Dynamics of Physical Systems
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Lecture - 23
Vector Field Around Equilibrium Points – II

In the last class, somebody asked me the question about the point at which be ended. We said that there are two ways of approaching the problem when you have complex conjugate eigenvalues. One was if you just say that the two eigenvalues would be complex conjugate, therefore they two eigenvectors will also complex conjugate. And as the result of which the two eigenvalues and eigenvectors do not contain independent information, they have the same information. So, you release one of them, the other is to say they no, we will go the same way as we did earlier, we will have the two eigenvalues, two eigenvectors.

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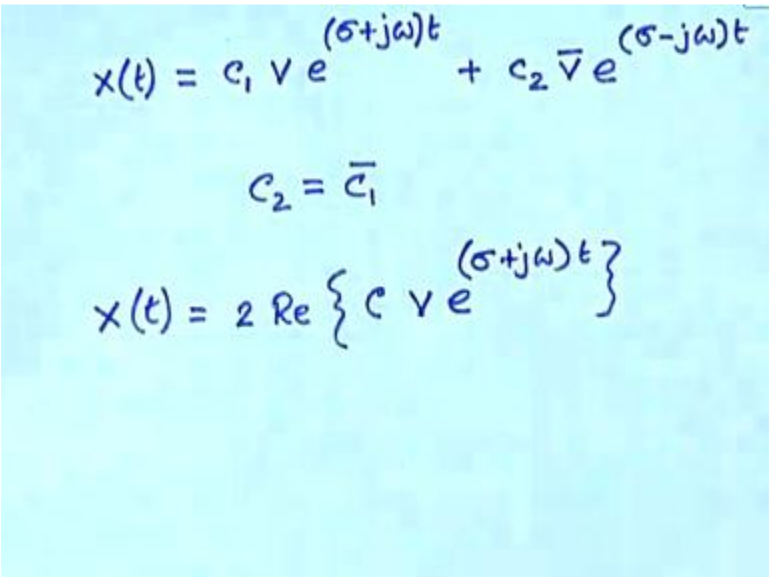

And will write the equation X t as c 1 V e to the power it will be sigma plus j omega t plus c 2. This will be the conjugate of V, so I will write it like this e to the power sigma minus j omega t. Now, then we say that this right hand side, in general will be complex, but the left hand side must be real, which is true only if the c 1 and c 2 are also complex conjugate of each other. So, c 2 actually is c 1 bar now when that is so, then the two real parts are the same, the imaginary parts one in the positive direction one in the negative direction.

So, if you want to extract the real value, then it is X t the imaginary parts will cancel each other. So, you have twice of the real part that is why we said twice. Twice of the real part of c which is both c 1 and c 1, V e to the power sigma plus j omega t. So, that was the logic, but here if you actually do it you will ultimately get the same result as you did with the other approach. So, let us illustrate this other approach with one example.

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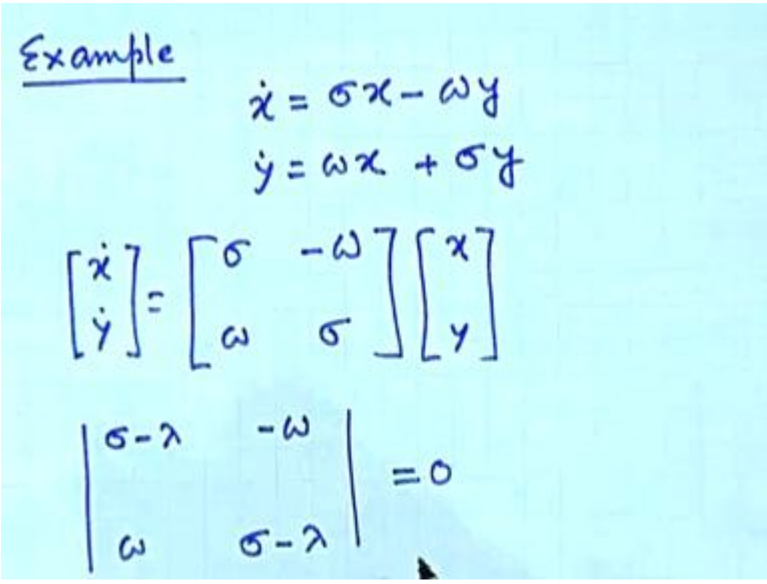

So, let us take an example to clarify the point, consider a system x dot is equal to sigma x minus omega y and y dot is equal to omega x plus sigma y. This omega and sigma could be any number, but we have just chosen one. The reason will immediately be clear, because what will be the next step, you say that the matrix equation is x dot y dot is equal to sigma minus omega omega sigma x y. And then, we will proceed to obtain the eigenvalues of this, yes do that. It will be obtained from the determinant of sigma minus lambda minus omega omega sigma minus lambda equal to 0, which will give, can you see yes.

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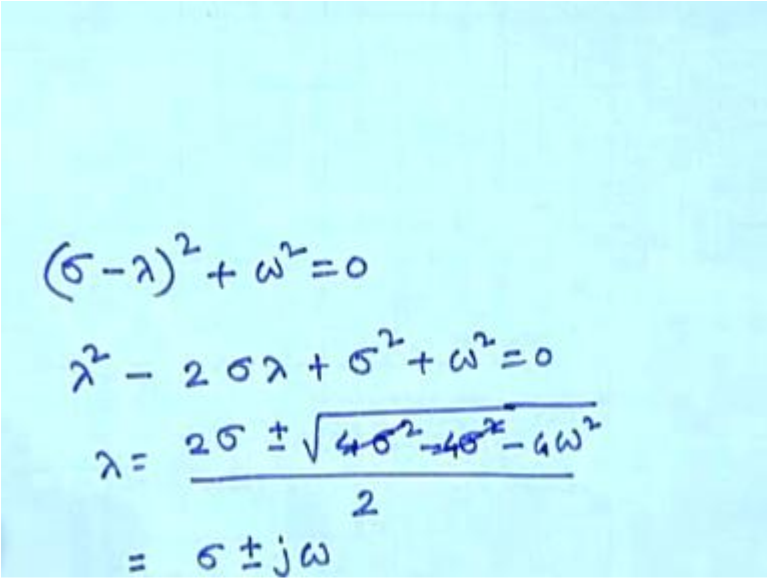

Sigma minus lambda square plus omega square is equal to 0, so if expand it you have lambda square minus twice sigma lambda plus sigma square plus omega square is equal to 0. Now, you obtain the solution lambda is equal to twice sigma plus minus root over 4 sigma square minus sigma square minus 4 omega square by 2, this cancels of you have left with sigma plus minus j omega.

That is why we chose that equation actually, because this is a nice complex conjugate eigenvalues. So, you do not really need to have this ((Refer Time: 06:12)) kind of symmetric equation, but whatever equation you start from, ultimately you will end of with a pair of complex conjugate eigenvalues that look like this, then we will go ahead. So, you have obtain this obviously, there are two eigenvalues sigma plus j omega and sigma minus j omega. And as we have say the sigma plus j omega are contain the information sigma and omega, sigma minus j omega also contain the same information sigma and omega. So, we will not consider them separately which start with only one sigma plus j omega.

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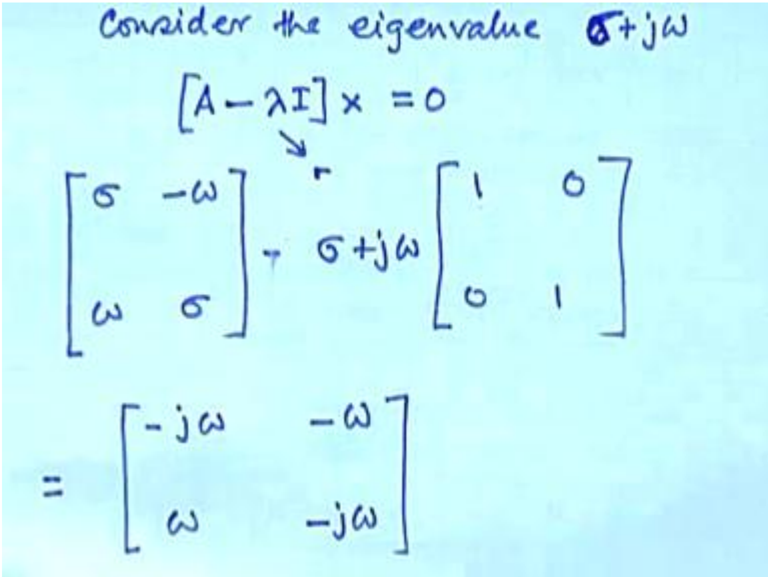

So, consider the eigenvalue sigma plus j omega. What will be the corresponding eigenvector, the eigenvector will be obtained from the equation A.

Student: ((Refer Time: 07:13))

Sigma plus j omega, it will be sigma plus j omega, it will be A minus lambda I times X is equal to 0 that will be the eigenvector equation. So, write down the eigenvector equation for this particular eigenvalue, where lambda is sigma plus j omega do that your A was this. So, I will write them separately sigma minus omega omega sigma A minus lambda I would be, I will write this separately.

Sigma plus j omega times 1 0 0 1, this is the A minus lambda I, so this will be equal to do that once, do the algebra once.

Student: ((Refer Time: 08:30))

Sigma minus sigma it will be 1, so it will be minus j omega here, here it is minus omega, here it is omega and here it is minus j omega. So, that is A minus lambda I.

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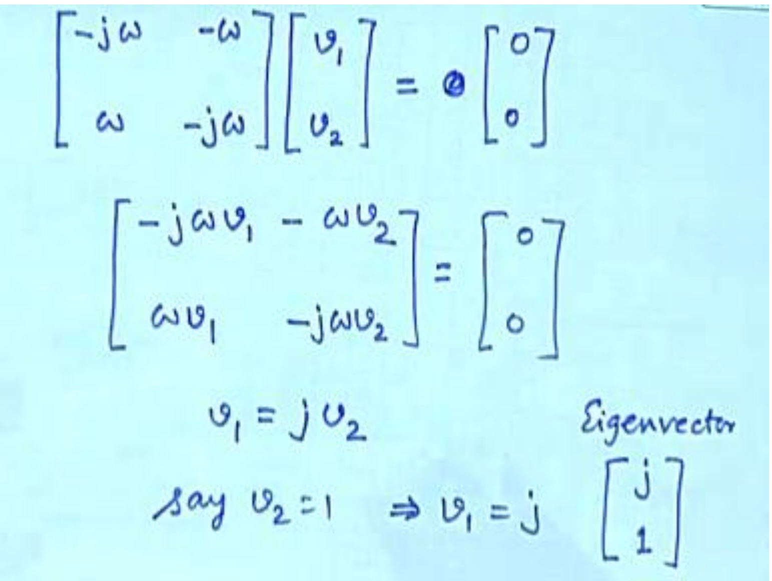

So, now the eigenvector equation would be then, A minus lambda I is j omega minus omega omega minus j omega times X essentially. So, it will be say V 1 and V 2 two components, it is equal to 0. 0 means of course, 0 vector. Now, notice this will give minus j omega V 1 minus omega V 2 omega V 1 minus j omega V 2 is equal to 0 0. What does this equation say, this minus this equal to 0, take one to this side you get V 1 is equal to what times V 2, j times V 2.

So, that is the Eigen direction, that is why it is a complex Eigen direction, V 1 is equal to j times V 2. But, that is a Eigen direction, you need to a vector, as I said it will not be unit vector you have to choose one point and then, on the basis the other will be given. Say V 2 is equal to 1 that gives V 1 equal to j, so the V 1 V 2 Eigen vector will be is j and 1, that is how you obtain the eigenvector. So, we have obtained one eigenvector which is this.

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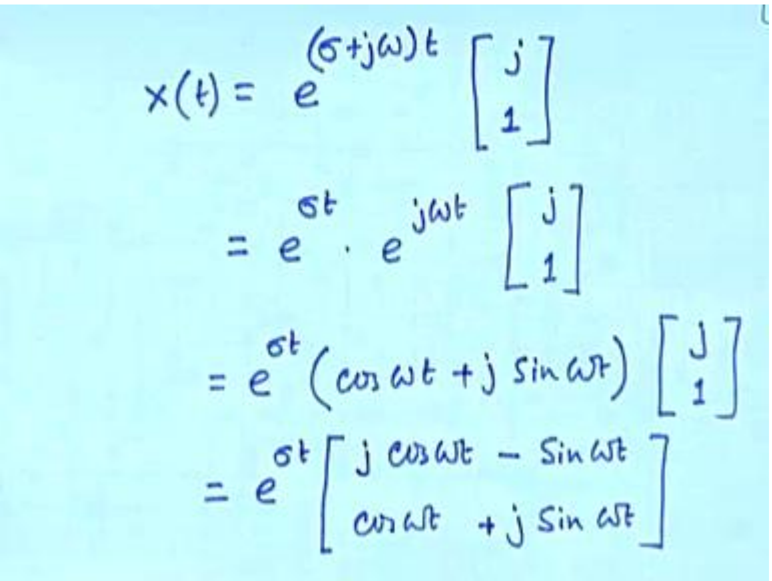

And therefore, the corresponding solution will be X of t will be e to the power the eigenvalue, sigma plus j omega t times the eigenvector which is j and 1. That will be definitely be a solution, because this is the eigenvalue and this is the eigenvector. Now, break it up, you have e to the power sigma t times e to the power j omega t times j and 1, now e to the power j omega t Euler equation.

Student: Yes sir

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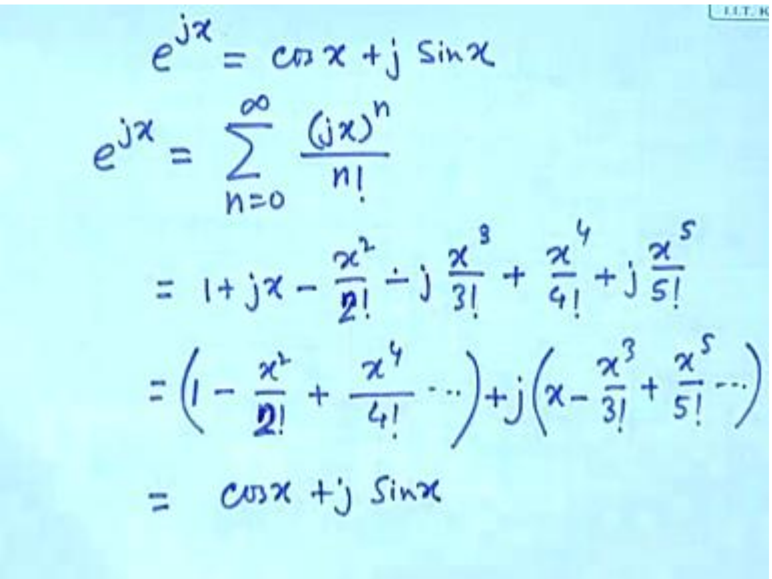

Yes, Euler equation is e to the power j say x is equal to cos x plus j sin x, so we will substitute here, by the way do you know how to obtain it, how Euler proved it?

Student: ((Refer Time: 12:54))

Yes, you have to expand it just expand this one, so e to the power j x when expanded will give 0 to infinity j x to the power n divide by n factorial, which is if you write it in expanded way 1 plus j x minus x square by n factorial minus j x cube by three factorial plus x 4 by 4 factorial and so on, so forth. Plus j x 5 by 5 factorial, then clubbed then together, club some of them together, so we will first put together 1, this, this, this the even terms.

1 minus x square by n factorial

Student: ((Refer Time: 14:12))

Two factorial, plus x 4 by 4 factorial and so on, so forth. We will put in one bracket plus will take the j out this the common factor is j. So, you have here x minus this x cube by 3 factorial plus x 5 by 5 factorial and so on so forth. This term is and this term is you know, so this is equal to cos x plus j sin x, it is a remarkable result, all the time used in engineering. And when Euler proved it, nobody knew that it will ever have any application in engineering.

So, where are we were here we will simply substitute it here, it will be e to the power sigma t times cos, it will be cos j omega t cos...

Student: ((Refer Time: 15:26))

T minus j

Student: ((Refer Time: 15:28))

Yes, you can do that, I showed that you can always do that no problem. There are two eigenvalues, you can write the equation with two eigenvalues, but there are actually two approaches. That approach uses both the eigenvalues, but actually two eigenvalues do not contain independent information. That is why we choose to have use one eigenvalue and I show that it is possible to extract to independent solutions out of that only, because there is a real part, there is a imaginary part.

Real part and the imaginary part independently contain a information, that is the point. So, you have plus j sin omega t and it is j 1.

Student: ((Refer Time: 16:18))

Plus, so our job is now to separate out the real part and the imaginary part which will proceed. E to the power sigma t, now we will multiply this what you have j cos omega t minus sin omega t and here it is cos omega t minus plus j sin omega t, now can you see it yes.

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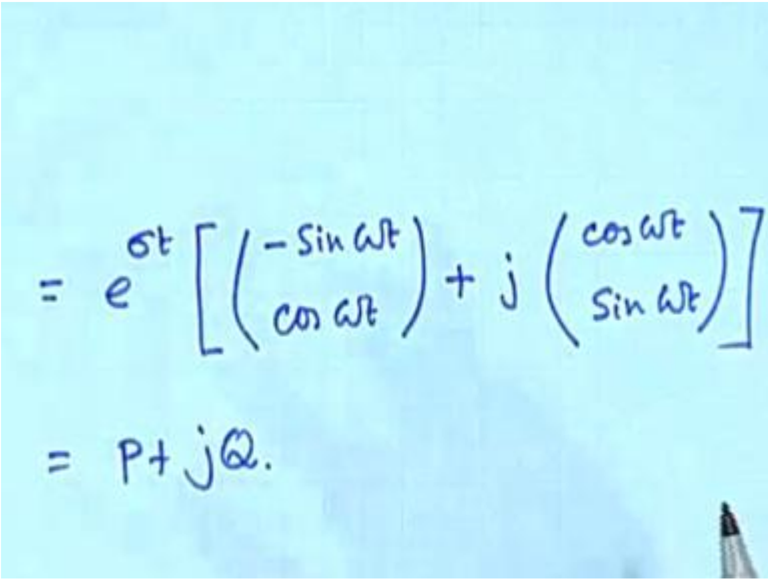

Now, this is equal to e to the power sigma t remains. And then, we will first write the real part which is minus sin omega t and here cos omega t plus j cos omega t sin omega t. So, we have separated out the two parts. Now, you see there are two components of it, one is this, the other is e to the power sigma t times this. So, it is actually P plus j Q and then, the argument was that I gave in the last class was that this is P and Q must both the solution, because this is nothing but a linear combination of the two components, multiplying by j is also linear combination. Therefore, these two must independently be solutions and these two are all real, so this must be a solution and that must be a solution.

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$$e^{\sigma t}\begin{pmatrix} -\sin \omega t \\ \cos \omega t \end{pmatrix}, \quad e^{\sigma t}\begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}$$

$$X(t) = c_1 e^{\sigma t}\begin{pmatrix} -\sin \omega t \\ \cos \omega t \end{pmatrix} + c_2 e^{\sigma t}\begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}$$

$$x(t) = -c_1 e^{\sigma t} \sin \omega t + c_2 e^{\sigma t} \cos \omega t$$

$$y(t) = c_1 e^{\sigma t} \cos \omega t + c_2 e^{\sigma t} \sin \omega t$$

So, we have then the two solutions as, one solution is e to the power sigma t times minus sin omega t cos omega t, that is one solution. The other solution is e to the power sigma t cos omega t sin omega t, good. If these two are solutions, then the final solution, the general solution X t will be c 1 this plus c 2 that, e to the power sigma t minus sin omega t cos omega t plus c 2 e to the power sigma t cos omega t sin omega t.

So, that is the solution what does it mean, it means that the individual solution x t can you write down, this is having two components x t is c 1 it is minus c 1 e to the power sigma t sin omega t plus c 2 e to the power sigma t cos omega t. And this is c 1 e to the power sigma t cos omega t plus c 2 e to the power sigma t sin omega t, we have got the solutions happy, I am not happy yet.

Why, because am never happy with solutions are differential equation till I have got by this by differentiating that. So, if you differentiate this you should get back the original equation just check that, I always just do at the exams. Whenever I have a solution of differential equations, ultimately I will have to differentiate it and get back the original equation if it is true, I am fine. So, differentiate this and check if you get the original equation which is, where did I keep it, here I should get back this ((Refer Time: 21:13)).

Check that once, this would be anyway you can do that, I should not bother about this trivial thing, but you should always check convince yourself and then, only go ahead got it. So, if you differentiate these two, you should get back these equations if you done or trusting me, do not trust me. Important thing is that you should not trust the teacher, you should always do it yourself, you are not doing, you have done it. So, we have the solution as this now what does the solution look like really, in order to see it will do a bit of algebra with this. Is it visible, yes so say we start with this equation y t is this, then y t will be written as this is equal to c 1 plus c 2. So, if I multiply and divide by c 1 square plus c 2 square.

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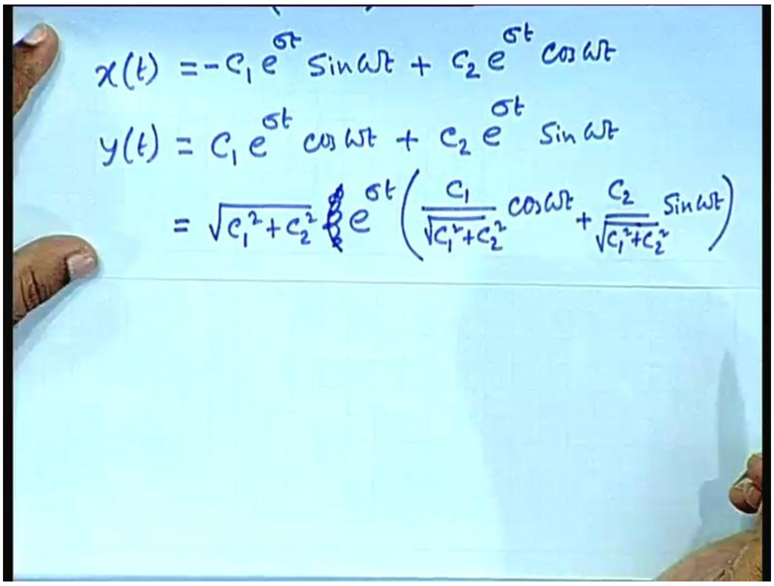

Then e to the power sigma t will remain, e to the power sigma t is common, then what you have is c 1 by root over c 1 square plus c 2 square cos omega t plus c 2 by root over c 1 square plus c 2 square sin omega t fully you visible. Now, if I define these things in a suitable way how will you define, because I want to extract the phase information from it.

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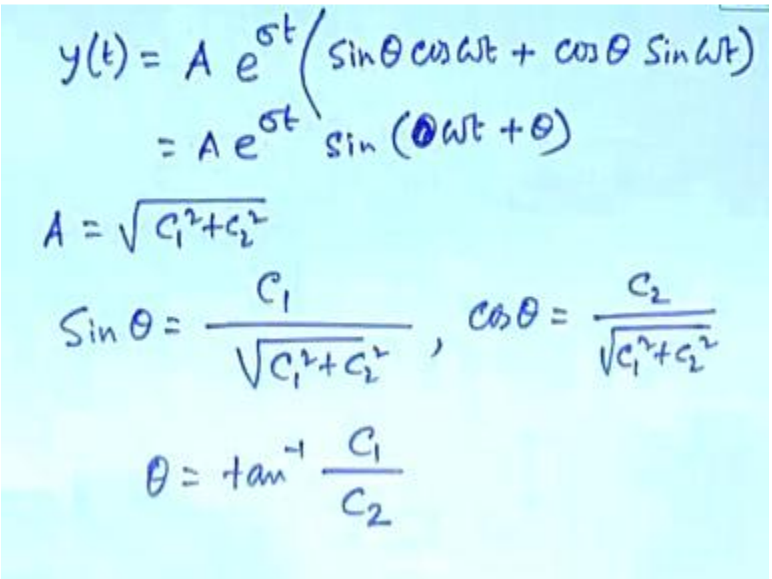

So, if I say what?

Student: ((Refer Time: 24:51))

Yes that will come, yes so it will be y t is equal to it will come to be this term will remain, so it is some magnitude A. So, we will write A is equal to root over c 1 square plus c 2 square. A e to the power sigma t, then it will be it will be cos omega t sin cos, we will we write it like that. So, it will be sin theta cos omega t plus cos theta sin omega t you can write it like that, which means this is equal to A e to the power sigma t sin omega t plus theta.

So, in obtaining this what have we assumed A is this, and we have assumed sin theta is equal to c 1 by root over c 1 square plus c 2 square cos theta is equal to c 2 by root over c 1 square plus c 2 square. So, that this plus this, this square plus this square is equal to 1 sin square theta plus cos square theta is equal to 1. Then the theta is, if you assume this then it is this, this is easier to see easy, easier to visualize than this ((Refer Time: 26:57)).

Why, because this tells us that it as a magnitude A, it has a sinusoidal component which has the phase theta. And the phase theta is related to whatever the excitation was there and then, you have this term which is either exponentially decay or exponentially increasing, depending on the value of the sigma.

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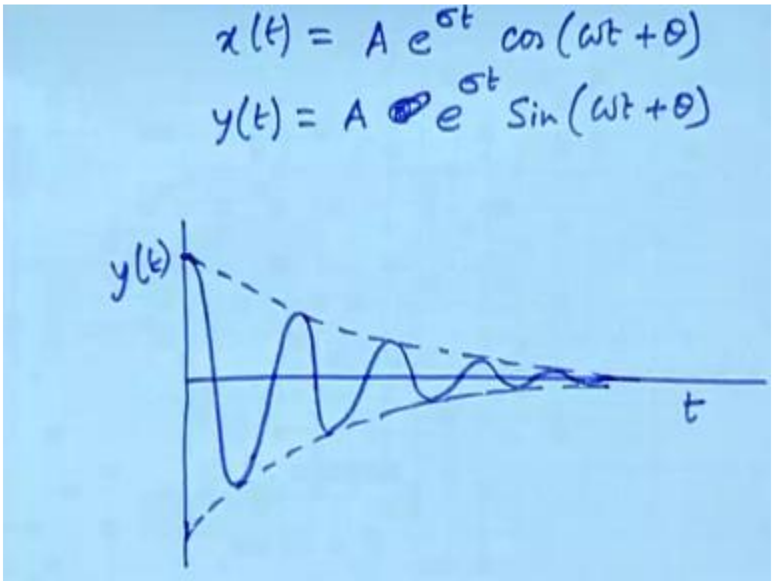

So, if you have y t is equal to A sigma, e to the power sigma t sin omega t plus theta. Then what will be the behavior will be like, by the way what is x t in that case, what will x t be x t was we started with this ((Refer Time: 28:15)). Do the same manipulation and check what is x t then, do exactly the same manipulation.

Student: Cos

Yes, so it is A e to the power sigma t cos omega t plus, so let us see how does this look like, as a function of time this component is a sinusoid. This component is exponential decay, so what will be the this whole component, it is damped sinusoidal. So, when your t is 0, this term is 1 and this term is 0, therefore it is sin theta A sin theta, so that is where we start.

It will start at a value something like A sin theta, theta gives the phase and then, after that what will happen, after that this will define some kind of a envelope like this. And your sinusoid will be this is y t, this is t, x t will be a similar thing, but 90 degree out of phase, because one is cos and another is sin.

Student: ((Refer Time: 30:16))

Well yes, if he asking this will happen only if sigma is negative, what if sigma is positive.

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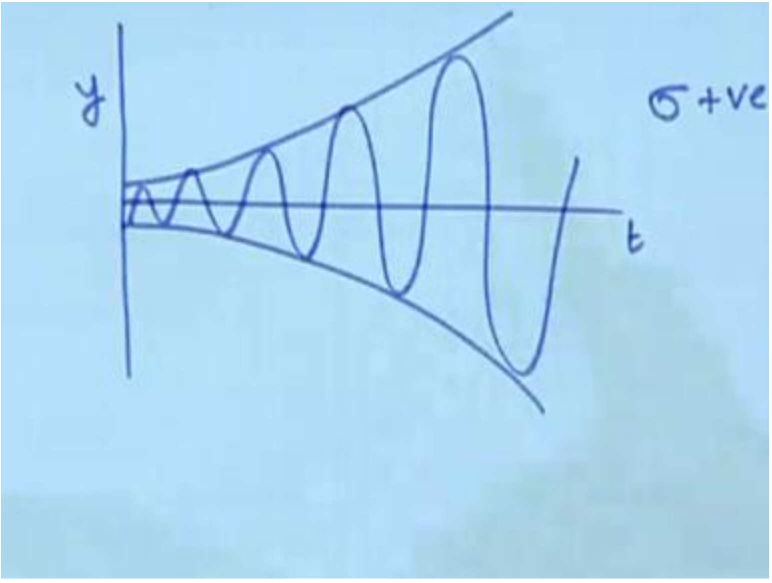

Good question, in that case it will be exponentially increasing envelope and you will have, for sigma positive t and y understood. And the x will also have the same property only thing is that x and y will be out of phase that is all, are you convinced. So, that is the result when you have complex conjugate eigenvalues. Now, there will be one case where, the eigenvalues are purely imaginary, let us understand that case first.

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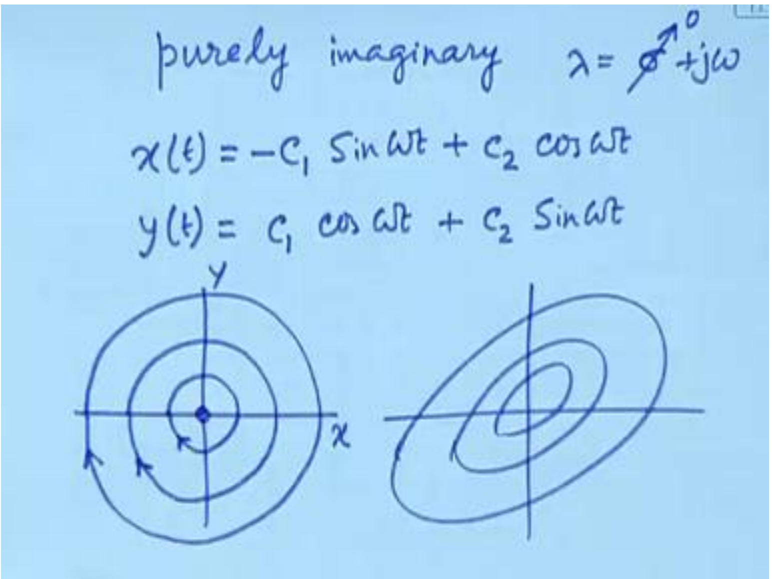

So, the complex conjugate eigenvalues with purely imaginary special case, which means in the lambda is equal to sigma plus j omega this term is 0. Then, your solution originally was x t is equal to this sigma is 0, therefore this terms will vanish. What remains will be I just copied from there, minus c 1 sin omega t plus c 2 cos omega t and y t is equal to c 1 cos omega t plus c 2 sin omega t. What is this, what kind of response is this?

Student: ((Refer Time: 32:24))

Pure sinusoidal, so this behavior will be a pure sinusoidal behavior, but now let us go a little ahead. What will this behavior be in the state space where you are drawing x versus y, in order to visualize you just imagine.

Student: ((Refer Time: 32:51))

Yes, there how to prove it

Student: ((Refer Time: 32:54))

So, this actually x square plus y square is equal to 1, this is that.

Student: ((Refer Time: 33:03))

That is right, so ultimately that yields a circular equation, which means the behavior will be like the circle. Now, where do get the c 1 and c 2 from the initial condition, so depending on the initial condition it will be a different circle like this may be, or like this may be. So, if the eigenvalues are purely imaginary you have circular equation, circular solution, but this circular solution is really what should I say it is topological circle.

It means that if y x and y are in the same scale then only you get circles, and x could be the derivative of y or y could be the derivative of x, therefore there is no meaning into the term that they are really the same scale. So, in general they would be a ellipses, that is why I said the same topologically circle, because its stretch and squeeze in order to make them circles.

So, in your mind you should carry with the idea that they are really circles, but with the pointer that the here is x and here is y. One fellow can be the angle of a pendulum, the other fellow can be the rate of change of the angle, how do you express them in the same scale. Normally you would not be able to in that case you will get ellipses, that is all. But, I mean where you go back with the concept go back with the concept of the circle, because of this.

And that is why this specific equilibrium point is called a centre, earlier when we are considering the real eigenvalue, we said it is a node or it is a saddle. That means, the equilibrium point is called a node if the eigenvalues are real and both negative or both positive. It is called a saddle if one is negative other is positive, but real, but if you have a purely imaginary eigenvalues, then this is called a centre with very obvious reasons, by the way in which direction will it rotate clockwise or counter clockwise.

Student: ((Refer Time: 36:07))

That depends on the…

Student: ((Refer Time: 36:16))

There is a easy way of checking it, suppose you have got this as the solution, suppose you take any point along the x axis. You have got the original solution equation as where is the original equation here, ((Refer Time: 36:42)) these are the original equations, takes this point where y is 0 and x is there. Y is 0 means this term vanishes, so x dot is something dependent on the value of x, x dot is something. So, x is changing x dot is there. Y dot is where this is 0 is omega times x which is positive.

Positive means y dot is positive means a vector like this, so that immediately tells that it rotates actually counter clockwise. Or in other words, if this omega is negative term, then it rotates ulta, opposite direction clockwise.

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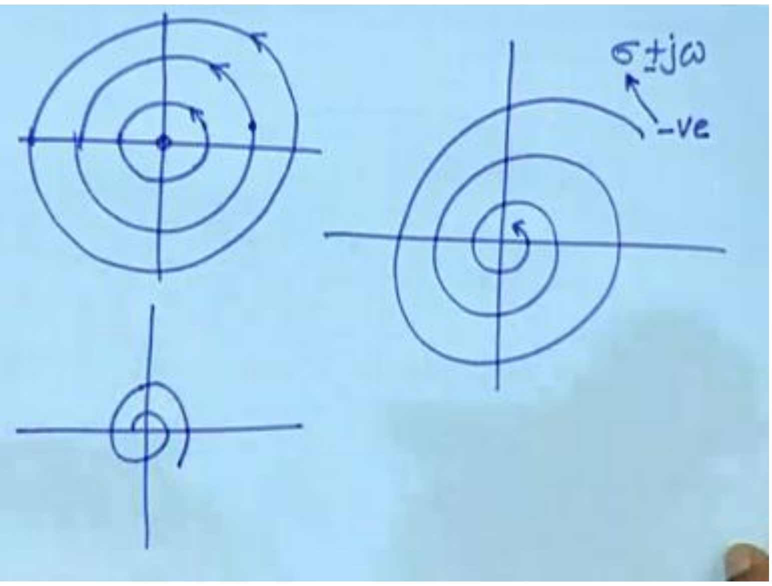

So, we have understood that it will be, if the omega is positive it will be omega is positive, then it will have vectors like this, so it will be like this. So, if you start an initial condition here it will go around it and comeback here. If you start a initial conditions here it will go around return comeback here. How about the time, how long does it take for it to come back here and how long does it take for it to come back here. What is depend on omega and since for both the c 1 and c 2 does not depend on omega, it depends on the initial condition.

And therefore, if all these the period of revolution should be the same omega. frequency should be the same. So, wherever you may start it will see the same period of revolution, same frequency does it follow from logic, because your equations ultimately what you have got are these, where is it are these ((Refer Time: 38:55)). That is why the frequency term actually appears here and if you get an initial condition say, if I say you start from 1 1.

You can easily obtain if I start from say 2 2, you can easily obtained and you will find that there, so those solutions c 1 and c 2 are independent of omega and so you have the same frequency for all this, that is one important concussion. The third important concussion is that here what does this term do, this term is the exponentially increasing or decaying term. Now, if you have this, so far we were considering purely imaginary eigenvalues, but now if I consider complex conjugate eigenvalues with this.

Then how will the behavior in the state space change, notice that we are actually drawing the vector field, vector field around the equilibrium point which in this case is the center. If you have sigma plus j omega plus minus j omega, then the behavior would be depended on this sigma say it is a negative value, then it is spiral towards the center it will actually go like this. When this is negative and when this is positive, it will spiral out probably, it will be counter clockwise direction.

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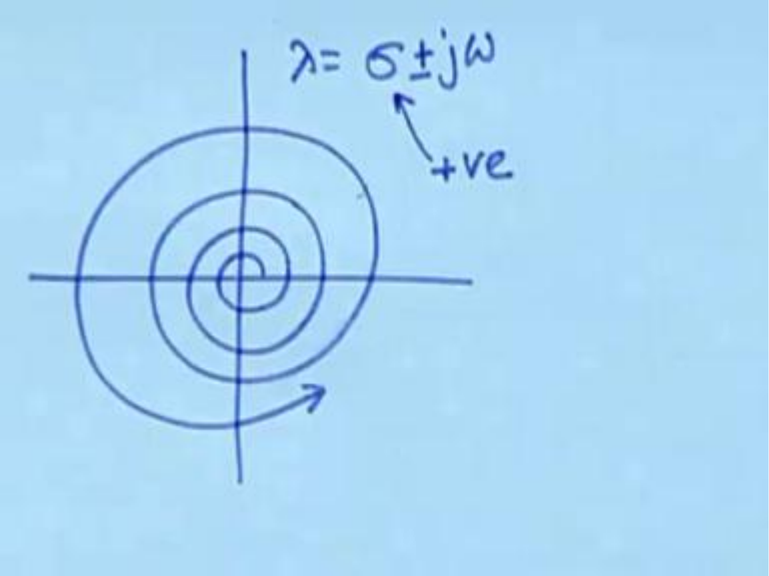

So, Sigma plus minus j omega is eigenvalue with this is as positive are you through it with understood it correct. So, you see we have more or less understood how such what kind of vector fields this complex conjugate eigenvalues give rise to…

Student: ((Refer Time: 41:29))

Initial condition, it will be dependent on the initial condition. See what we are trying to do is that, we are trying to take a snapshot at the region in the neighborhood of the equilibrium point. And we have understood that the region in the neighborhood of the equilibrium point it is vector field will be dependent on the eigenvalues. And so for different possible types of eigenvalues we are trying to obtain the character of the vector field.

Let us just do one problem, Galileo made a very important observation, he was sitting in a church and he found a big something hanging from the tope, that is swinging oscillating like a pendulum. And he uses his heart beat to make the measurement and said that it has the same period of oscillation. Independent on the initial condition, so can you prove it, now by deriving the equation for the pendulum, on the base of this you should be able to, let us do this. Since you have done the Lagrangian method quite some time back and may have forgotten, that is why let us go back to the Lagrangian method and do it.

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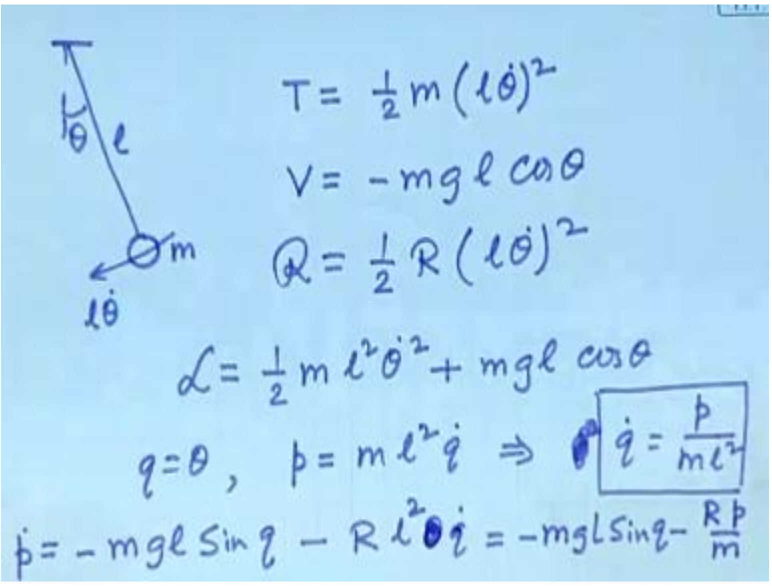

So, you have a simple pendulum the one the Galileo saw, it has a angle theta l mass m that is what he saw. So, in this case your velocity of this one will be l theta dot and therefore, the kinetic energy will be half m l theta dot square. The potential energy will be from this point how much is gone up, so it is minus m g l cos theta, so this value. Then your friction term will be half R l theta dot square, where R is the air friction, there will be a friction due to air R is the friction.

So, that is the Rayleigh term, your Lagrangian will then be T minus V half m l square theta dot square plus m g l cos theta. So, since we were not considering the theta as the variable, our variable was position coordinate was q, which is in the case theta. And the momentum coordinate was p, which is p is the derivative of the Lagrangian with respect to theta dot. So, what was p, m l square q dot, we need to obtain two differential equations, so that is why I am writing as p and q.

So, you have these two, so this gives immediately q dot is equal to p by m l square which is differential equation. So, this is one differential equation, the other differential equation p dot is equal to, how do we obtain p dot from the Lagrangian equation. Remember from the Lagrangian equation, so the first term in the Lagrangian equation is p dot. And the other two terms are derivative Lagrangian with respect to theta or in this case q and the derivative of the Rayleigh with respect to theta dot just do that.

This term will have a negative sign, so minus m g l it will be sin and representing theta by q, so sin q that gives the first one and second one will be it will be plus no, minus R l theta not in theta, and q dot.

Student: ((Refer Time: 46:57))

l square, you are right, so that is the equation for p dot I will substitute here q dot, so that the right hand side is completely divide of dotted terms. So, this gives minus m g l sin q minus l square actually cancels off, so you have R p by m, so this is one equation, this is one equation. Now, let us write these two in matrix form, just write the equations have we written now, so from there you write it in matrix form.

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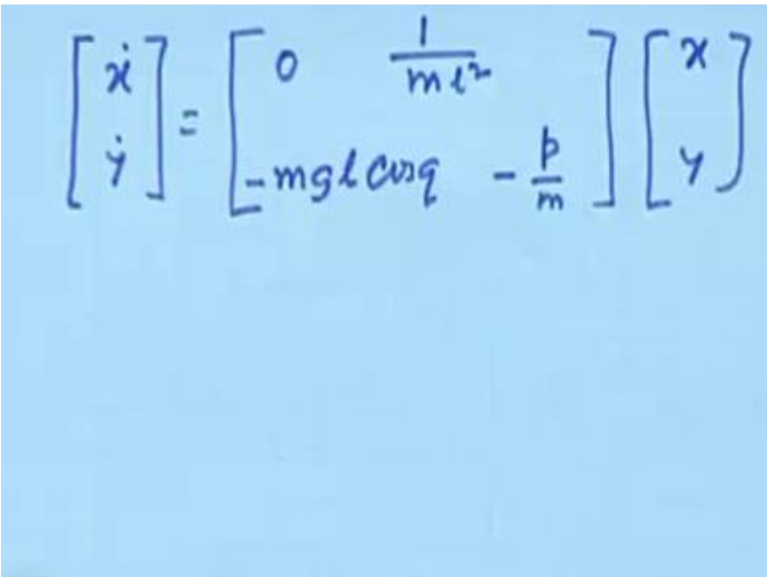

Since, we will be now trying to obtain the vector field, let us write it as x and y, p and q. So, you have x dot y dot is equal to the first term, it is not there, so it is 0, then 1 by m l square here it is minus m g l, we are writing the Jacobean matrix, here we have.

Student: ((Refer Time: 48:52))

We have to differentiate with respect to q, sin q will become cos q and here it is minus x y. So, the original equations where these, now we are locally linearized it and we have written the Jacobean matrix equation like this. So, this is the local linear description of it, not only that, what? No, let we write it once again.

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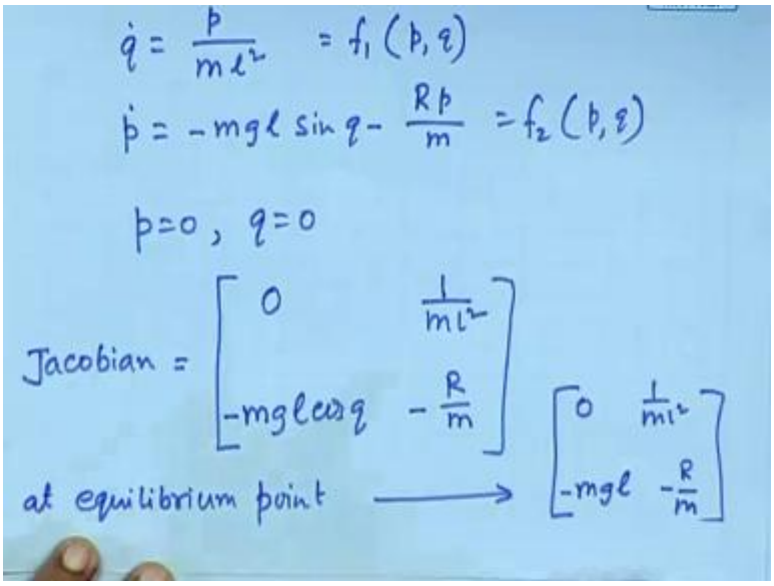

Your q dot was p by m l square, your p dot was minus m g l sin q minus R p by m, now this is non-linear equation I can see that. So, first will need to obtain the equilibrium point, and then we will need to local linearized along the equilibrium point, where is the equilibrium point that is obtain if you said the left hand side to 0, q dot p dot is 0, q dot is 0 immediately means p is equal to 0 and p dot is 0 immediately implies, this p was 0.

So, this whole thing should be 0 and you can easily see there are many solutions of it, because sin q being 0 will happen at 0, 180 degree, 360 degree and all that, so there will be many solutions, many equilibrium. We will come to at later, but at least q is 0 is one equilibrium, so p is equal to 0, q is equal to 0 is another equilibrium point. So, the next step is to obtain the Jacobean, the Jacobean matrix is then the derivative of the right hand side, this is f 1 of p and q and this is f 2 of p and q.

The first term in a Jacobean would be this as differentiated with respect to q. So, this term will be 0, second would be this differentiated with respect to p this term will be 1 by m l square, this term will be this differentiate with respect to q. This whole thing differentiated with respect to q will be this term will vanish and this term will be minus m g l cos q. So, minus m g l cos q and there last term will be this whole thing differentiated with respect to p which is...

Student: ((Refer Time: 52:15))

Wait, I am yet writing only the Jacobean matrix, from here I am obtaining. And then, will say that now that we need to change the coordinates, because these are the local coordinates. Define in terms of deviation from the equilibrium point and then, will say x and y. Presently we are writing only the Jacobean matrix, the Jacobean matrix this is right and here it will be R by m, so that is the Jacobean.

Now, the Jacobean has to be a evaluate at it, at the equilibrium point, equilibrium point is p is equal to 0, q is equal to 0, here there is nothing like p and q, but here there is. So, we have to substitute cos 0 here which is 1, therefore it becomes m g l. So, at equilibrium point it becomes 0 1 by m l square, here it is minus m g l it is minus R by m. So, that is the Jacobean as evaluated at the equilibrium point, and therefore when you have evaluated the Jacobean.

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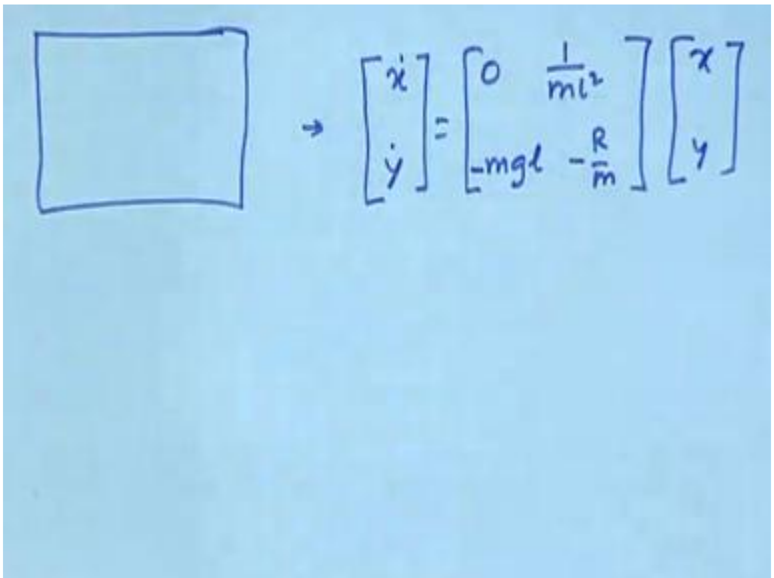

Then you say that here is the total system, but now we will define a coordinate system, that is valid in the immediate neighborhood of the equilibrium point, let that be x and y. So, there the equation would be x dot y dot is equal to this 1 by m l square minus m g l minus R by m x y. What are the eigenvalues of this, obtain the eigenvalues of this, I leave it to you, since the time is coming to an end.

Obtain the eigenvalues of this matrix and then from there what can be the solution of this set of differential equations starting from any initial condition. And then from there try to if you can deduce Galileo’s observations. That I leave for you to be done in the meantime, we will meet in the next class.