Dynamics of Physical Systems

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Lecture - 24
Vector Field around Equilibrium Points – III

In the last class, we were talking about the simple pendulum with some friction and we had proceeded to some extent, but there was a question asked at the end, regarding the signs. So, let us just have you continued with it have you completed the problem, no, let us complete it today.

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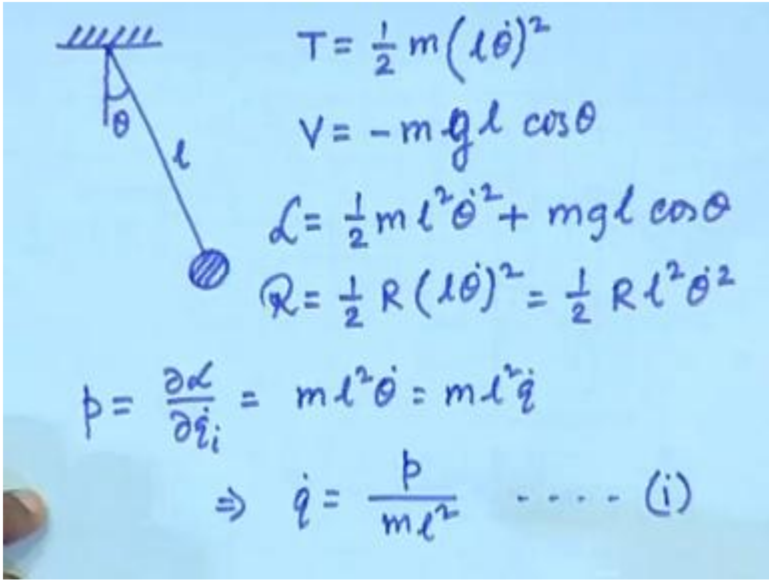

So, we are considering a simple pendulum with a normal support, your angle is theta. So, in terms of that, we had written the kinetic energy as half m, the linear velocity is l theta dot square. The potential energy is minus m g is l, l cos theta. So, the Lagrangian is half m l square theta dot square plus m g l cos theta. And your Rayleigh function is half R l theta dot square is equal to half R l square theta dot square.

In it is term, when you write the p, that is the momentum, p was the derivative of the Lagrangian with respect to in this case theta dot. So, it will be this one, m l square theta dot, so you notice that the momentum comes in terms of m l square theta dot; that is fine. So, this immediately gives, since we wish to write things in terms of p and q, let us say theta is q, so this is m l square q dot.

So, you are this immediately tells you q dot is p by m l square, so this is the first differential equation. The second differential equation has to be obtained from the Lagrangian form, which that is where the confusion was.

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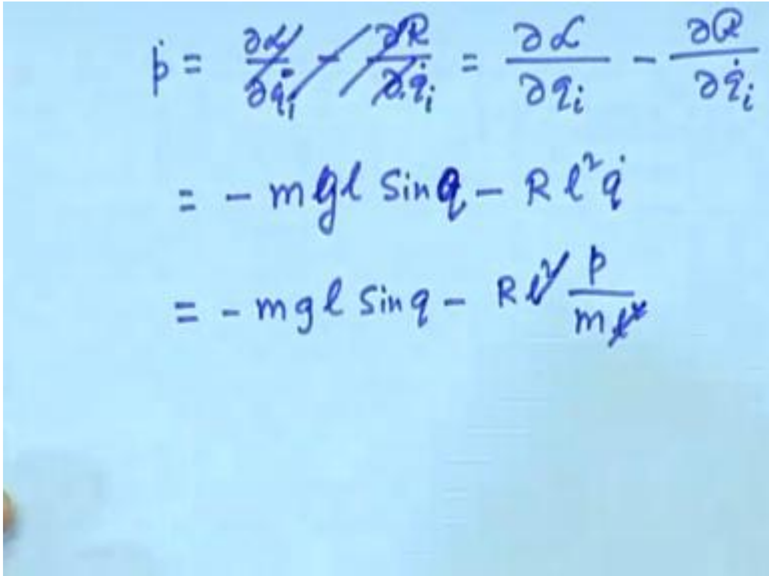

We have to write p dot is equal to, so what was the equation that we used; it was derivative of the Lagrangian with respect to q i dot minus derivative of the Rayleigh with respect to, so in this case also we have to do the same thing. So, the first one would be

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I will correct it, so that the error does not carry through, fine. So, when you evaluate this with respect to q, q i which is this is q, so it will be, this term will l is derivative with respect to q, this is q will be minus m g l sin theta. And the derivative of the Rayleigh term with respect to q i dot, which is R l square theta dot. So, minus R l square q dot sorry, this will be q, because we are writing things in terms of q and p. So, m g l sin q minus R l square q dot, fine.

In case of q dot, we have to substitute this in order to express the right hand side in terms of m g l sin q minus R l square q dot is p by m l square, fine. So, we have got an expression with q dot and expression with p dot. We would now like to express that, so let me write it completely, so that things are clear to start with.

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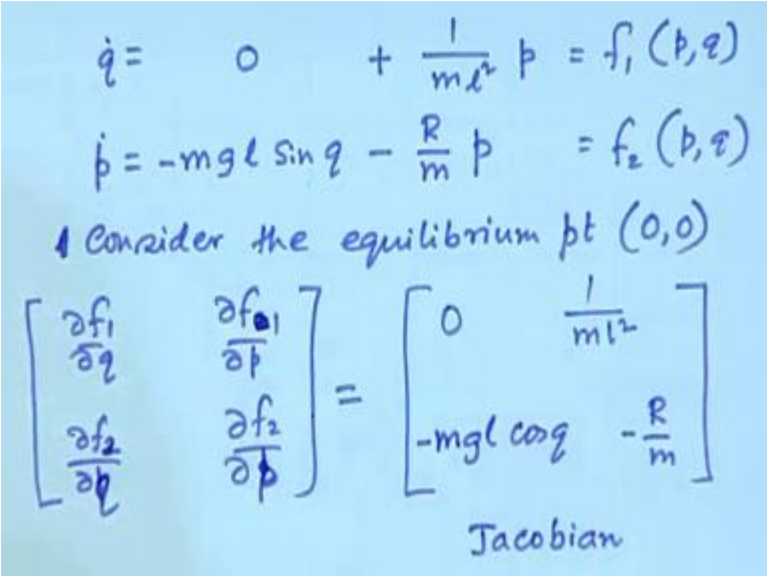

We have q dot is equal to, we are expressing in terms of something times q, which is 0 in this case, so I will write it here plus 1 by m l square p, this is 0 and p dot is equal to here minus m g l sin q minus R by m p, fine, so this term is 0. So, you had two equations, two differential equations and the next step as I told you is to locate the equilibrium point. In this case, what is the equilibrium point, you can easily see, that if you substitute q dot and p dot equal to 0, a solution is p is equal to 0 immediately from this.

And then, p is equal to 0 this will have infinitely many solutions, at 0 pi twice pi minus pi minus twice pi and all that, because it is sinusoidal. So, let us take one of them 0 0, so consider, remember there will be another equilibrium point at pi 0, there will be another equilibrium point at minus pi 0. And all that, we will consider them, but first let us consider this.

What is the next step? Next step is to locally linearize this equation; it is a non-linear equation, because of the sin. We have to locally linearize, around that equilibrium point and for that we will have to take the Jacobian. So, the Jacobian is, so this f 1 of p and q, that is what I have written, this is f 2 of p and q, that is why, I wrote it this way. So, this is doe f 1, doe, the first one is q, doe f 2, sorry q and p.

Now, what we will get here, doe f 1, doe q is 0, doe f 1, doe p is half 1 by m l square, doe f 2 with respect to q, you have and this one is minus R by m, so that is the Jacobian matrix. Now, this Jacobian we will have to be applied in the neighborhood of an equilibrium point, which is this. So, the moment you apply q is equal to 0, you have this term unity, so this is only minus m g l.

So, you have an overall set of differential equations an overall vector field and we are trying to locate, what is happening, what is a vector field around this equilibrium point and for that we have substituted.

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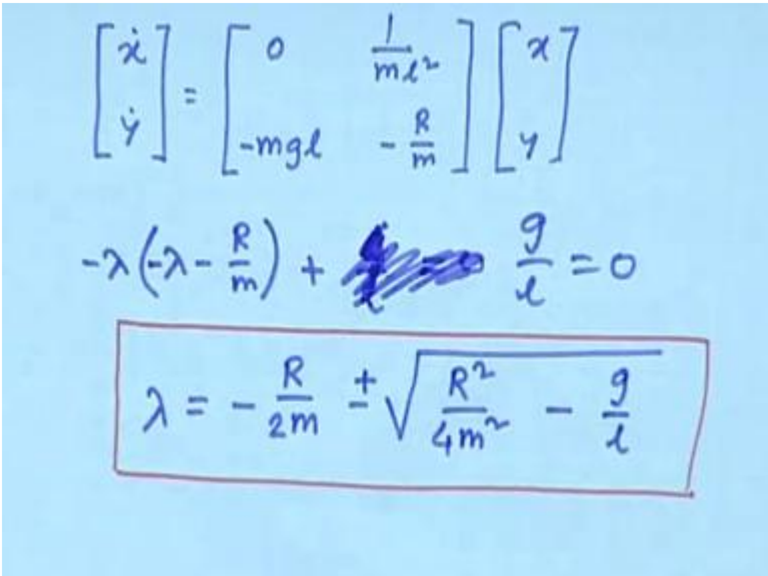

And in that neighborhood, what is the equation, it will be now we are writing it as x and y, because this are the local coordinates x dot y dot is equal to what is it, 0, 1 by m l square minus m g l and minus R by m x y. Who was asking the question, the last time about the signs, yes have you understood now. So, this is the equation in the local neighborhood good.

So, what is the next step, to try to solve it and for that we have to obtain the Eigen values and Eigen vectors, so obtain the Eigen values of this. Now, do it yourself, obtain the Eigen values of this matrix, it will be minus lambda times lambda minus R by m plus m cancels of this and so g by sorry, l be…

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This is a minus, no, here, so this is g by l, so that is equal to 0, I will write it clearly g by l is equal to 0. So, what are the Eigen values of this, lambda is equal to it will be minus R by 2, that will have to be, I will put it here plus minus root over, what is there in the root be…

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R square by 4 m square minus l, so these are the Eigen values of the system. Let us see, what does they imply, first; if there is no air friction, what does it tell, if there is no air friction, this term is 0 and the square root term is only minus g by l minus. And therefore, it will be purely imaginary Eigen values, if you have purely imaginary Eigen values what will be the behavior like in the state space circles, circle behavior means oscillation. So, it will be a normal oscillation of the undecaying oscillation of the pendulum. So, a normal oscillation of the pendulum, where theta goes from the positive side to the negative side and theta dot also does so... That is represented as a circle in the state space, clear.

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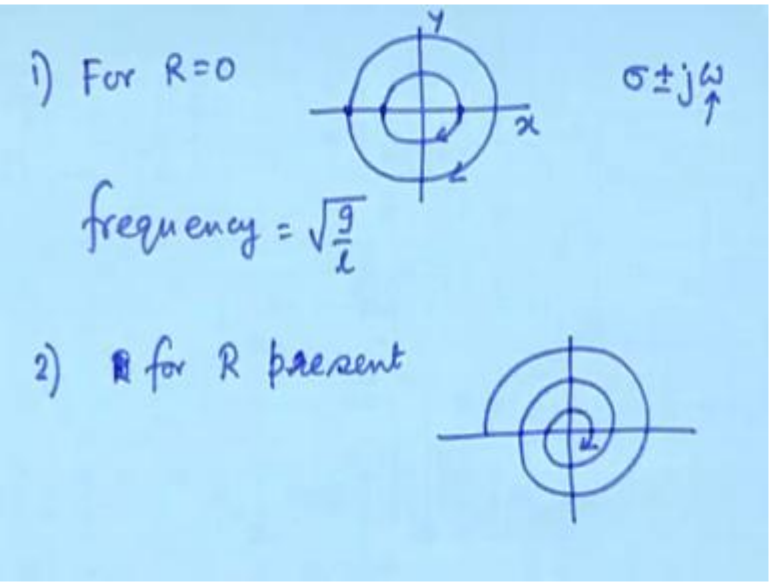

So, for R is equal to 0, your state space representation will be, no I would not immediately say right, because I had put arrows, how do you know, it is in this direction. Now, that needs to be obtained from this equation really, how you say that I have taken some positive value of x and no, y as 0, which means x dot should be 0, 0 times x plus 0 times something times 0 is 0, so there is no component of x dot in this direction.

What about the other one y dot, y dot is minus something times a positive quantity, which is a negative quantity, minus this times 0, which will be negative quantity, negative quantity should be in this direction, so this is right. So, these arrows are correct, but you should convince yourself that these arrows are correct, because they could have been opposite also. So, the moment I something does not immediately accept, first question check and then accept, that is one thing, that is I am something happy about.

What is the frequency of oscillation? Frequency of oscillation is where we express this at as sigma plus minus j omega, this omega is the frequency and omega in this case would be just g by l root over g by l. So, frequency not only that, for this one and for this one, the frequency would be the same, so if you start from here, the time it takes to come back here is the same as that you start from here and it go around it and come back here. Frequency is the same for all of them, that is one of Gallous classical observations and that follows from these, this way that.

Secondly, if there is air friction, what will happen, yes if you consider air friction, this term will be there and this term is a negative term. So, it will be sigma plus j omega, that sigma is a negative term, so e to the power sigma t times is a sinusoidal term. So, that sinusoidal term will decay and it will be a spiraling inward orbit. So, for R present, it will be, still there is a question of frequency, because starting from here, it takes some time to come back here, starting from here, it takes some time to come back here.

So, what is the frequency now, yes now the frequency is changed, it is not the same thing and this changed by this thing and if R is small it is change is also reasonably small, can you see that. So, the frequency will almost be same root over g by l, but change slightly by the extend of this. What is the damping factor? Damping factor is the sigma, which is this, so depending on the value of R, it will decay.

The other thing is that, important thing is that the frequency of oscillation does not depend on the amplitude of oscillation, here of this. So, all this follow, simply if you follow, whatever we have talk from the first class to here, because we have talk to how to obtain the differential equation, but the Lagrangian method. How to obtain the first order, how to obtain the Jacobian, how to solve the differential equations, that is what ultimately follows, so just to give you some idea about numbers.

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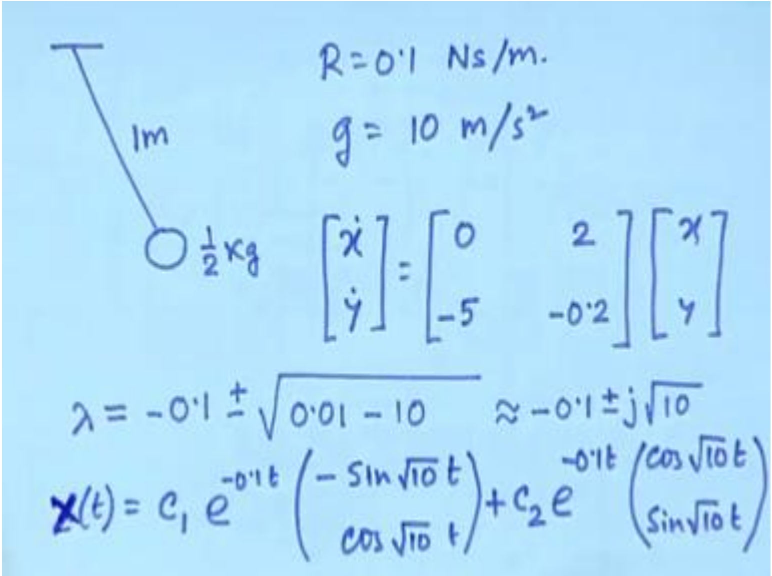

Consider this equation just as do this, you consider a very standard pendulum, what could be a nice weight of a ball, say half a kilo, this will half could be a nice length of A 1 meter and what could be nice value of R, 0.1. I am not trying to give you something for which you have use the calculator, so R is equal to 0.1, what is the unit, Newton seconds per meter. Is there any measures needed g, g is 9.81 and you will have to use the calculator. So, take it as 10, let g be 10, what is the unit.

Now, do this calculation, firstly what are these, what will be the equation be x dot y dot would be equal to something times x y. This is obviously 0 and 2nd term is 2, 3rd term is minus 5 and the 4th term is minus r by m, minus 0.2, nicely guess. So, what will be the Eigen values, Eigen values would be minus R by twice m, which is 0.1 minus 0.1 plus minus root over g by l is 10 and the other term is

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So, 0.01 minus 10, can easily see that this is negligible in ((Refer Time: 22:25)) 10, so we will ignore it. So, this is approximately equal to minus 0.1 plus minus j root over 10 correct. So, if you have this as the Eigen values, what is the solution of the system, you already know that. So, solution of the system x of t will be in this case theta will be c 1 the first solution, plus c 2 the second solution.

So, what will it be, you already know…

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C 1 e to the power minus 0.1 t times, yes.

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Let us is as the vector x t, so that it is x and y, this will be something, something. What was that, just open that page minus sin omega t, which is root over 10 t cos root over 10 t plus c 2 e to the power minus 0.1 t, this is cos root over 10 t and sin root 10 t, that is the solution. So, when you on investigate about theta, which is in this case x, first 1, you will take the first line and when you on to investigate the velocity, angular velocity theta dot, which is y, you will investigate next line, this capital X, yes, this is the vector X.

So, you can see there is a sinusoid with frequency root over 10 and here is damping term with magnitude minus 0.1, so this is how we normally solved. But, that is true for only the equilibrium point that is at 0 at 0. What happens at the other equilibrium point, immediately the moment you try to do it, you would realize that it is different, do that just once.

Where was it here, this was the equation, this was the Jacobian matrix in which we are substituted 0 0. What is the other equilibrium point, pi 0, pi 0 at q is equal to pi and you also get a p dot is equal to 0, so pi 0. If this is pi, what you have minus 1, so this becomes positive, this become positive means, let us go back straight to the example, that we have doing, where is the example values here. This value becomes plus, because then it will be easy for you to see this become, this value becomes plus, this value becomes plus means plus 10.

What is it imply, no more complex Eigen values, real Eigen values, what else do you infer from here, there will be one Eigen value, which is 0.1 plus root over 10. Another Eigen value 0 point 1 minus root over 10 roots, over 10 is how much; 3 point something, minus 0.1 is negligibly small in comparison to it. So, we can one is plus 0 point something other is minus 3 point something and other is minus 3 point something.

What is the character of that equilibrium point, it is several, because there is one Eigen value which is positive, another Eigen value which is negative. So, if we see the same system as simple as a simple pendulum harbors equilibrium points of different characters, what would that mean, I will come that to latter. But, the point is that for every equilibrium point, you have to calculate it and you cannot assume for that another equilibrium point, the same equation will hold, the local linearization might be different. So, we have understood, what happens if there are real Eigen values, which are different, which are unequal Eigen values? We have also understood, what happens if the complex conjugate Eigen values and pure imaginary Eigen values.

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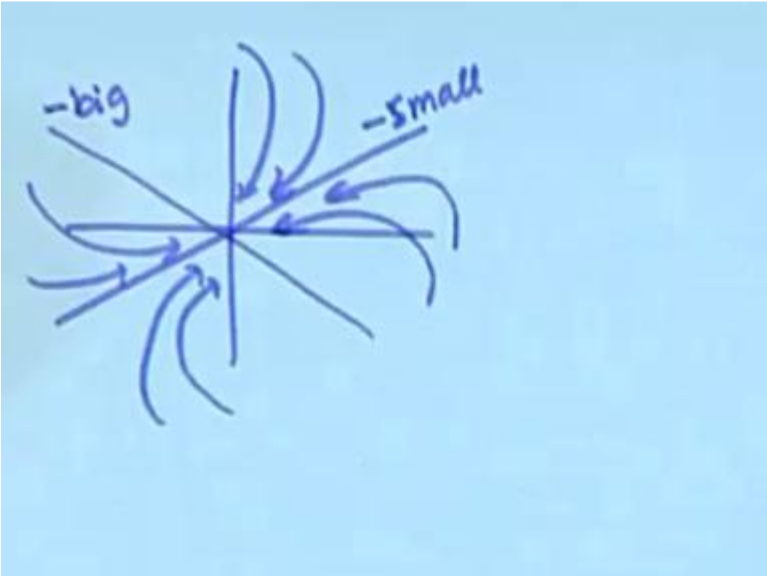

In the first case, we have understood that, it will be like this, under which condition this will happen, when there are two Eigen vectors and the decay rate along this direction is larger than the decay rate along this direction, which means e to the power something t. That Eigen value should be minus a larger quantity minus big and this one will have minus small, the two Eigen values.

Is it possible to have, let me ask the question the other way, this will be true, when there are two Eigen values negative, but they are different, what if they are equal, what may be the angle, how do you know, it is 45 degree, but he was asking a question be..

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Normally, here one Eigen value, since you have first writing the Eigen value and for that Eigen value, you are extracting the Eigen vector. If you have one Eigen value, you will be able to extract one Eigen vector; you will not get two Eigen vectors. So, one Eigen vector only, so in that situation, you will have one Eigen vector only.

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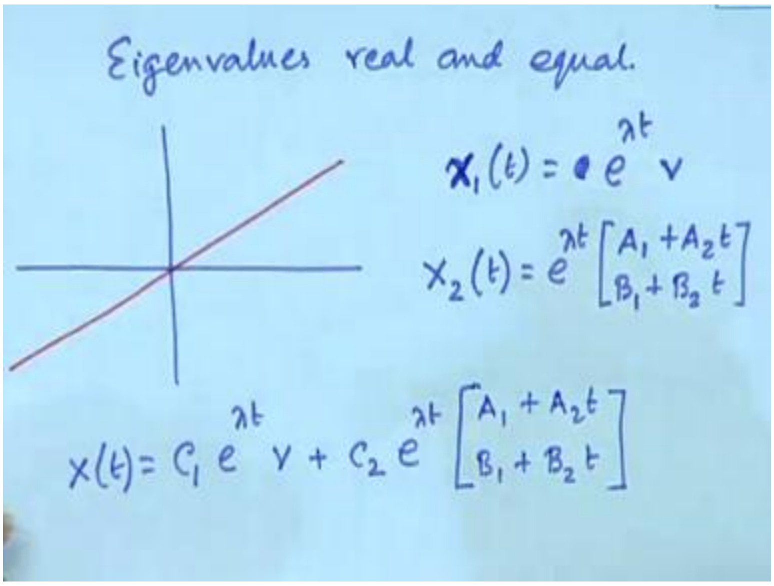

So, Eigen values real and equal, first point is that you will get for one Eigen value, you will one Eigen vector, so imagine that is something like this. The other Eigen vector, we have not found the whole idea of writing in terms of Eigen values and Eigen vector was that I wanted two independent solutions. And in terms of that, I am composing a general solution, but here I have problem, I got only one.

So, what is to be done in that case, what is to be done is, I am not going into theorems and all that, one looks for another solution of this form. The first one would be X 1 of t; it is X 1 of t is one solution is e to the power lambda t times, the v, the Eigen vector that is true. When a compose, then c 1 will come, when you compose and write down the general thing, then c 1 will come. At this stage c 1 will not come, c 1, this one is one of the general equation.

The second one, one looks for of this form X 2 of t is equal to it will be e to the power lambda t all, but another thing of the following form is that will be A 1 plus A 2 t and B 1 plus B 2 t. So, it is a time dependent term, but there is a proof, that you always getting a solution of this form, which means general solution X t will be then c 1 e to the power lambda t times v plus c 2 e to the power lambda t times this thing, A 1 plus A 2 t and B 1 plus B 2 t. Let us do one problem and then this will be clearer.

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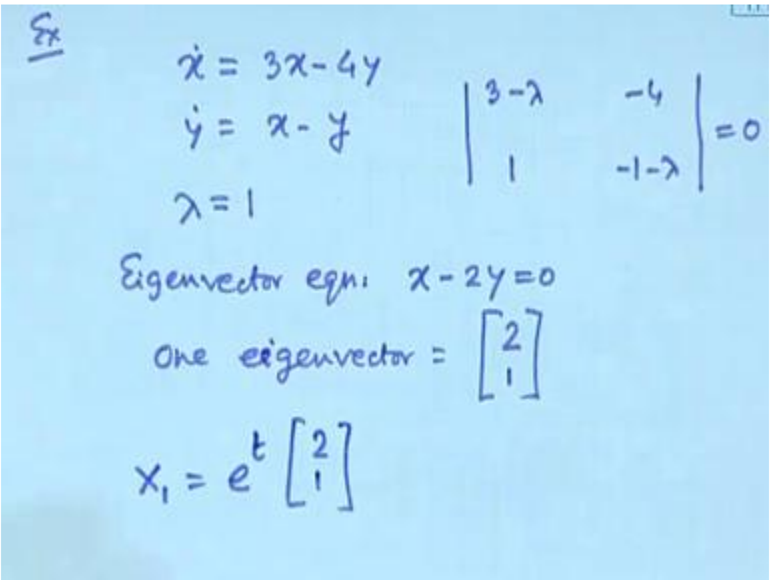

Let us take an example system, where your x dot, this is example, x dot is equal to thrice x minus 4 y and y dot is equal to x minus y, obtain the Eigen values, the matrix is 3 minus 4, 1 minus 1. So, it would be lambda plus 3 minus 4, 1 minus lambda and you have to obtain it 2, see it carefully, yes. In this case, the lambda will come out to be 1, for both, it will in both cases turn out to be the same; have you done that convinced yourself.

What you get, you get 3 minus lambda minus 4, 1 minus 1, minus lambda and then you obtain it, got it, For this, what is the Eigen vector, well you cannot say V is equal to 1 2 we can say that there, I have identified the direction and indicator is not a Eigen vector. So, the moment you say 1 2, I will say why not 2 4, so yes it could be, no, be careful

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Well, I would tell me the Eigen value equation. For this one, Eigen vector equation, the equation is x minus twice y is equal to 0. Yes, in that case one solution, one possible Eigen vector would be 2 1, if it is 2 1, then it is 0, nice let us take it. So, one Eigen vector good, so the X 1 solution is e to the power 1 t, e to the power t 2 1, that is no problem. Let us trivial, let us try to work out the next one. In that case, the solution is of this form, this solution must satisfy this equation. So, let us let us do this.

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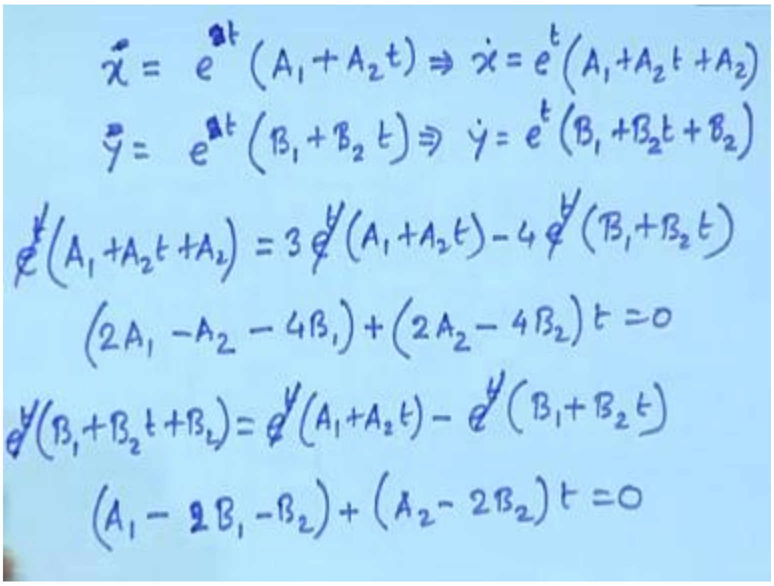

We have, if you use this, then you have x dot is equal to e to the power lambda t, in this case 1, A 1 plus A 2 t and y dot is equal to e to the power lambda t B 1 plus B 2 t. No, this is x, sorry this is x and y, this gives x dot is equal to this lambda is 1. So, I can easily eliminate it, what is x dot e to the power t this will remain, because e to the power will differentiate as e to the power t. So, it is A 1 plus A 2 t plus e to the power t A 2 plus A 2 and this implies y dot is, do it e to the power t B 1 plus B 2 t plus B 2. Now, that should sub, that should satisfy this equation.

So, we would write e to the power t we can see cancels off. Anyway, we will write x dot is e to the power t A 1 plus A 2 t plus A 2, this is x dot is equal to thrice x, which is 3 e to the power t A 1 plus A 2 t minus 4 y, minus 4, y is this e to the power t B 1 plus B 2 t. Similarly, this gives what, this gives cancels off and ultimately you get from here, can you just simplify this, can you simplify this, it will be A 1 here thrice A 1.

So, bring it here, it is twice A 1, here it is A 2 here, so minus A 2, then is there any term divide of t, yes, minus 4 B, minus 4 B 1. So, this is the t independent term, plus what are the t dependent term, here is the A 2, we are taken to this side, it is here thrice A 2 t minus A 2 t is twice A 2, then you have yes minus 4 B 2 t is equal to 0. Similarly, from this equation y dot, y dot is this, so e to the power t B 1 plus B 2 t plus B 2 is equal to it should be y dot is x minus y x minus y.

So, you have e to the power t A 1 plus A 2 t minus y is e to the power t B 1 plus B 2 t. This cancels off, now club together the t dependent and t independent terms, what you have mean, then A 1 minus twice B 1 minus B 2 plus. Then the t dependent term A 2 minus twice B 2 t is equal to 0. So, we have these two equations and we intent to find out A 1, A 2, B 1, B 2 from there, can you find, try.

See the line of logic is that, these two equations must hold independent of t, which means the individual terms must be equal. So, this term will be 0, this term will be 0, this term will be 0, this term will be 0 as been, else you do not get it 0, independent of t. So, write this equal to 0, this equal to 0, this equal to 0, this equal to 0 and then try to extract result.

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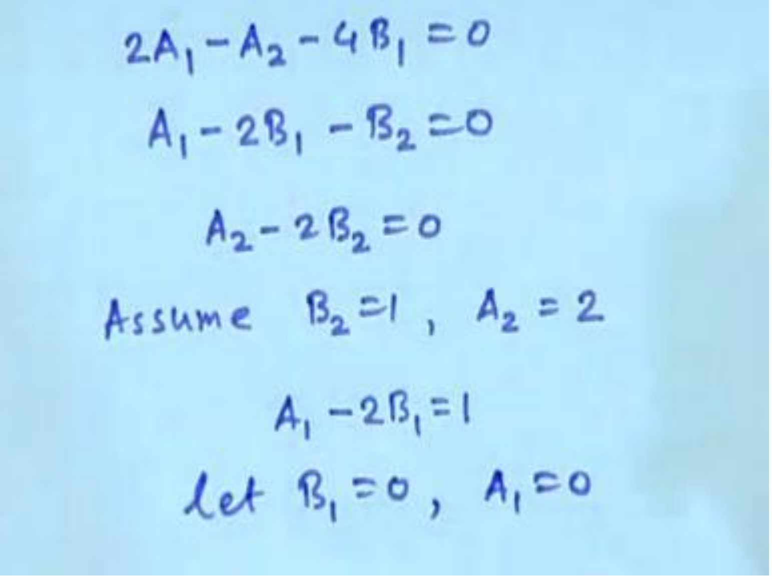

So, we have equations like twice A 1 minus A 2 minus 4 B 1 is equal to 0, first equation. Second equation, I will write this one, A 1 minus twice B 1 minus B 2 is equal to 0. Next two are the same, yes, there is the point, next you are the same, but that is not a problem, because you are trying to find out any A and Bs, which will satisfy this equation; that is it.

So, now the next equation is A 2 minus twice B 2 is equal to 0. This equation, I will write down and this equation, this equation does the same thing, so I have there is no point writing down. So, we have three equations, no, ultimately we have three equations, from this three equations, we are trying to identify, how many are really determinable from here, how many are determinable.

See if I multiply two with this equation and subtract from here, this cancels off and this cancels off, you have left with A 2 and B 2, A 2 and B 2 check, it gives a same equations, so we are needed to assume 1. So, out of that A 2 and B 2, we need to assume 1, what if we assume, it will be simplest and then so let us assume B 2 is equal to 1 and A 2 is equal to fine.

Now, substitute here, can you determine the other things now, no, you cannot do it, yes. So, out of which equation, if you substitute you have A 1, this you have substituted B 2 is substituted A 1 minus twice B 1 is equal to 1. Now, here you need to substitute something, so what is the simplest, let B 1 equal to 0 A 1 equal to 1. If you put B 1 as 0, let B 1 equal to 0, then that gives A 1 is equal to 1.

See, after all these four things satisfy the equations, so long as it satisfies, we are fine. So, have we have assume, so what that is what we do, whenever we are trying to identify an Eigen vector. We assume one thing and then we get the other same thing, so we have obtained it.

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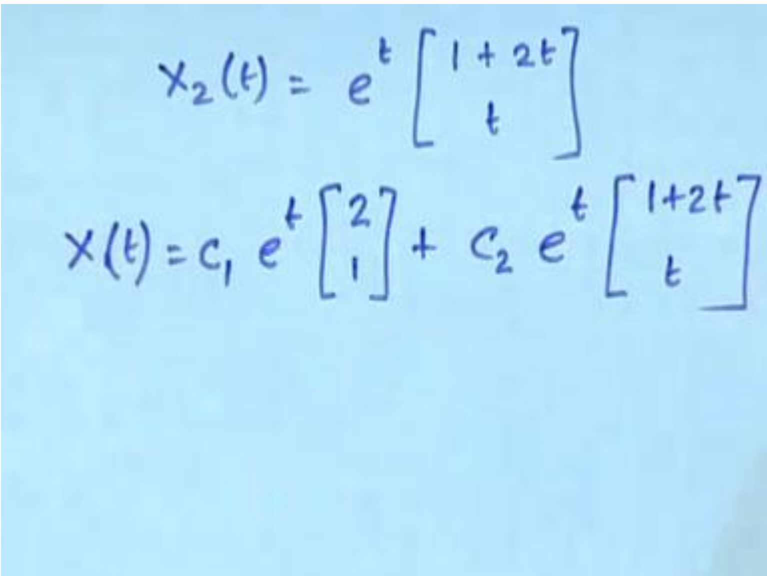

Then, we can write down the equation, finally as the X 2 t will be e to the power lambda t, which is e to the power 1 t times A 1 plus A 2 t, A 1 is 1 plus twice t and this is B 1 plus B 2 t, which is t. You are the total solution is X of t is c 1 e to the power t, what was the first one, 2 1 no 2 1 plus c 2 e to the power t 1 plus twice t; that is the solution. From here, it is not easy to visualize the vector field, so let us do that exercise.

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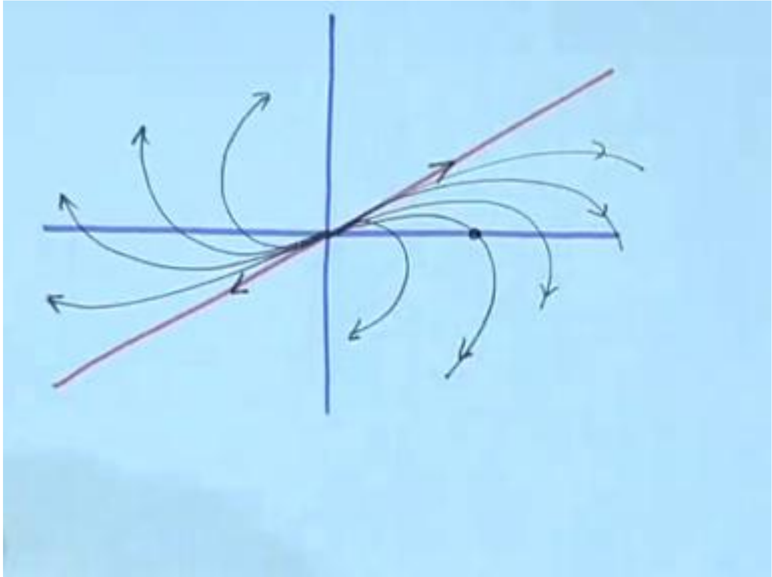

What will the vector field would like? The vector field will have one Eigen vector not 2, say this is the Eigen vector and what is this term, Eigen value is plus 1. So, if it is plus 1, then these are outgoing directions, it will go out along the direction, but there is no other one. So, how do you visualize, what will happen in the other places, what will happen in the other places is something like this, it will it will move in one direction like this and here like this.

So, you can see around this, all these things actually emerge along that direction, that Eigen direction, but after that, they change. How would we know this, how do you know this, there are two ways of knowing it. First, look at the equation, there is one term, which is dependent on what we already know. But, there is another term, where the vector is dependent on time; vector itself is dependent on time. So, it will be independent thing, so as it goes out the vector will change like this.

The other way of looking at it is that, since you do not have another direction. Try in whatever possible way of drawing it, this is the only way, ultimately you will be able to draw. So, that all the vectors emanate go out in this direction, this is the only way, you can do it. Try, any other possible way, what is necessary is that if you give a slight part of version from here along this direction, it will move along that that direction.

Slight part of version in other way, it will first move along that and then it will go out, the part of version in the other direction will slowly build up. That slowly building up will take it away from here, but it should always, all the curves will be tangent to this line at the equilibrium point and this is the only way you can write. Try drawing it your way, any possible way, you will ultimately see that this is the only way you can draw it.

What you can you try it in something else be…

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Mirror image, yes you could, wait, that is exactly where, I was expecting this question and there are two possible ways.

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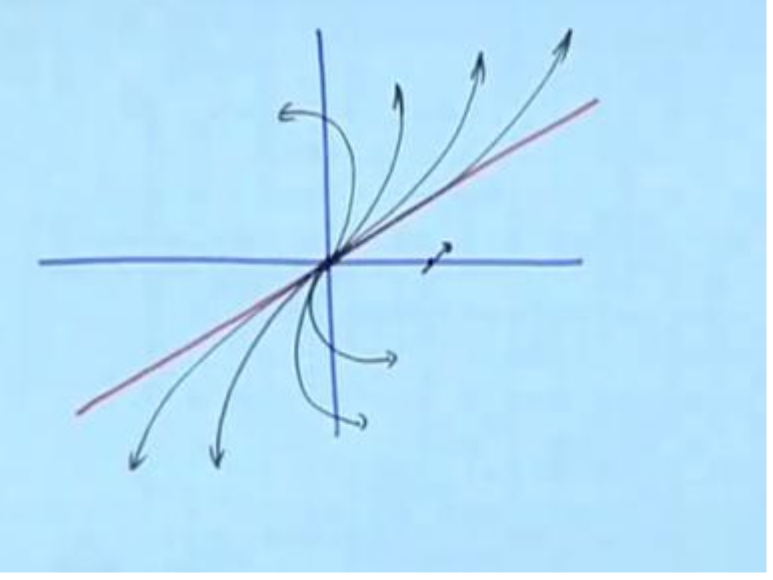

So, there are two possibilities, she has correctly pointed it out. Well, when we draw the circles, there are two possibilities, either it could go this way or could go that way, how do it we take the decision. On the basis of the direction of the vector at a very convenient chosen point mostly at some x value with y 0 or some y value with x 0. Here also, there are two possibilities out of that, which is true, will have to decide on the basis of the same consideration.

So, how will you do take a point on the x axis, y 0 and then substitute it in the original equation, where is it in that So, what do you get as the direction of the vector, both are increasing, if x and y both are increasing that at this point, it should be like this, which is possible only if this is true. That will be the vector field, there are two possibilities out of which we are chosen because of that consideration, now let us summarize what we have learned.

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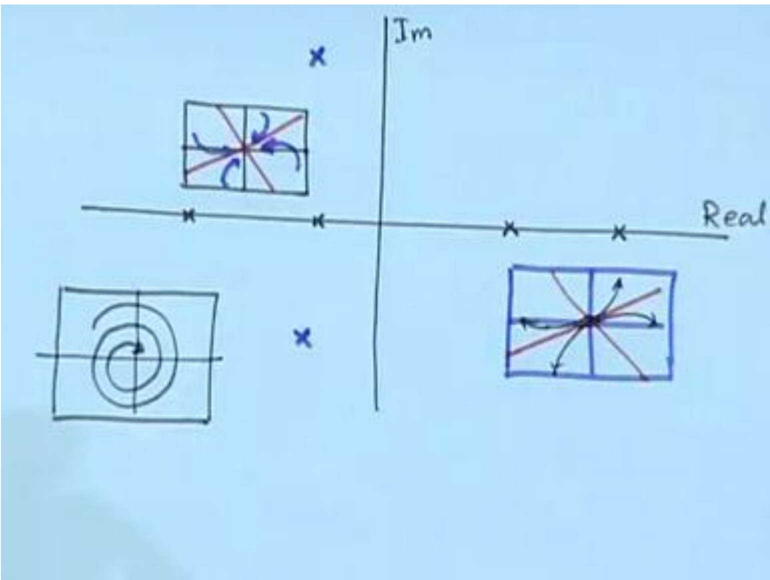

In a system, you can have the Eigen values, Eigen values will have positions, supposing the Eigen values are like this. What is the behavior like; in that case the behavior is like this is the real axis, this imaginary axis and stable. So, for this, I will just make a box and draw, there will be Eigen directions and the vectors will be incoming. For this, if the Eigen values are here, it will be opposite of that, so it will be fine.

If the Eigen values are, say here, what will be the behavior like, it should be able to visualize and that is what I am telling. It should be able to visualize, then the behavior will be invert, because the real part is negative. If it is here, spiral outward, nice, no problem, but suppose you have a system in which you are changing the some parameter and the Eigen values move as this.

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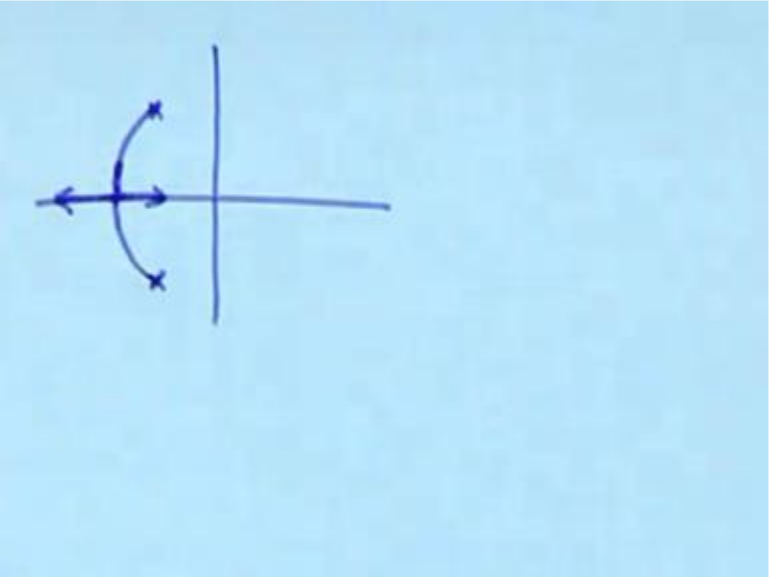

Suppose, you have the Eigen values like this and then as you change the parameter, they move towards each other and at some point, they collide. And then, they move away from each other or the opposite. Initially, they were here as you change the parameter they come closer and collide and then move out.

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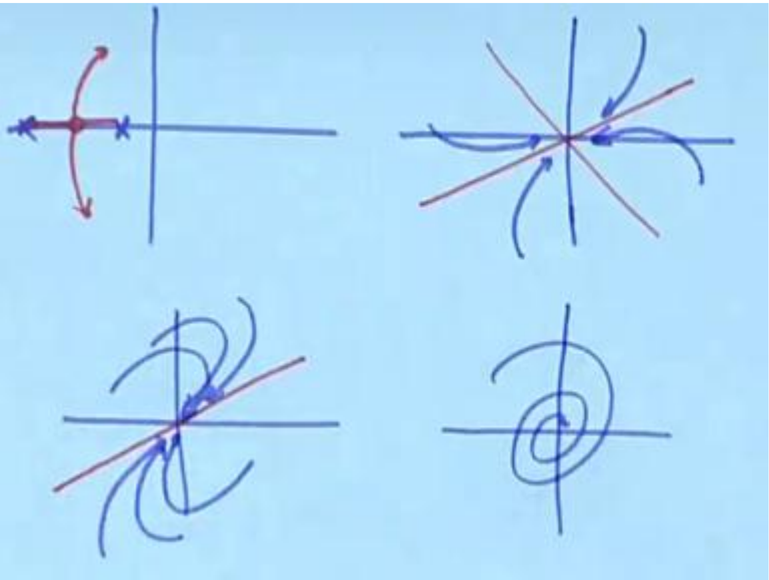

Consider that it will be easier to visualize that starting from the real thing, it was like this and then they move like this. At this point, they collide and then move out like this, can you visualize, how the vector field will be changing, how draw these three situations, one after the other, draw these three situations. First 1, 2 Eigen vectors and then negative real part, negative real Eigen values, so you have like this.

What is happening as you move closer, what is happening with the Eigen vectors, this Eigen vector will move an ultimately coincide with this. That it does not just vanish, as you change the parameter as the Eigen values change, this will change and at some point, it will coincide with the Eigen other Eigen vector. Then, at that point, it will become, suppose this one as moved and as coincide with this only this remains and it was like this, so it would be, can you see that, it is almost on the verge of turning spiral. This is the critical situation, where it is about to turn spiral and then as it exceeds the real line, it becomes can you visualize, how the vector field is changing smoothly. That is end of it today, we will continue in the next class.