint is dependent on time. Say, there is a bowl in which you have released a marble to move and the bowl is being moved around, so these are the situations where you have a holonomic constraint all right, but that is dependent on time. Now, when that happens we need to go into another stage of abstraction, we said that we will not consider the actual motion, actual velocity we will not multiply the force with the actual velocity.

Rather, we will multiply the force with a specific conceptually or emergent velocity, emergent motion which is consistent with the constraint at every instant of time. That means, if you have a pendulum with a oscillating support, then at every point the chord has a particular angle and the bob is here. In that case, the admissible motion at that position would be orthogonal to the direction of the chord.

So, the moment we conceptually imagine that kind of a possible motion admissible motion. Then, we have a work the moment you multiply the forces with the, that admissible motion we get a work, but that is what the actual work we done by this system. That is again admissible work, you might ask, what gain do you make by considering this, yes we do will it will be clear to you later.

But, the what we did was we say that originally the equation was, originally the Newtonian equation was j is equal to 1 to N, j is the number of mass points and it was m j r double dot j that is mass into acceleration term minus F j this is the applied force or the given force, this was equal to the constraint forces. So, that was our logic that was the initial starting point, because this was the, what Newton said.

And then we said from here, that let us multiply both these sides by the admissible displacement. There by we got this is a vector equal to again the right hand side is equal to 1 to N F c this vector j times, now this term will; obviously, vanish because for each of this N bodies, the constraint force times the admissible motion is 0. So, is equal to 0 that is how we eliminated the constraint forces, so keep this final equation written in your copy, we will refer to it later.

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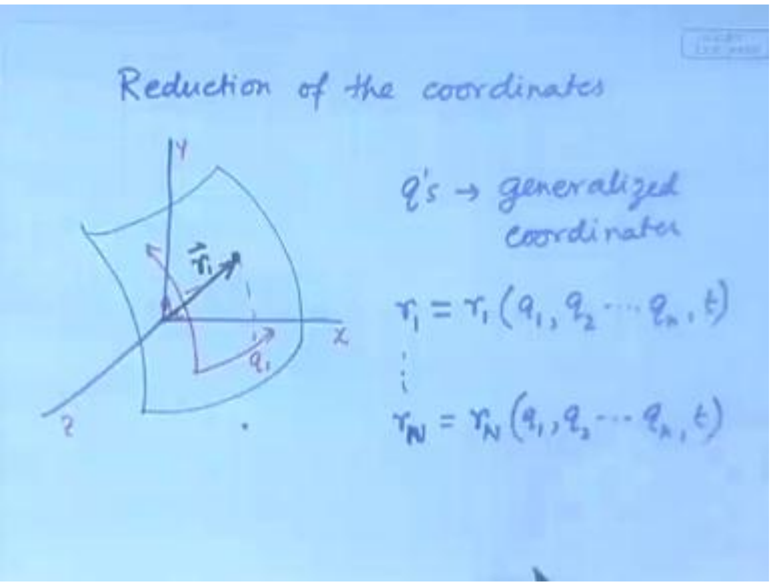

The second problem was the reduction of the coordinates and in that what we did we have do, we said that after all in the thrice n dimensional configuration space, each of the holonomic constraints define a surface. And if there are two holonomic constraints imagine for the sake of simplicity in visualization, the original configuration space is three dimensional, in which you have defined one holonomic constraint that will define a surface.

So, that reduces the dimension by one, if you have another holonomic constraint that will define another surface. So, ultimately the body will be constrained to move in intersection of the two holonomic constraints and that reduces the dimension by 2, so each holonomic constraint is reducing the dimension by it is number. So, since ultimately the body is constrained to move on the intersection of holonomic surfaces, there is no reason why we should still continue to define the coordinates in terms of the old x, y, z coordinates.

We said, that now that we have identify a subspace, where the body must lie the configuration point must lie not the body really, because we are now talking in terms of configuration space. If there are three bodies, there will conceptually thrice n three times three number of coordinates, where still I would advise you to still visualize or extend the visualization. We cannot visualize in nine dimensional space all right, but still try extend the visualization, that it is some kind of a space in which the moment you have defined a constraint it defines a conceptually a surface.

And the body this the configuration point must lie on that surface, must move on that surface. And therefore, we said that now let us define a new set of coordinates on that constrained surface and these are marked as q, the generalized coordinates, so in case of the spherical pendulum what would we do there is only one body. So, original set of coordinates were x, y, z and then since the body is constrained to lie on the spherical surface.

Therefore, on the spherical surface the most convenient coordinates are the spherical coordinates theta and phi. So, we define theta and phi as the new set of coordinates consistent with the constraints and in a system there could be some motions that are constrained to on a spherical surface, some motion that are constrained to be on a linear side. In that case those motions will be we would try to define the minimum number of coordinates consistent with the constraint that completely define the positional status of the system and that will be the ultimate our definition of the generalized coordinates.

So, we said that if we have a original thing like this and we know that the behavior is constrained on this kind of a surface. Then, we would define a new set of coordinates and then any point which was earlier expressed in terms of x, y, z, now would have to be expressed in terms of q 1 and q 2. Now, extend that vision this x, y, z could be thrice n and this q 1 and q 2 could be again a large number of variables, just the idea is that we have now defined a lesser number of variables that are consistent with the constraints.

So, q’s are, so suppose a body was there at this position, then it is position is given by this r vector in the original coordinate. If it is given by the r vector, then for this body this r can r was earlier expressed in terms of x, y, z, but now it has to be r has to be expressed in terms of q 1 and q 2. New set of coordinates, similarly there was another body suppose here it is coordinate say this would be r 1 and this one would be r 2. And all those bodies’ coordinates now would have to be expressed in terms of the new set of coordinates.

And that can be easily done, because we can say r 1 is equal to r 1 of the new set of coordinates q 1, q 2, q n and may be time, it might also depend on time, because the surface itself might depend on time, similarly that will go to r n capital N q 1.

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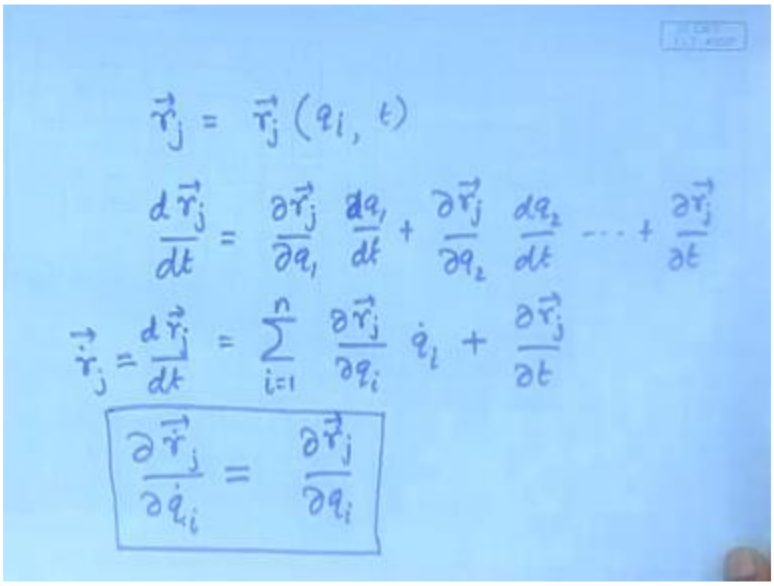

So, in short we can write then t j th mass point this is a vector as r j of q i and time, where i goes from 1 to n. Now, once we have defined the old set of coordinates in terms of the new set of the coordinates, we can also define the velocities, the velocity would be r dot j or d r j d t. So, if r j is dependent on all these, when you take the derivative it will be expressed in terms ((Refer Time: 17:30)) you will be able to write and all that plus dived depends on time.

So, this is nothing but, sigma i is equal to 1 to n delta r j vector delta q i and this is q dot i plus r j. So, this is how the velocity would be expressed velocity of each body fine, notice in the left hand side what we have, this body is moving ((Refer Time: 18:46)) bodies velocity expressed in terms of the new set of coordinate, we will use that. Now, if I ask you how does the r dot j this is, so this is r dot j how does this fellow depend on the generalized coordinates.

So, if i ask you del r dot j del q dot i what you get, suppose you want to express this derivative, look at the right hand side this is a generalized velocity in one direction i. So, if we differentiate it with respect to another generalized coordinate you get 0, because it general coordinate is a independent quantity. So, the moment you want to do this derivative with respect to q dot i, then you will find only that term will remain where this and that are the same.

All the others will vanish and this will also vanish, because this tells how does the constraint surface vary with time and that also does not depends on q dot i. So, ultimately what remains is the set clear, so we keep this for future reference, we have argued that this would be valid and we keep this for future reference fine. So, there are two things we are keeping for future reference, one is this equation another is this equation we will come back to that.

Let us keep somewhere, so that we can again refer to that fine ((Refer Time: 20:58)) now let us get back to this picture, this was the position r of a body, if I ask you what is it is admissible motion it is admissible motion is on the surface.

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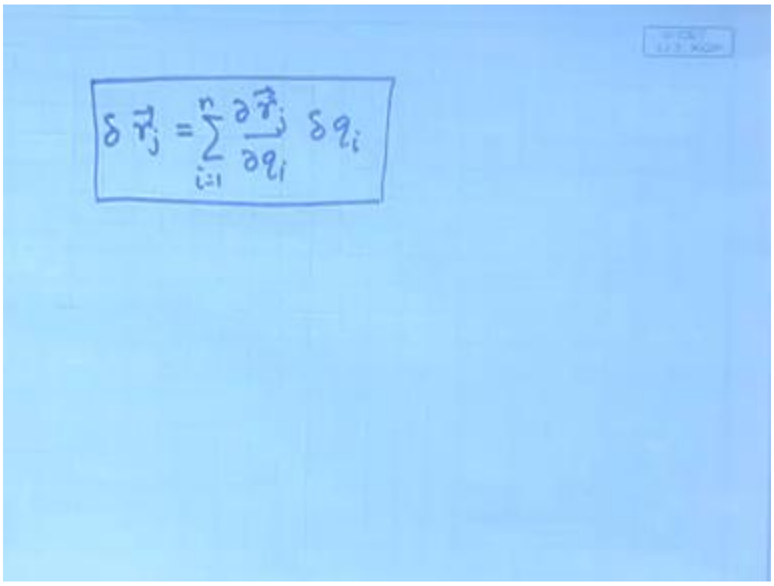

So, your admissible motion delta r j this now has to be on the surface that now has to be expressed in terms of the new coordinates and how would you expressed that in terms of the new coordinates. Exactly the same way we will say that this is delta r j del q i times delta q i sum over i is equal to 1 to n correct I am not writing the broken up form, because that will be the same for each body j the derivative of that r j in terms of the ith coordinate times the admissible motion along the ith coordinate.

So, this has to be summed up, so this is another thing that we keep for future reference, getting too complicated know what you have essentially done is, that what are we interested in we are interested in the transformation of the coordinates ((Refer Time: 22:37)) and who do we transfer. First this vector to the derivative of the vector; that means, the velocity position and the velocity how do they transform into the new coordinates. And then the admissible motion that is the three things we wanted to express in terms of new coordinates, that is what we have written now fine ((Refer Time: 23:11)).

Now, let us start with this equation that we wrote, we had multiplied by the generalized by the admissible motion and obtained the right hand side 0. Let us start from there ((Refer Time: 23:28)) at this time our objective is number 2 all the given forces were vectors. So, everything in the Newtonian framework has to be expressed in terms of vectors, and then in any given system what is the absolutely general scalars that we have the energies energy is a scalar.

So, we would like to express that was the ingenuity of Lagrange, that he said that everything can be expressed in terms of energies. So, let us try to frame the whole set of equations in terms of energies, which are scalars and what are the energies kinetic energy and the potential energy. So, that is what we are trying to arrive at, but you we will have to go through a route, because the way Lagrange himself did and the way you are to understand, they are a bit different. Because, now we have the hindsight of how this thing can be derived far easily, that is why we are going this way.

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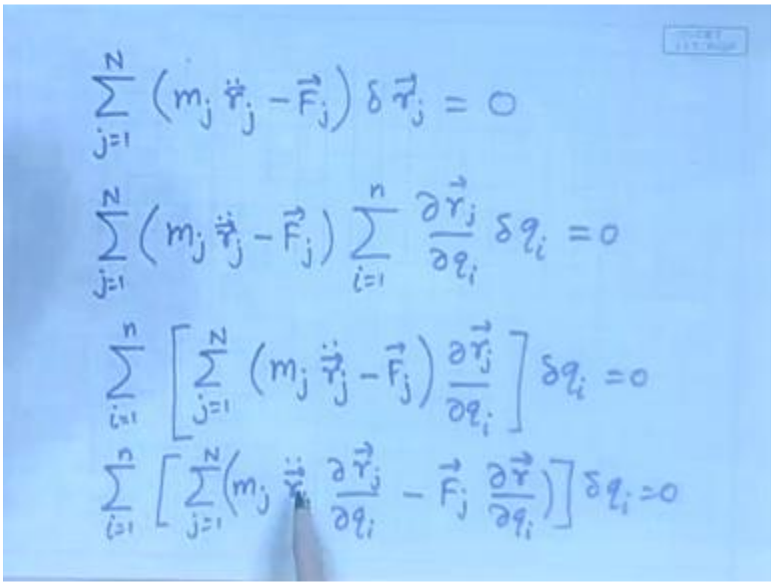

So, we start from here j there are n number of mass points, so N m j r double dot j minus F j I am just copying this, this is a vector, this is a vector delta r j is equal to 0 that is where we were. Now, we have obtained something about how the generalized admissible motion transforms, we ((Refer Time: 25:13)) that and that is what we need to substitute here the first step. So, we substitute what we have, you can also substitute on your copies, you will see N m j r double dot j minus F j is a vector times all this stuff i is equal to 1 to n I have just substituted this.

Now, there are two summations here and is immaterial in which order you sum them, so we bring this summation over i forward. So, let us do that we bring this summation over i forward, what remains at this term, so you have the summation over j, summation over i just to recall what is summation over i the number of generalized coordinates and summation over j is the number of bodies two things.

Then, we this term comes m j r double dot j minus F j this and then this term delta r j delta q i. So, this term and then you have to sum over fine, so this term is summed over j and that term after that it is summed over i let us put this term inside, so what do we have is this part visible yes i is equal to 1 to n just put this inside you have sum j is equal to 1 to N m j r vector double dot j times r j delta q i I have multiplied it here minus F j vector r delta q i, just keep check that I am doing it right fine done.

So, we have this term we need to put a bracket here, because the whole the summation is over the whole thing. Now, let us see what each term means, here we have r j that is the position of the nth or jth body is double dot times this derivative, what does it means physically just look at it.

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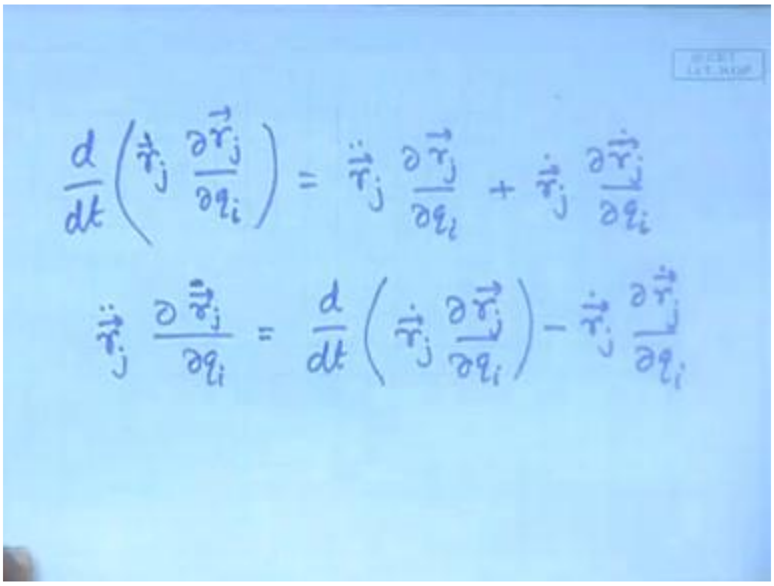

First suppose we have we take r dot j delta r j these are all vectors, suppose we start with this term and take a d d t of this can you do it, there are two terms here we will go by the one the standard way of differentiation. So, we first differentiate this, so that would be this is dot r dot here r double dot j times this plus this remains now and this term is differentiated with respect to time.

So, we have r dot, so del r dot you can do it this way, so we have just written the derivative like this. Now, you notice why did we do all this, because this term is this term, so this term is expressible as r this is what we are interested in expressible as this is not there r j del q i is equal to this minus this r dot j derivative of j with respect to q i minus r j is a vector dot. So, we have understood the meaning of the first term in terms of two terms, which we are yet to understand leave it at this stage, but let us substitute it here, let us substitute it here.

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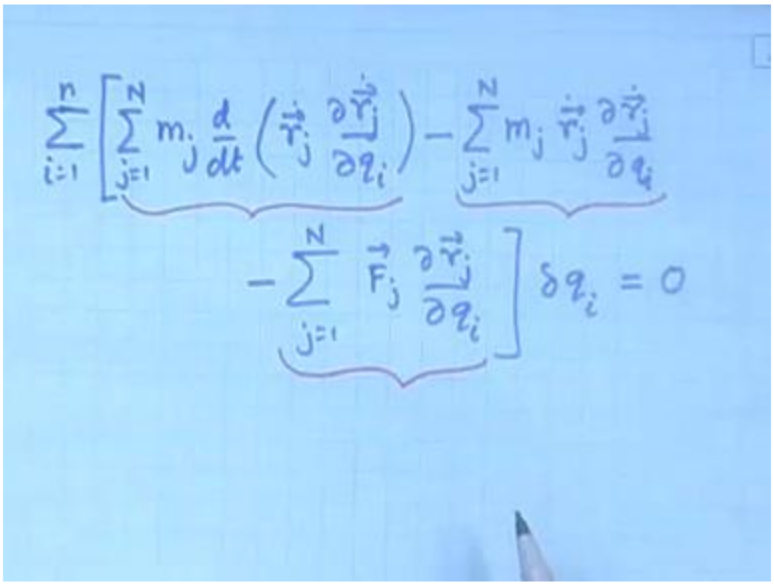

What do we have after substitution we have, let us start from further left hand side, because it will be rather large i is equal to 1 to small n, then we have a bracket then it is in terms of j is equal to 1 to capital N, this is what we are writing m j then this is to substituted. So, it will be m j d d t of r dot j dou r j dou q i fine minus let us break up the summations again j is equal to 1 to capital N, the next term m j remains now comes this term r dot no this is vector dot j dou r dot j dou q i.

So, that substitution is over, but there was another term remaining here, that has to be written. So, we will again write it as a separate summation I will write here minus j is equal to 1 to N it remains as it is F j dou r j dou q i, now this bracket will be closed times your generalized coordinate generalize admissible motion is equal to 0. Now, after all these business we have come to three terms, one term, second term and the third term three terms.

And these just by looking at these terms it is very difficult to physically understand what they are. So, let us try to physically understand what they are, we will come back to this, so let us keep this here and let us write down individually, we said that we want to express things in terms of energies.

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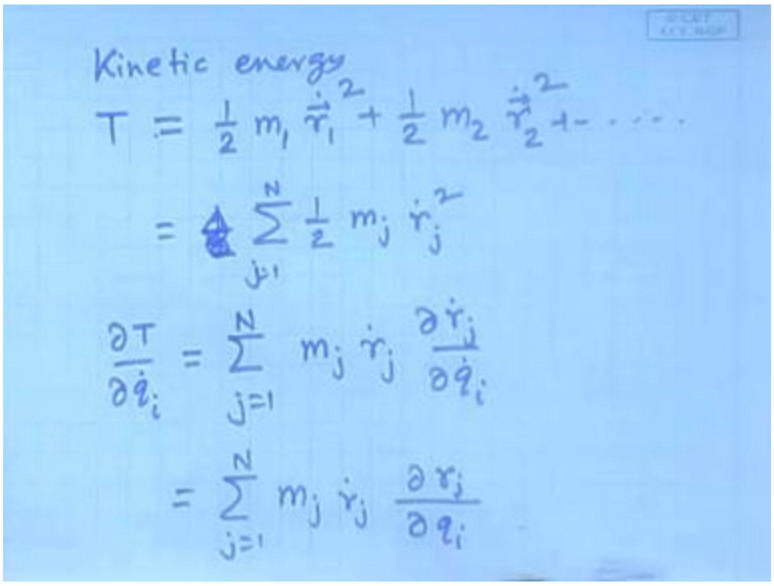

A first energy is kinetic energy, what is the kinetic energy total kinetic energy, kinetic energy expressed written as T. So, kinetic energy is half m v square for each body, so half m 1 that is r dot 1 square plus half m 2 r 2 vector dot square plus, so that is the total kinetic energy, that is half no I will keep the summation outside half this is summed over j is equal to 1 to N, N number of bodies half m j r dot j square fine do we have this term around here no not really not exactly this term.

But, we have what could possibly be obtained from it is derivative, so let us take the derivative of this term, this kinetic energy. So, this is the kinetic energy, so the derivative of the kinetic energy, kinetic energy is after all dependent on the velocity, so we will derive take the derivative in terms of the generalized velocity T del q dot i that is the generalized velocity.

If you take what do you have in the right-hand side, we have the summation first j is equal to 1 to capital N, when you take this derivative this two comes forward and gets cancelled with the half. So, it will be m j r dot j and then the derivative of r j del dot j dot q i, this is how would you express fine, so we are better off because we can identify this term somewhere here. So, we know that here this term was actually talking about the derivative of the kinetic energy with respect to the generalized coordinate good at least one thing is settled.

Now, where did I keep the expressions here, ((Refer Time: 37:06)) see we have already obtained that the derivative of the velocities would be equal to the derivative of the coordinates, we had already done this we had kept it for future reference that is where we will use it. This term is then equal to m j r dot j, but now we do not need to write the velocities just the positions, so this is something that we will keep and then let us also differentiate the kinetic energy with respect to the position.

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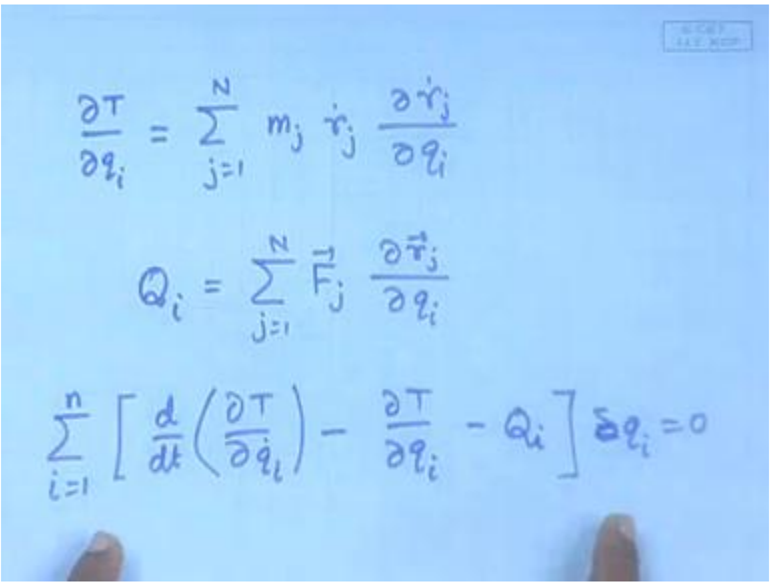

So, we have derivative of the kinetic energy with respect to the position not q dot i it is q i, then we will again have to write is as the same summation of partial derivatives. So, it will be summation of j is equal to 1 to capital N and in that case what will remain m j see this is what we are trying to express or take a derivative of in terms of the q i. So, it will be same way m j r dot j times the derivative of r dot j times the derivative i, you can possibly identify that this is this term.

So, we are slowly homing on to what we wanted to express, ((Refer Time: 39:09)) finally if there is a force acting on this body, that force if now we expressed in terms of the new coordinates how what shape will it take, in the new coordinates suppose that generalized force is called Q i will be expressed as summation of again j is equal to 1 to N F is the vector j r vector j. So, all we are doing we had the actual forces applied F j, now we are converting into transforming into the new set of coordinates Q i.

So, these are the forces generalized forces that are applied in the direction of the generalized coordinates, so this will be the force. Now, suddenly we see that all the terms that we wanted to know, ((Refer Time: 40:14)) they are all clear what is this, this is actually the original forces acting on the bodies converted into the generalized forces, what is this term, this term we have just indentified as this term.

So, this term is same as this term, so d d t of this is d d t of this clear, so and this term is already we have derived. So, all we can now express this equation in a much simpler and elegant form as i is equal to 1 to small n and then we had written, now the all the summation go away, because this term summed together is some simple term, this term summed together is some simple term, this term summed is sum together is some simple term.

So, all these summation go away over j and we get d d t of del t q dot i, because that is what we had written, this term were expressed as this and that is what we have in that particular term. So, this is the first term minus the second term here is what we wanted to express, so we have del T q i dou Y dou q i that is the second term and the third term is minus simply Q i, this whole thing simple it has to be multiplied by the admissible motion is equal to 0.

So, true that we went through a lot of you know partial derivates and messy stuff, but ultimately the objective was to simplify stuff and here is the simple thing. And you notice, that here is something that is multiplied by the admissible motion and the admissible motion along the first coordinate and the admissible motion under second coordinate are all independent quantities. And this term summed over i would be 0 only if this term is 0 always can you see that.

So, here is the term for the ith thing it is multiplied by the ith admissible motion and all that summed over will has to be 0. And that can be 0 only if since this cannot vanish, this is something that we have assumed that it can be just any motion, therefore this term must vanish. Not only that this term must vanish for all the generalized coordinates, not the summation, just individually along all the generalized coordinates this term must vanish.

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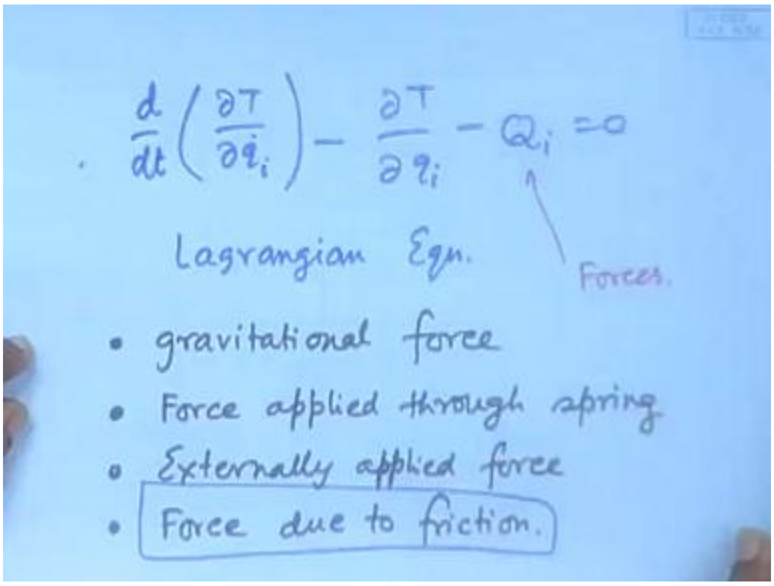

So, that gives a very simple equation d d t of dou T dou q i dot minus doh T doh q i minus the generalized forces equal to 0. Now, this is the Lagrangian equation in the most generalized form, in the most general form, you might wonder that, so far we were initially talking about energies, energies would be kinetic as well as potential. But, so far we have talked out only kinetic energy, some where does the potential energy come or where does the potential energy go.

Now, you notice that here we have got a generalized force, what does that mean, it means the forces on the bodies in the direction of the generalized coordinates. So, in case of the pendulum, if you apply a force in the direction of the theta, so that will be the generalized force. Now, what does this force consist of what could be the different types of forces acting on a body just think, these are the forces, so these are the forces that we need to consider.

What kind of forces could it be, it could be one forces like the gravitational force it could be force applied by one fellow on another fellow by means of a spring in between. So, a force could be also externally applied somebody pushes a body, so externally applied force and there could be also the force due to friction. So, there could be all these types of forces have we taken it into account everything is there any other type of force that you can imagine, that can be applied on a body, no we have taken everything in.

So, if you drop this the last one, then all the other types of forces are actually derivable from a suitable defined potential energy I will show that. But, ((Refer Time: 47:03)) this will have to be separately treated, but other forces gravitational force we know that there is something called a gravitational potential and the gravitational force is the gradient of that potential you learned in school. But, yes you may not know how that can be done for a force applied through a spring or the externally applied forces.

But, for now take it for granted I will illustrate all of these separately, take it for granted that all these forces can be treated the same way we treat the gravitational force. That means, in the gravitational force we said that there is a gravitational potential and the force that is applied is actually the derivative or the gradient of the potential.

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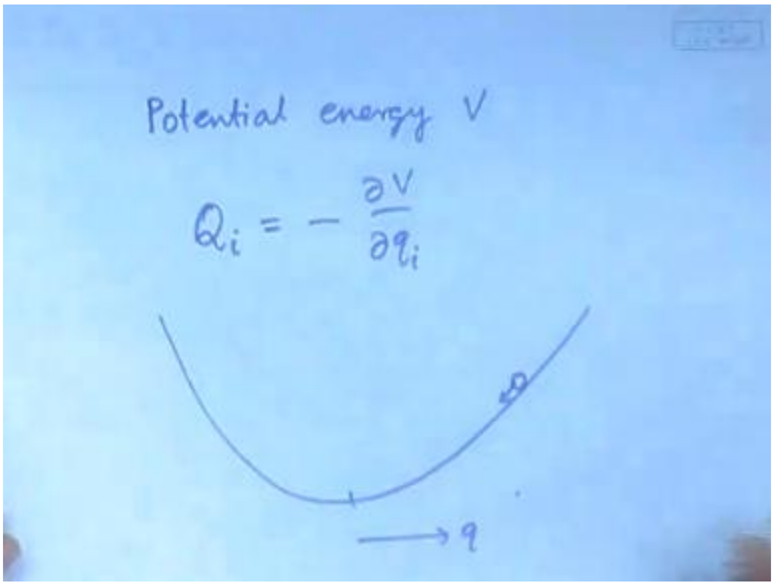

So, the potential energy is expressed as v, so the generalized force can be obtained as the gradient or negative gradient clear and that satisfies your concept of the gravitational force. But, in general in order to visualize imagine in this way, the potential is like a potential surface, say suppose you have got a potential surface like this and the body. So, this is the q q i coordinate and here is the body sail, so body’s position in q i is this much from here to here, what will be the force acting on the body the gradient.

And you might imagine a three dimensional surface and in that if the body position is such, then the force acting on the body is nothing but, the gradient of the potential surface. So, potential is conceptually a surface, you have got the q 1 coordinate, q 2 coordinate generalized coordinate and in that it is a surface and the position of the body, if you know then depending on the position it applies a force which is a gradient.

And how it also applies to other kind of forces I will come to, but that at least satisfies your concept of the gravitational potential does it not. Gravitational potential concept was exactly this, even electro static potential's concept is exactly the same, what was the concept of electrostatic potential, that if you have a particular body at a particular place how much force is applied and the force is obtained as the gradient of that potential. So, always the potential's concept is the same that the force is a gradient of the potential negative of the gradient like here, how it applies I will to the other kinds of forces I will come to that later.

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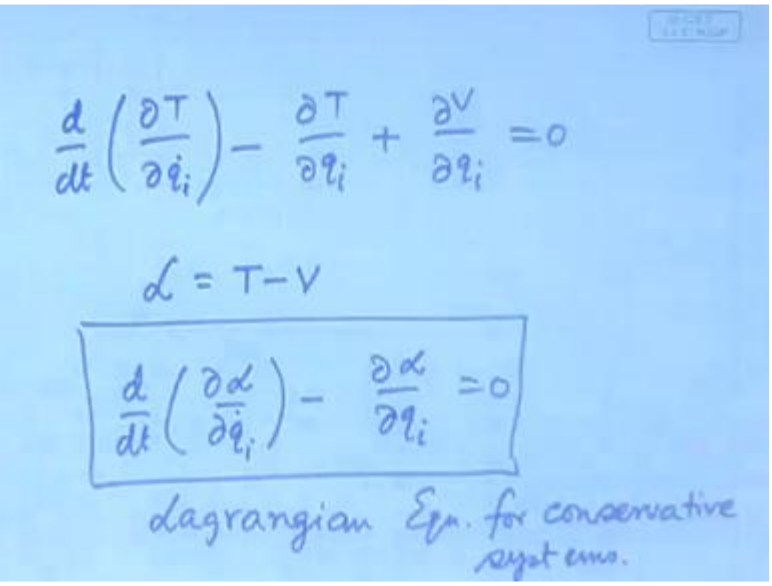

But, once you assume this, then this can be substituted from there we will substitute and write d d t i dot minus here we substitute minus dou v dou Q i, so this will be plus. Now, Lagrange defined a new term called, now it is called Lagrangian we write it as script L which is the kinetic energy minus the potential energy. So, this can easily be seen that this is a derivative of the Lagrangian with respect to q i, can you see that this term the whole term.

But, this term well here you were you are taking a derivative of the kinetic energy with respect to the velocity and the potential is normally not depend on the velocity. So, we can very simply write it as d d t of derivative of the Lagrangian with respect to minus derivative of the Lagrangian with respect to, so simple after all this day's labour, we have obtained such a simple equation.

That is the actual form of the Lagrangian equation for a non dissipative system, ((Refer Time: 52:08)) because we have yet not considered this term, how to consider this term we will come to that later. But, this is the Lagrangian equation for non dissipative system for conservative system. Let us stop here and we will illustrate how to apply this to physical system in the next class.