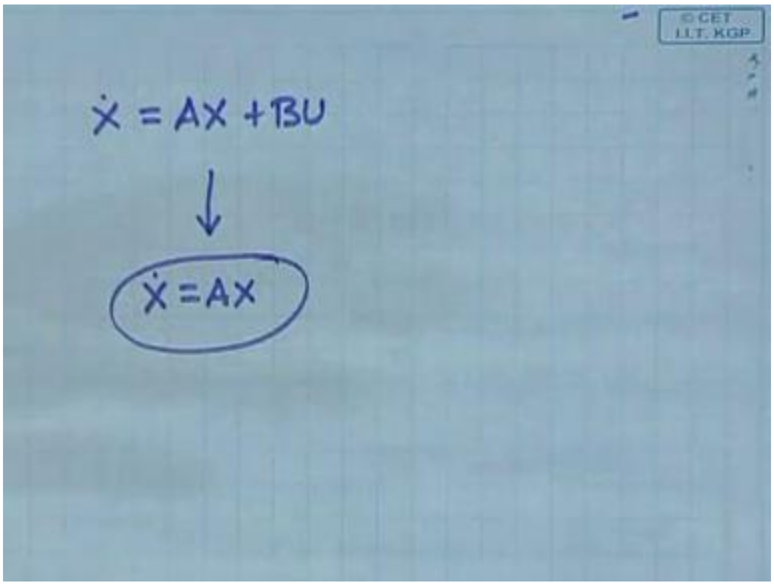
Dynamics of Physical Systems
Prof. S. Banerjee
Department of Electrical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 27
Linear Systems with External Input – I

We were dealing with we started by saying that in the neighborhood of the equilibrium points.

(Refer Slide Time: 01:00)

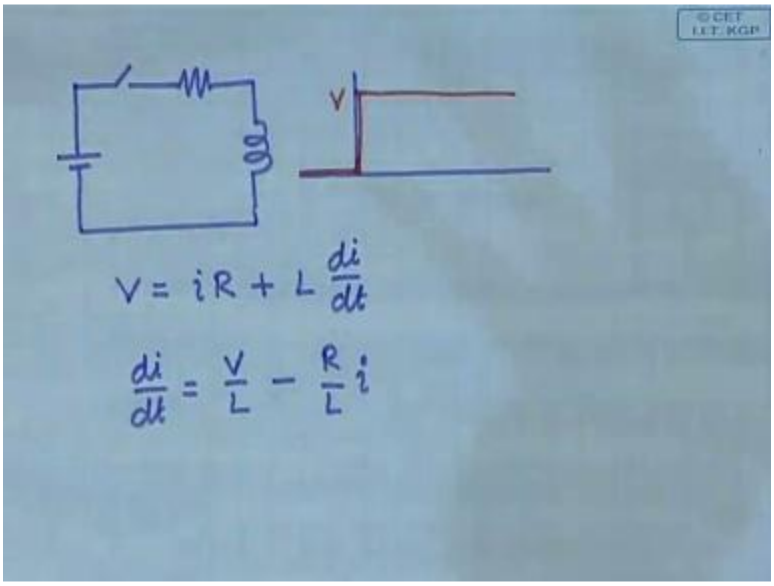


The equation would be of this form, but then we had reduced it to the form X dot is equal to A X and went ahead with that solution, why what was the logic. So, for all the solutions that we have obtained were of this form, so why not of this what was the logic. That we first tried to understand what happens in the neighborhood of the equilibrium points. And we said that the effect of this is basically to decide the location of the equilibrium point this would be 0.

So, the location of the equilibrium point will actually be given by this term B U, can you see that. So, if this is 0 X the solution would depend on the B U vector, so if this is not there it is the origin, if it is there then it will be something other than the origin. So, the effect of the external forcing the external term would essentially be to decide the location of the equilibrium point, and then we said that let us first put that out of our consideration.

First let us try to understand, what happens in the neighborhood of the equilibrium point. So, we considered this and then we have obtained we have understood what happens if this X is a two dimensional vector, what is if the X is a three dimensional vector and all that. But, now let us take this into consideration, so what will be the effect of this; obviously the effect will be nothing but a non origin equilibrium point that is all.

(Refer Slide Time: 02:59)

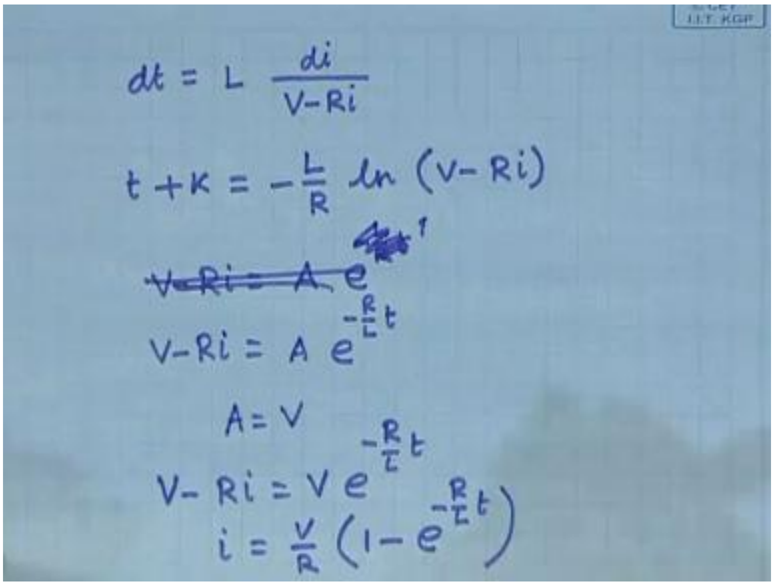


So, if you have a, let us take an example circuit for example, if you have a circuit something like this a battery here connected with a switch to a R L circuit. So, if this the switch is closed the applied voltage will be what, it was earlier 0 at this point it will rise to some value and then it will be like this. So, the applied voltage on this R L block will be initially 0 and then at some point it will be applied.

So, this is the applied voltage V, in that case how would you obtain the solution. Firstly, this problem is trivial and you have all learnt how to obtain the solution in your math classes. So, but here we will do it in a slightly different way to obtain a bit of conceptual clarity the way we want it to do in this particular course. So, let us first do it the way you have already learned, so what will you do, you will first write down the differential equation, the differential equation is V is equal to the Kirchhoff's voltage law plus that is the Kirchhoff's voltage law.

So, you have d i d t is equal to V by L, so we need to obtain the solution of this, this is a one dimensional first order equation. So, it is very easy to obtain the solution of this can you all obtain, the way you have learnt to solve it is by separating the variables, if you separate the variables d t comes in one side and I and d i comes in the other side.

(Refer Slide Time: 05:26)

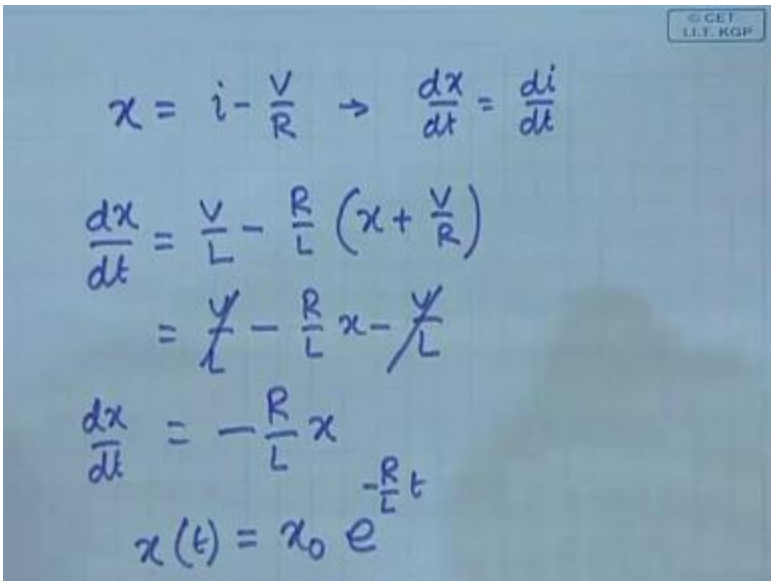


So, you will write d t is equal to L becomes common L d i by V minus R i, see we started from here. So, you have d t in one side L and d i by V minus R i and then you would integrate this, the integration would mean it would be t plus an arbitrary constant K, this will be integral of this l n of... Now, you have to obtain the or we can write this in the other side V minus R i is equal to exponential.

So, some constant time exponential minus L by R t I will write once again V minus R i is equal to some A e to the power minus. This A, then has to be obtained from the initial condition, suppose the initial current is 0 which is natural, so in that case how will you obtain A, what is A t is 0. So, this term becomes 1 this is 0, so you have A is this is 0, this is 1, this is 0 this V, so the solution becomes i is equal to V minus R i or is it visible yes 1 minus that becomes the solution.

Now, this is the standard way of solving it given an equation like this, this is how you have obtained you have learnt how to solve it in your math courses, separate the variables integrate them and then write. In this course we will take a slightly different route, only for the sake of conceptual clarity, we will say there what is happened. Here, we have an equilibrium point somewhere else, my starting point is somewhere else, we know how to solve the equation if the equilibrium point is the origin.

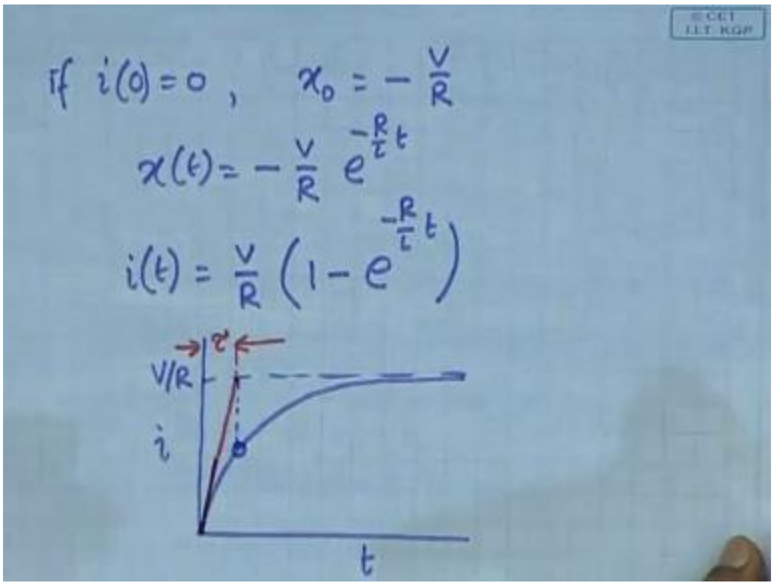
(Refer Slide Time: 08:59)



So, we can do that easily by introducing a new variable say x which is say x is where should I move it, where is the equilibrium point now, V by R. So, we will make the coordinate transformation as i minus V by R, so for x that is the equilibrium point, if this is the new variable x, in terms of that how does the original equation look, original equation was this. So, this gives d x d t is equal to straight.

So, you have the equation to be written as d x d t is equal to V by L minus R by L i is x plus, you can see that V by L minus R by L x minus consists of V by L consists of is equal to minus. So, the equation is simply a x dot is equal to something times x. So, d x d t is this, it is solution you already know what is the solution x of t will be x naught e to the power. What is your x naught, x naught see if the initial current is 0, the x naught is the initial deviation from the final equilibrium point.

(Refer Slide Time: 11:15)



How much is that, what is x naught then if i at 0 is equal to 0, then x naught is...

Student: ((Refer Time: 11:23))

Yes x naught is minus V by R. So, that is the initial deviation, so the solution immediately comes out to be x of t is minus V by R e to the power minus R by L t. Now, you substitute what is x, x was i minus V by R, so i of t becomes V by R 1 minus e to the power, that is the solution we had already obtained it. But, now we did it is slightly different root.

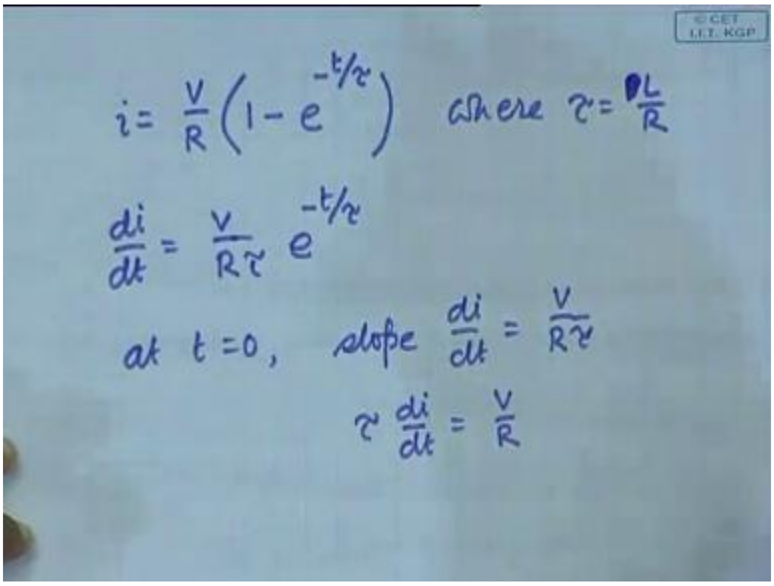
What is the advantage of this root, no algebraic advantage really, it is only the conceptual advantage that what we do is, we know that what has happened is essentially that the equilibrium point is somewhere else, my initial condition is here. So, I will do everything around the equilibrium point, so we simply redefine the coordinate and obtain the solution and then this immediately comes out. The advantage will not be immediately apparent from this very simple case, where it will be apparent when we consider for example, a square wave as the input.

What happens then, if you try to obtain the solution algebraically it will be unnecessarily complicated. In terms of concept what is happening is that, the equilibrium point is moving from here to here and back, equilibrium point is oscillating between two different positions. So, long as it is here it will try to approach it before it reaches this fellow comes here, so it will again try to approach it and so on and so forth.

So, if you look at it from the point of view of the solution around the equilibrium point, that also will become rather simple. But, at this stage it will be necessary for us to introduce the question of how fast or how slow it goes there. For example, what will be the solution like, the solution how will be the graph like at t is equal to 0 it will be 0 1 minus 1, 0 has to be 0 and when t is equal to infinity V by L.

So, it rises like this natural question is how fast does it rise, here is t, here is i, here is V by R. So, it could be like this, it could be like that, so how fast does it rise, so the standard way of answering that question is to write these in the form solution is written in the form.

(Refer Slide Time: 14:40)



I is equal to V by R 1 minus e to the power minus t by tau, where in this case tau is equal to, so t by tau yes. So, L by R and this term tau is the time constant, now you probably most of you have done this right or is it all, time constant the simple concept of time constant. You have all done just to recall what is happening is that, what does this say tau, consider the solution the slope of the solution exactly at this point.

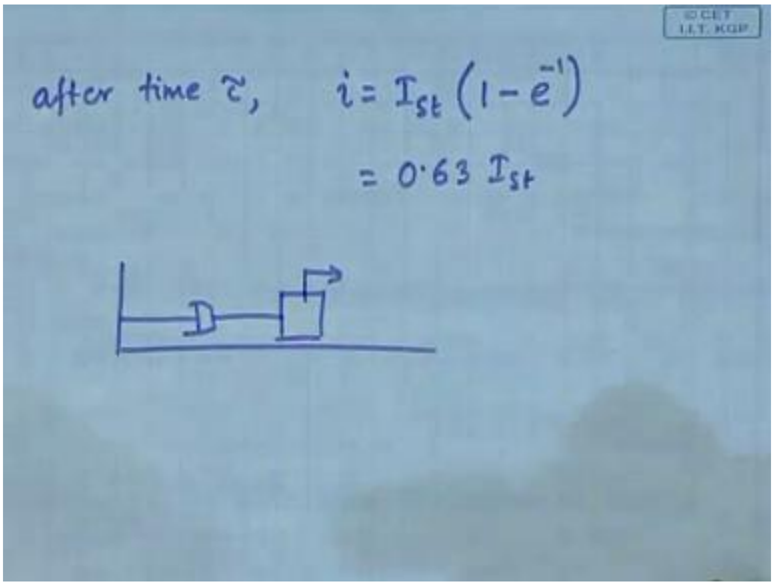
So, what is the slope of the solution d i d t is from here just do that this goes off it is V by R then in the denominator there will be tau, and you have e to the power minus t by tau at t is equal to 0 this goes off it is only this, it is only this much. So, at t is equal to 0 slope is V by R tau, so you have tau d i d t is equal to... This means ((Refer Time: 16:43)) that if this one were to continue, then this means that this slope times the time tau gives V by R.

So, after a time tau it will reach the V by R. So, this time is tau, so tau essentially says that if the initial slope continued, then after a time after a lapse of a time tau it will reach the final value. But, it actually does not it actually goes like this, but what does it say if the tau is small then; obviously, it rises fast, if the tau is large it rises slow. So, how fast or how slow this rise is given by the time constant tau, smaller the time constant, the faster the rise, the larger the time constant the slower the rise. So, that is the idea of the time constant there is some books in which it will be given in a slightly different way, they would say that if after a time tau where does this reach?

Student: ((Refer Time: 17:52))

Yes, it will just substitute here substitute tau here, you will get do that no, no you know that is I do not want you to say, it I want you to get it just for the sake of recapitulation.

(Refer Slide Time: 18:16)



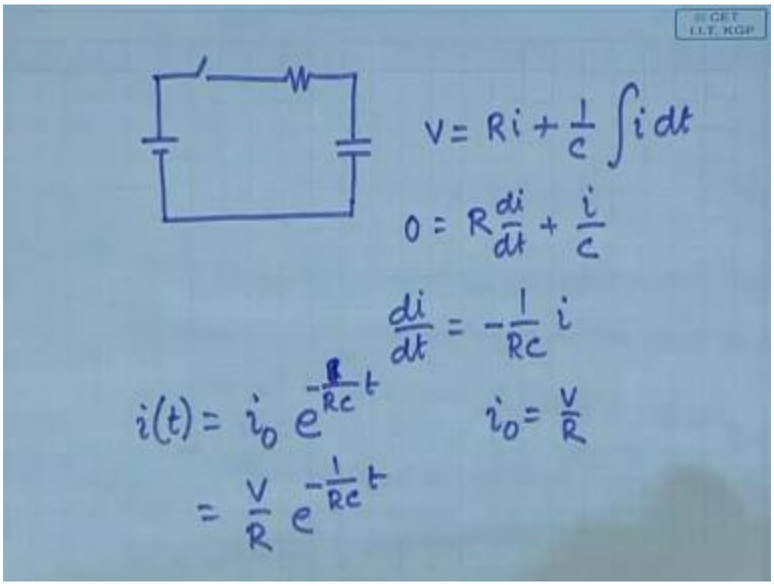
So, after time tau your i is the steady state value is V by R. So, I s t times where is it here times this becomes minus 1, 1 minus e to the power minus 1 this is comes to zero point. So, in some book that is why it is given, that the tau is a value after which the current reaches 0.63 times the final value. And this concept carries on to the capacitive circuits and all that.

How would you translate to mechanical systems, this is something that is generally dealt with in relationship to the electrical systems. How would you translate this to the mechanical systems a mechanical R L circuit means what?

Student: ((Refer Time: 19:27))

Mass and a damper. So, a system which is something like this a mass and then it is been acted on by a force, that starts acting at a certain time. So, it will keep moving can you see that, it will keep moving, but the speed will slowly increase and then will reach a certain value. And then, all that whatever you have learnt will be applicable in the same way.

(Refer Slide Time: 20:14)



In case of the R C network how will you do that in case of the R C network, again writing the equation first, it will be V is equal to R i plus how much is the current here 1 by C integral i d t how is the voltage here. So, you have you just differentiate it you get 0 is equal to R d i d t plus i by C, so the equation is d i d t is equal to minus 1 by R C times i. Now, in the same way if you obtain the equation you have obtained the solution, it will be i of t is equal to some initial condition.

Remember that it has already come in a x dot is equal to A X form without the additional term why, because we have obtained in terms of i. If you obtain in terms of voltage it will not come in that form, whatever it is. So, this is convenient, the solution is i naughtt e to the power minus 1 by R C t, what is i naught?

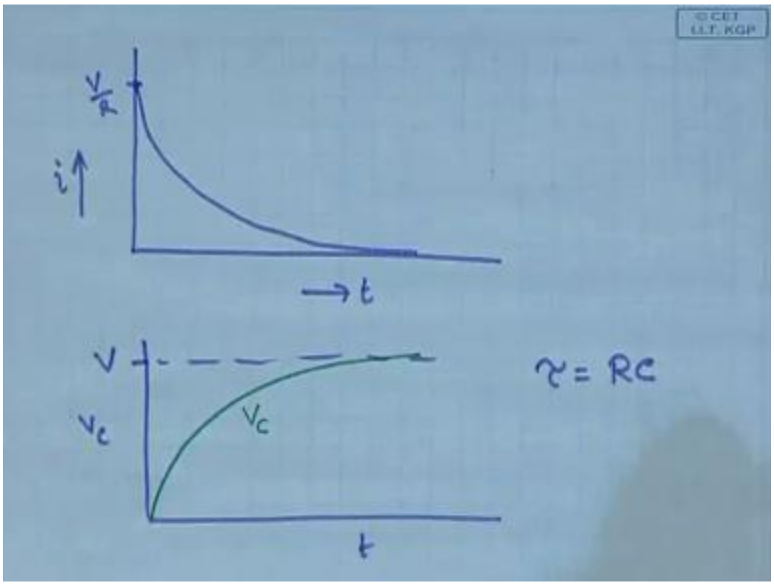
Student: ((Refer Time: 21:59))

Initial current.

Student: ((Refer Time: 22:01))

If there is no charge initially, then the voltage here is 0. So, the whole current passes through this V by R. So i naught is equal to V by R, so this is equal to V by R e to the power minus 1 by R C t. In that case what will be the wave form of this, what will be the wave form of this.

(Refer Slide Time: 22:33)



Obviously, it will be starting from. So, this is i, this is time, this is V by R, so exponential decay starting from V by R, ((Refer Time: 22:47)) what will be the wave form of say V R.

Student: ((Refer Time: 22:52))

Yes, V R will be proportional V C.

Student: ((Refer Time: 23:00))

Opposite increasing, so it will be let us let us draw the V C, V C will be will ultimately convert on to what value V C.

Student: ((Refer Time: 23:12))

[FL] how V.

Student: ((Refer Time: 23:16))

It will go to convert into V, so this is the current axis. So, we cannot draw it exactly on this it will be in a different graph t this is V C here is V. So, it will be, this is V C, the current graph is like this V R graph is similar, V C graph is like this and in all that will the time constant be different in the graphs or will be the same.

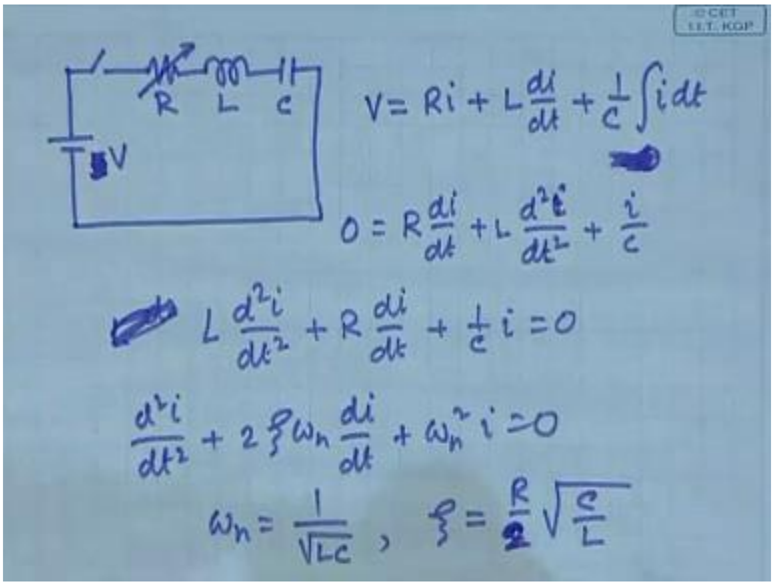
Student: ((Refer Time: 23:56))

Yes that is the point. So, a circuit will have the same time constant for all the different variables in question and in this case what is the time constant.

Student: ((Refer Time: 24:09))

R C, because the wave form is like this tau is minus t by tau, tau is R C. So, you have learnt how to solve this.

(Refer Slide Time: 24:32)



Now, how do you then apply this concept to R L C circuit you have got a switch R L C how will you apply this I would be calling V. So, again write down the equation V is equal to R i plus L d i d t plus 1 by C integral i d t is equal to 0 V is equal to, this integral is problematic. So, let us differentiate it through you have 0 is equal to R d i d t plus L d 2 i d t 2 plus i by C, so you have L d 2 i d t 2 plus R d i d t plus 1 by C i is equal to 0.

How will you obtain the solution of this, you have this is homogeneous, this is right hand side is 0, there is no forcing function appearing in this equation it will appear. But, when you write in form of i it does not appear, that is a advantage good. So, we do not want to change the coordinate, we would have to change the coordinate if you wrote the equation in terms of something, where the equilibrium point is not at origin, but in this case it is good.

How do you solve it, you already know that you will have to write it in the form d 2 i d t 2 plus twice zeta omega n d i d t plus omega n square i is equal to 0. The moment you write in this form, then these values will tell how the solution is good, so you write down what will be omega nth square?

Student: ((Refer Time: 27:08))

Yes, L will have to be divided through, because this has to be divide of this L. So, it is omega n is equal to 1 by root L C. If omega n is this and twice zeta omega n is R by L, then what is zeta R by 2 root over C by L check R by 2. So, this is how we are obtaining this and then we do not really need to solve it, because we already know the solution.

We simply decide in terms of these values. And you can easily see that the natural frequency of oscillation in the absence of zeta will depend on root over L C. And this zeta will be dependent on R L and C all of them, but you can see that zeta will increase for increasing R and increasing C by L ratio. So, if you have this parameters and say you have this as a rheostat and if you smoothly increase it from a small value to a large value then what will you see?

Student: ((Refer Time: 28:52))

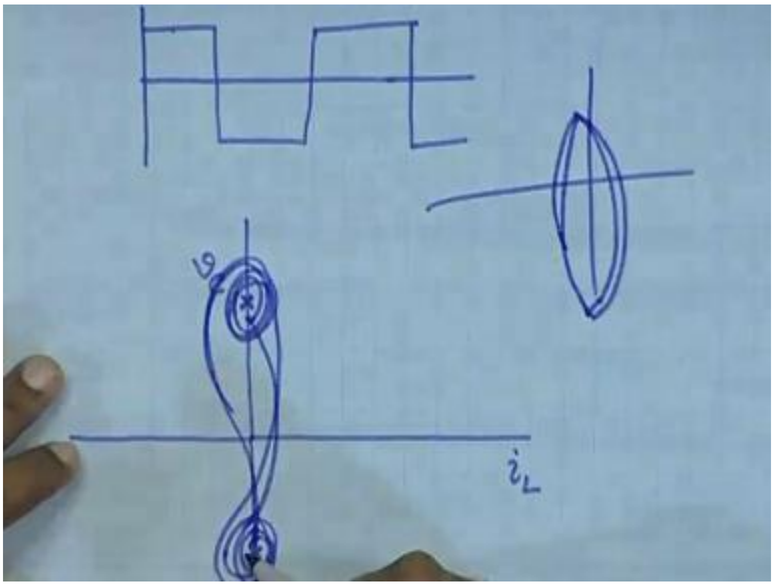
No, initially it will be under damped it will be oscillatory solution, because the Eigen values will be complex conjugate. And as you change it, it will slowly become firstly, a critically damped situation when this is equal to exactly one and then it will become over damped. When it is become over damped, you can still write down the time constant, you can at that time write down the time constant, because it will then be a exponential solution can you write down.

What will be the critical value of R at which the changeover will take place R critical, substitute this as 1 and then obtain it. So, when it is over damped what at that time will be the time constant, go back to the pages where we did this and you will be able to obtain it. Anyway, I leave it for you to do, but remember most of you often visualize the time constant only in relationship to R C network or a R L network.

But, when it is a R L C system then also there will be a time constant depending on when it is over damped system. And that also you should obtain get it, what is the equivalent mechanical system in this case RLC series circuit.

Student: ((Refer Time: 30:52))

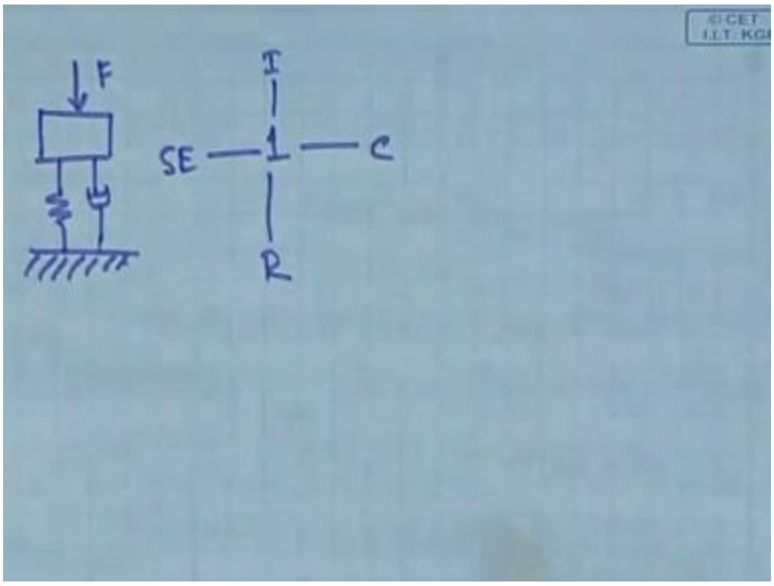
(Refer Slide Time: 30:56)



Yes mass spring damper we have already done that mass that is it with a force in. It will have the same bond graph, if you obtain the bond graph it is 1 S E, then I C R a series circuit. So, it is behavior this kind of a system’s behavior will also been determined by the same kind of dynamics. And depending on the value of the damping it will either be under damped or over damped, not only on the value of damping remember it also depends on this term.

So, if you keep the R constant and vary this; that means, you suppose slowly increase this spring constant. Then, also it will exhibit the same kind of behavior, yes I knew that you have done the time constant business earlier. But never the less it was necessary to place it in a general format.

(Refer Slide Time: 32:19)



Now, suppose you have got a square wave as the input voltage, then what happens, then remember if it is a R L C circuit the state space is two dimensional. The state space two dimensional with what are the two state variables, the current through the inductor and the voltage across the capacitor. And then, as you obtain the equations one thing I forgot to deal with, ((Refer Time: 33:00)) suppose you have a oscillatory solution or a over damped solution whatever it is, this is the solution of the i variable, what about the voltages, say V L or V C, what is V L?

Student: ((Refer Time: 33:20))

L d i d t, so you have to differentiate it and then this will give this value. Then, can you draw supposing this gives a over damped response, can you draw all these variables before I go into this I will have to just clarify the concept there, can you draw all the variables over damped. So, what happens to the current here, current starts with...

Student: ((Refer Time: 33:02))

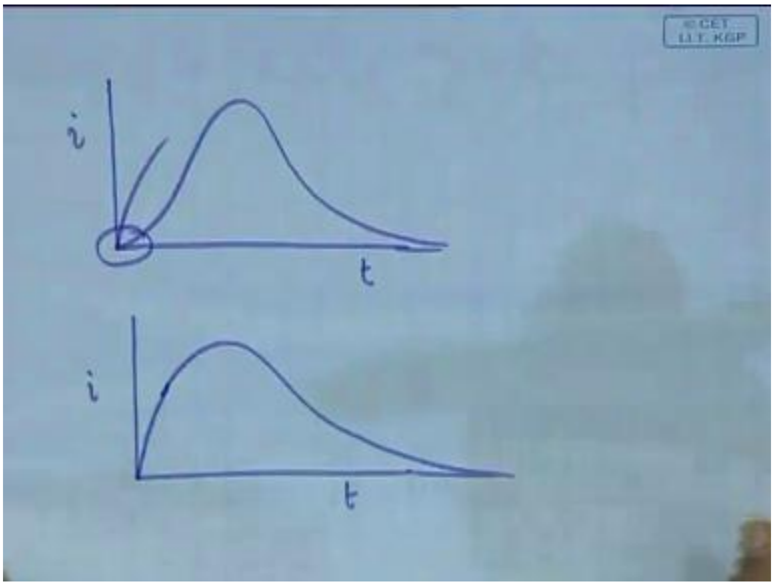
0 it is steady state value is...

Student: ((Refer Time: 34:06))

0, because the capacitor in steady state it blocks in starting condition this blocks. Therefore, the current starts with 0 and ends with 0, so when we are talking about the time constant it will notice this I was trying to get you at that.

Student: ((Refer Time: 33:30))

(Refer Slide Time: 34:30)



Yes, so it will be essentially a rise and then a decay, it will be rise and then decay i is t, what will happen to the voltages. Voltage across the R will be; obviously, proportional to the current, what will happen to the voltage across the L.

Student: ((Refer Time: 35:16))

You are talking about the voltage across L.

Student: ((Refer Time: 35:24))

You start from 0.

Student: ((Refer Time: 35:34))

Starts from where.

Student: ((Refer Time: 35:56))

Initially the current is 0 therefore, the voltage across this is 0, the capacitor voltage is 0, because there is no charge in it. So, it will start with V, so L d i d t is V, so d i d t is V by L, so d i d t initially starts from some value, which means I was somewhat surprised did not do protest this did not do protest this, the slope should not be 0 when it starts, it should start like this.

So, it should start like that at some slope which is V by L. And then it will go like this, yes now logically obtain the other waveforms. Initially the current slope d i d t you have decided they d 2 will start from some value V by L or the voltage across this is V and then it goes to...

Student: ((Refer Time: 37:34))

So, 0 ultimately when it becomes steady state.

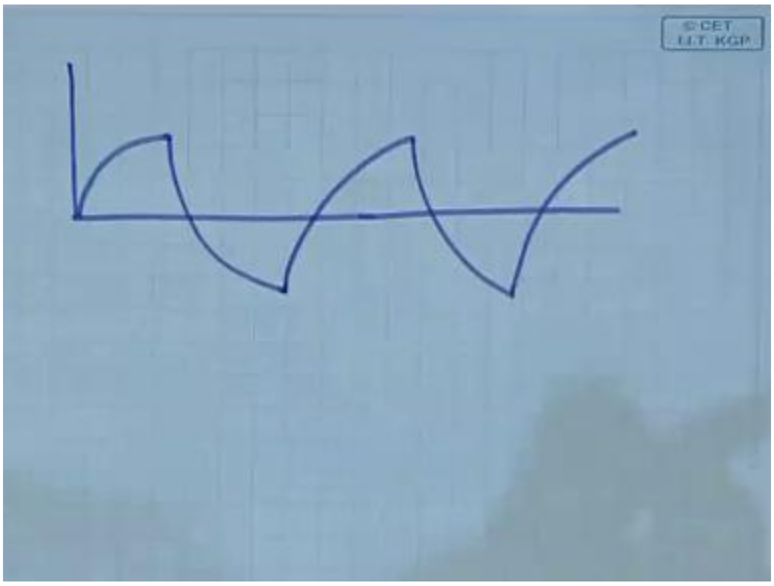
Student: ((Refer Time: 37:47))

The current is 0 slope has to be 0, if the current is 0 the slope has to be 0. So, there also the slope is 0, the current is 0. That means, it reaches like this, but from the positive side or the negative side does it go negative.

Student: ((Refer Time: 38:07))

Now, slowly you are trying to organize your thought, organize it, draw it and submit it to me. You have done R L C series circuits for how long I do not know, still I knew that this would give you trouble good, that let that be an assignment, draw it correctly and then submit it to me. ((Refer Time: 38:46)) Now, we were talking about what happens if you have the R L C circuit and you have or say may be R L circuit, simply R C circuit whatever and you have the forcing function as a square wave.

(Refer Slide Time: 39:05)



If it is a R C circuit, then what will be the waveform be the capacitor tries to charge to the applied voltage. But, then the applied voltage changes to a negative value it is tries to follow it again it tries to rise it tries to follow. How did I obtain without solving i, simply by understanding, that all the time it is trying to reach the equilibrium point and it has to do, so exponentially. So, all these will be exponential things, there is no other way of for it to do.

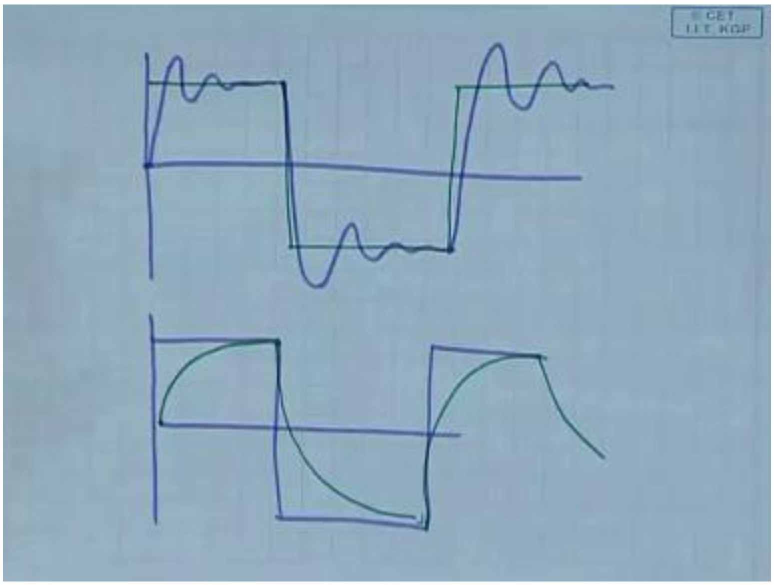
If it is a R L circuit, then what happens then the current tries to reach and the current wave form will be similar. So, let us talk about more important and visually beautiful thing, ((Refer Time: 40:03)) that what happens if it is a R L C circuit, in a R L C circuit if you apply a voltage like this, where is the R L C circuit equation it was here ((Refer Time: 40:18)) R L C circuit equation is this. So, the currents equilibrium point is at i is equal to 0, what about the voltage’s equilibrium point it is not and the state variables are this and the current is one state variable and V C is another state variable.

So, what is the equation for V C, equivalent equation for V C or in other words, the way we did it was if you have this equation. Then, you can write it in equivalent first order form in which I will be one variable and d i d t integral i d t, we see will be another state variable. So, you have two state variables out which one will be the V C, the other will be the i and therefore, V C you can easily see, if you consider V C what will be the equilibrium point for V C V, so it is different.

So, if it goes like this from plus V to minus V in the V C there will be one equilibrium point here and another equilibrium point here, can you see with i value 0 clear. So, once it will try to reach this particular equilibrium point, when it is here again before it reaches actually, because it actually reaches after infinite time, before it reaches. The equilibrium point jumps to this value this fellow and then it tries to reach here.

So, what will be the orbit like, first it will try to reach like this, say it is oscillatory solution, it will reach like this. And then, suddenly it will go there and it will try to reach like this again it will go will there it will try to reach like this, then it will go there try to reach like that there is time to reason. So, in the state space you will find that the equilibrium point is shifting and this fellow is trying to follow it, but before it can reach jumps back. So, it comes back and it goes around it, if it is a non oscillatory solution it will go straight like this, it will try to go straight. And then, it will try to come here and it will try to go straight and try to come here like that. If it is oscillatory solution it will be going around it for some time, but before it reaches it will have to jump back to the other.

(Refer Slide Time: 43:09)



In time domain what will be the behavior like, in time domain how will it look. Suppose, I am drawing the wave form for V C, then it will try to reach V C comes very close and then it jumps. Have you seen this waveform in your experiment on the C R O, on the R L C circuit.

Student: ((Refer Time: 43:57))

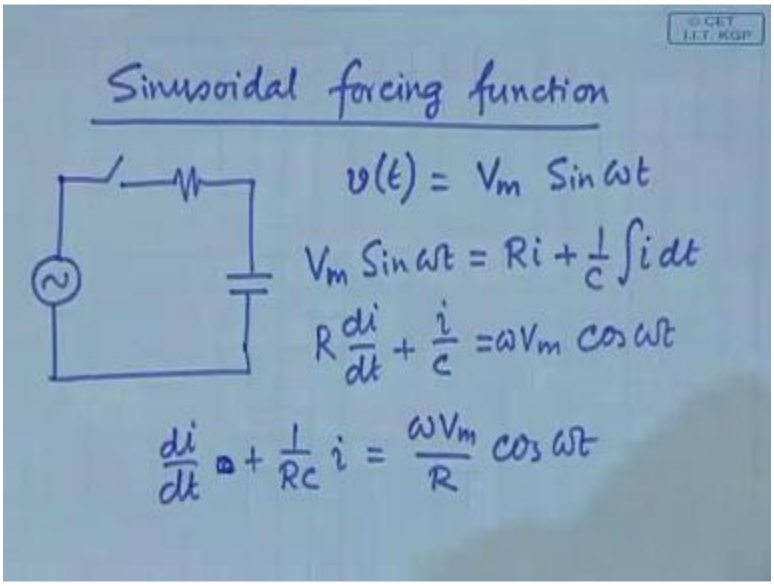
Yes you have done that, you have seen that, this could be either like this or it could also be like this, like that, this will happen in case of non oscillatory solution and this will happen in case of oscillatory solution. See, all these are coming in place the moment we look at it from the point of your without doing a blind solution of the differential equation. What are you doing, we understand that the equilibrium point is shifted, the way we showed a different way of solving the equation, we are actually conceptually doing that way.

It is approaching that and then the equilibrium point is moving. And now without even solving the differential equation I can talk about it, under which this will happen under which this will happen, that depends on the zeta and omega n. So, this is done, what other type of forcing function can you visualize, we have done a single step function, we have done a square wave function, what else could be...

Student: ((Refer Time: 45:17))

Sinusoid, let us take the sinusoidal forcing function.

(Refer Slide Time: 45:21)



Let us take the sinusoidal forcing function. Now, here in this case notice you have the equation in a written in the same way, if it is a first let us do with R L circuit, R C circuit and then go ahead to R L C circuit; otherwise it will be the difficult. But, never the less, what is happening is that, the sinusoidal forcing function will cause the equilibrium point to move continuously in the state space. And your actual state if the system is stable, then will try to follow it, it will never reach it will try to follow it around.

And so it is not difficult to see, that if it has the certain type of wave form the following thing will also have a similar type of wave form, it does it stand to reason. If I have got something moving and the other fellow is trying to follow it close the heels, they will have move in the same way more or less with some difference, it will not be exactly the same, but will have the same type of wave form.

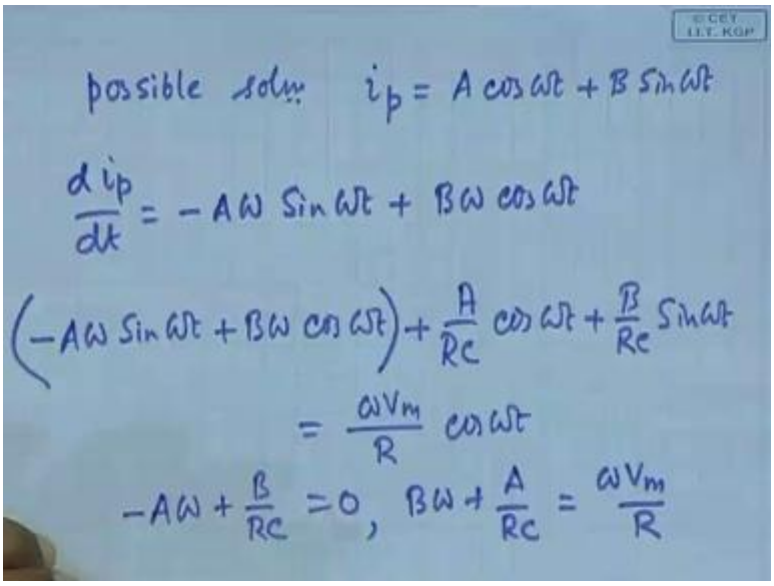
So, if the forcing function is the sinusoid, logically we expect the solution also to be a sinusoid. So, let us do it for today at least with the time that we will have remaining, let us do it for a R C circuit, we have a switch and then a R C network, write down the equation. Here, in this case V of t is some maximum voltage sin omega t, that is how we have applied the voltage.

So, the equation is this voltage is V m sin omega t is equal to R i plus 1 by C integral i d t this is uncomfortable. So, let us differentiate just write this in the left hand side, you have R d i d t plus i by C is equal to V m cos omega t. So, in order to write these without this R we divide both sides by R, we have d i d t is equal to or plus 1 by R C i is equal to did I write.

Student: ((Refer Time: 48:45))

Yeah, omega will come here yes omega will come here, you are differentiating this is R in the denominator. So, omega V m by R cos omega t, so this is the equation, the question is what will be the solution, the logic that I proposed was, that in this case the equilibrium point moves sinusoid ally and naturally the current will also follow it and you can expect a sinusoidal function as the solution, what is the general form of a sinusoidal function A sin omega t plus B cos omega t.

(Refer Slide Time: 49:38)



So, a possible solution is we will check whether it to be a possible solution is I will write it as i p equal to A cos omega t plus B sin omega t, why did I say that this is a general form of a sinusoid. Because, cos omega t has a 0 at pi no pi by 2, pi by 2, C by pi 2 and all that sin omega t has a 0 at 0 pi 2 pi and all that. So, there exactly 90 degree out of phase, their addition depending on the values of A and B can yield any possible sinusoid in between, that is why this is called the general form of the sinusoid.

Student: ((Refer Time: 50:34))

No, it will be lagging behind that. So, there will be a phase, but if something is going like this and the another fellow is trying to follow it, it will follow the same kind of, but this is a roundabout logic I am sort of hand waving to give that logic. But, we will have to mathematically check if it can be really solution, yes you are this right hand waving logic is intuitively good, but after all we have to prove that it can be a solution.

Student: ((Refer Time: 51:19))

No, no, that his point is that, if there is a equilibrium point you have to give some time for it to reach there, no it will never reaches there by the time it is start and by the time it goes this will has already moved. So, the equilibrium point is moving like that and the solution point, the state in the state space is following it. And while it follows there is no time for it to reach, because the that fellow is already moved that is moving all the time.

So, it never reaches, but what happens is that have you ever seen the flight of bees, one bee flies like it the other bee, follows it and then move in the same way. Talking about waveforms, they will have the same wave form with some difference we will talk about those. But, yes this is a hand weaving logic and we will have to prove it, so how to prove whether this can be a solution of this, just differentiate and substitute it here and if it can be obtained it is fine.

So, let us do that differentiate this and substitute here, so d i p d t is minus A omega sin omega t plus B omega cos omega t. So, I want it to be a solution of this equation, so we substitute it here, it is minus A omega sin omega t plus B omega cos omega t is this term. Then, plus 1 by R C times this it is 1 by R C times where is the solution yes here, so, 1 by R C times A cos omega t. So, A by R C cos omega t plus B by R C sin omega t is equal to, so a this I have done equal to omega V m by R cos omega t.

So, I can see that this equation has sinusoidal terms, this and that and cosine sinusoidal terms here, under what condition will the left hand side and the right hand side become equal.

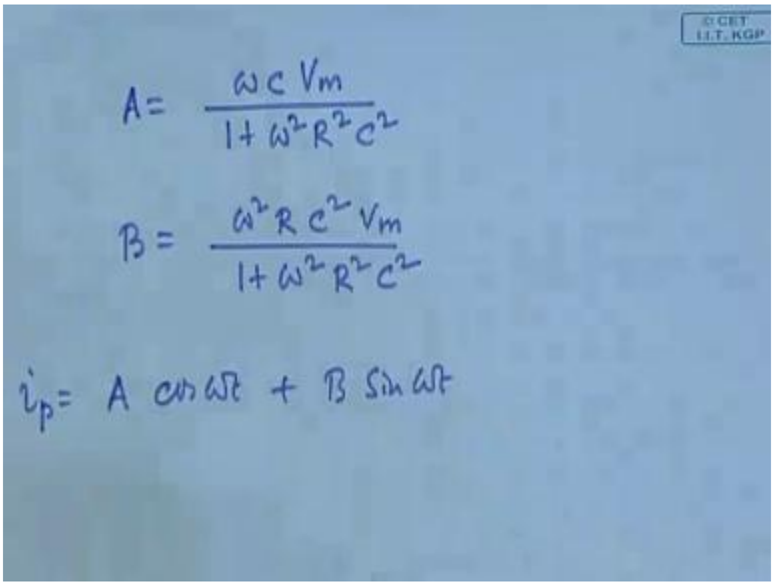
Student: ((Refer Time: 54:15))

Yes the coefficient must equalize. So, this coefficient plus this coefficient must equalize the right hand side coefficient of sin omega t which is 0. So, we can write minus A omega plus B by R C is equal to 0 and the coefficients of the cosine sinusoidal term this it is B omega plus A by R C is equal to omega V m by R.

Student: ((Refer Time: 55:01))

No, no, no the point is that is they are independent terms, because they are 90 degree out of phase, they are independent terms. That is, why you can write it this way, they will this equation can be is possible only if this is true in that sense. Now, you can see that from here, we can is a two equations two unknown A and B, we can extract the values of A and B which will satisfy this. That means, if you obtain the A and B from here and plug it in here, you will get an equation for i p which is a solution of this equation. So, from here just obtain the values of A and B.

(Refer Slide Time: 56:23)



They, will turn out to be just check A is equal to omega C V m by 1 plus omega square R square C square. And B is equal to omega square R C square V m by 1 plus omega square R square C square. So, your solution is i p is, this cos omega t, this sin omega t that's it, so A cos omega t plus, we will continue with this in the next class.

Thank you.