Dynamics of Physical Systems
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Lecture - 22
Vector Field Around Equilibrium Points – I

Well, in developing the understanding about this behavior of systems govern by a set of differential equation, we have taken as specific route.

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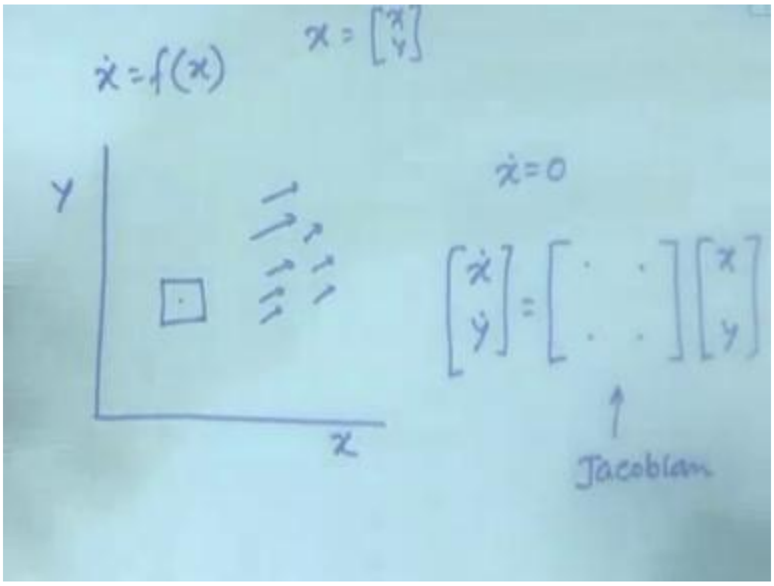

It was that when you have a set of differential equation x dot is equal to f of x. Then, it gives essentially the state space where x and y are the coordinates, where x vector is x y. Now, every point here would be every state is just a point in the state space, and whenever you say that this equation defines a vector. So, at every point in the state space there will be a vector and the solution will essentially follow these set of vectors that was the logic.

And then, we extended that to formulate the methods of solving the differential equations numerically. But, when we come to the analytical solution or developing an analytical understanding we said that, let us not consider the whole space and the vectors in it in one row. Let us feed the problem in stages, where we first read what happens in the neighborhood of some special points, these the equilibrium points where x dot is equal to 0.

That means, these points as the property that if this is the point, then if the initial condition is there it will forever remain that. So, we said that, we first let us understand the character of the system in the neighborhood of that. So, in the neighborhood of such points we had locally linearized it by obtaining the Jacobean matrix, so ultimately we are obtaining the equation of this form, say if it is 2 D. Then it will be x dot y dot is equal to some matrix times x y, square matrix with 4 terms here, so this will be the Jacobian matrix, this we have seen that. And then we said that if this is the set of equations on which we are further developing the concept.

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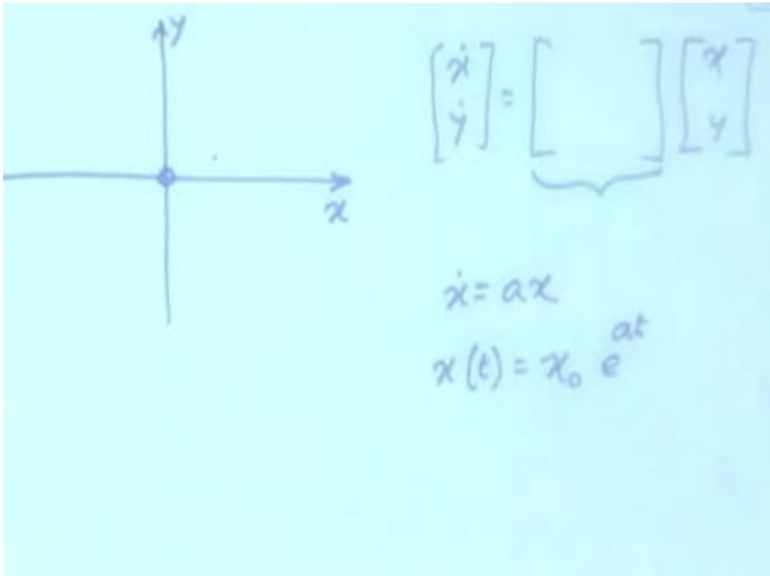

Then our state space is where this is the equilibrium point, that means the equilibrium point essentially has been moved by the coordinate transformation into the origin. So, you have this new x and this new y, which are actually the deviations from the equilibrium point, if it is placed somewhere else, so keep that in mind. So, these are the new x and y, when seen just in the neighborhood of that equilibrium point and then, the equation govern it would be x dot y dot is equal to x and y.

Then we say that if that is so, then the character of the system, the evolution of any initial condition, say it starts from here, how it goes all that will then given by just this matrix. Because, there is nothing else in the system, so this matrix will contain all the information necessary in order to understand the dynamics of the system. So how, then we said we have a method of solving differential equations, for 1 D systems which is if you have simply one dimensional system x dot is equal to some a x.

Then the solution was x at t was x naught e to the power at we know this, somehow we ne one to use this in order to obtain the solution of this differential equation. But, then we had use, we have said that will use the property that if it is somehow obtain two solutions, then any solution would be a linear combination of those solutions. So, what or how to obtain these two solution was the problem that we will concerning in the last class.

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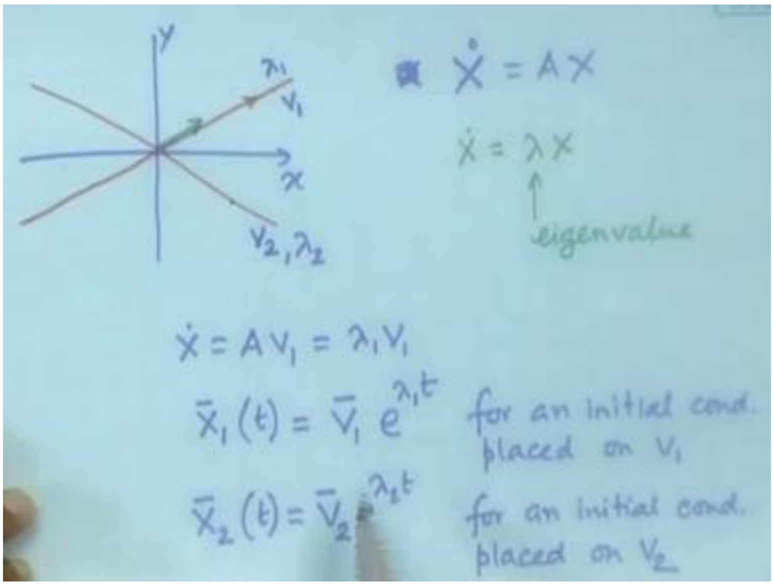

And then, we say that in this space you have x and y space, we have some special directions that have the property say, this is the one direction say and this is another direction. So, that have the property that if any initial condition is placed there, that means if your x vector x I will denote with capital X, the X vector if it is here, then X dot is equal to A times X.

So, this is the vector, this is the operator which is operating on X is giving X dot that will also be in the same direction, that means like this. This special direction we said or called the Eigen directions, any vector along an Eigen direction is Eigen vector. And in that case what happen is if it is in the same direction you can say that X dot is equal to nothing but, a number multiplied by the old vector X, this number is the Eigen value.

And then, in the last class we had understood how to obtain the Eigen values and Eigen vectors.

Then the crucial point is that, if an initial condition starts on an Eigen vector, that means an initial condition placed on an Eigen vector. The X dot vector will be in the same direction, and therefore it will move in the same direction. If it X dot is the opposite direction it will still move in the same direction in the opposite direction, but in the same along the same line, along the same Eigen direction.

Which means that if an initial condition is here, or anywhere on this Eigen direction it will forever remain confined to that Eigen direction, which means this is the one dimensional equation, it will effectively become a one dimensional system. If it is a one dimensional system, then we can use this solution that we already obtained. Now, there will be one such solution, then obtained along this direction, another such solution obtained along that direction.

And therefore, any given solution will be nothing but a linear combination of these two solutions, so that was the simple logic that we are proceeding with. So, is the method of obtaining the Eigen values or Eigen vectors clear in the last class, so we will do examples, so it will be more practice. So, from here we take off, now suppose your Eigen direction is say V 1 and say V 2, these are the two Eigen directions. And Eigen field again Eigen value along this is a lambda 1 and the Eigen value along this is a lambda 2.

So, associated with each Eigen direction there will be one Eigen value. Now, suppose you have placed an initial condition here which means our X is V 1, in that case we will write that X is the V 1. So, X dot is equal to A into V 1, now since this is on the Eigen direction, therefore your equation actually reduces to lambda 1 V 1.

So, what is the solution of this X dot is equal to lambda 1 is the number times V 1 what is the solution of this, X 1 of t is equal to, so it will be V 1 times e to the power lambda 1 t. For a initial condition that is placed along this the solution will be following this, that is one of the solutions. Similarly, if you placed another initial condition on this, so this is for an initial condition on V 1, V 1 is the Eigen direction.

Then the for an initial condition placed on V 2, it will be X 2 of t is equal to V 2 e to the power lambda 2 t. These two are two possible solutions, the moment we have obtained this any solution will be just a linear combination of this.

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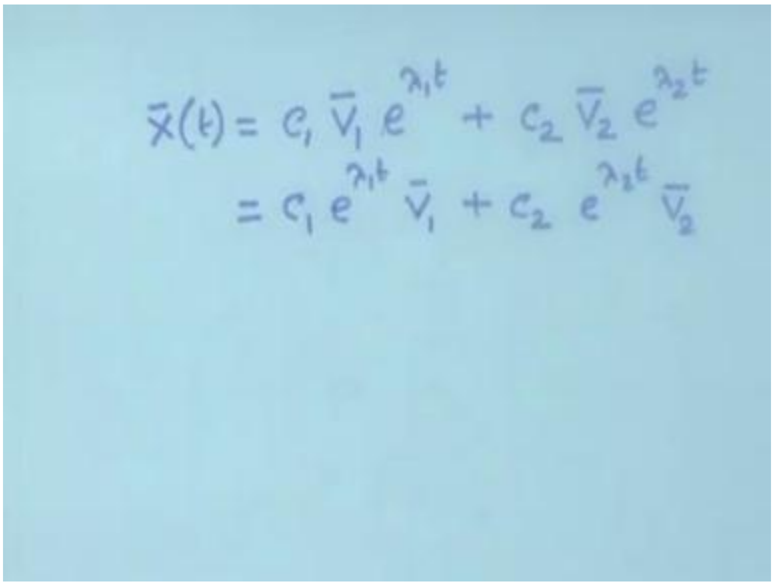

So, any solution will then be simply X of t is equal to some C 1 constant times V 1 e to the power lambda 1 t plus C 2 V 2 e to the power lambda 2 t. Normally C 1 is a number this is also a number, but this is a vector. So, if I want to distinguished between the numbers and the vectors I should better put some kind of a symbol, meaning that these are vectors.

So, X t is equal to this is the vector, this is the vector or normally one would like to write it as e to the power lambda 1 t times the vector V 1 plus C 2 e to the power lambda 2 t times V 2. Let us just do at one example. What does it mean, will understand that after we workout the example.

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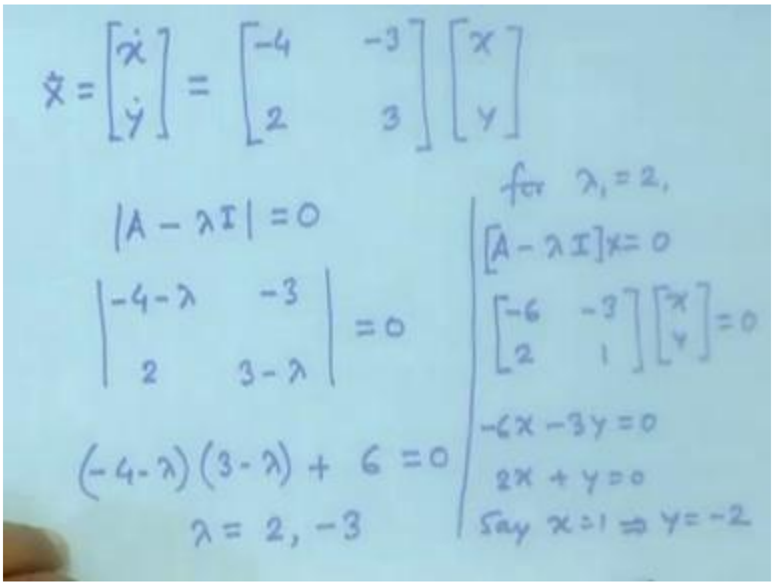

Suppose you have an equation given as x dot y dot which is nothing but, X dot X vector is equal to minus 4 minus 3 2 and 3, x and y. What will you do, we will first obtain the Eigen values and Eigen vectors of this matrix, just obtain it you know how to, we will say A this matrix minus lambda I is determined is equal to 0. A minus lambda I would be A minus 4 minus lambda minus 3 2 3 minus lambda equal to 0.

Minus 4 minus lambda 3 minus lambda plus 6 is equal to 0 thus obtain the Eigen values here minus 3 and 2, so lambda is equal to there are two things 2 and minus 3, so these are the two Eigen values. So, we will say that here in the picture the two Eigen values are 2 and minus 3. Now, let us obtain the corresponding Eigen vectors. So, let us in order to do that, will first start with the value 2 for lambda 1 is equal to 2, what will you write the matrix A minus lambda I is equal to 0, which will say A minus lambda X is equal to 0.

So, from there you obtain what are the two vectors, so this is the matrix A minus lambda i, i substitute 2 it is minus 4 minus 2 this minus 6, this minus 3 2, so it will be minus 6 minus 3 2 and 1 x y is equal to 0. You see the first equation is minus 6 x minus 3 y is equal to 0 and the second equation is twice x plus y is equal to 0, they are actually the same. So, what we say the equation is, the Eigen direction here it is Eigen direction not the Eigen vector, it will give the Eigen direction.

It says that the Eigen direction is where twice x is equal to minus y, in order to placed an initial condition on that Eigen direction, you have to A prior is specifies some value of x and find out the value of y. Say x is equal to 1, then y is equal to this gives y is equal to minus 2.

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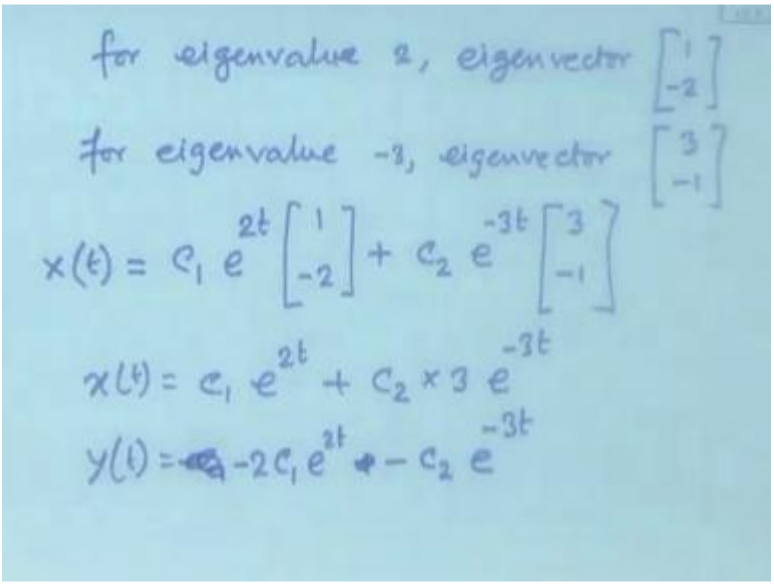

So, the vector for Eigen value 2, the Eigen vector is what 1 and minus 2, similarly obtain the other Eigen vector for minus 3, tell me what it is in the same way. For Eigen value minus 3 Eigen vector is depends on where you have placed the initial condition minus 3 1 or 1 minus 3 same things really, depends on where you have placed the Eigen vector. What I am trying to mean that that minus 3 are not sacrosanct, it could be the anything that is important.

So, if one of you say that no it is minus 3 1, another fellow say no 3 minus 1, so long it is in the same Eigen vector. So, let us take one of them 3 minus 1 may be check that it is on the same Eigen vector, just to measure since you are said 1 minus 3 as 3 minus 1, because it the same Eigen vector. Now, you have to write it like this, which means we can say that X of t, which is x t y t is equal to c 1 e to the power twice t times this Eigen vector plus c 2 e to the power minus 3 t times that Eigen vector.

Which means, it is actually if you write it in individual equation form x of t is equal to c 1 e to the power twice t plus c 2 into 3 e to the power minus 3 t. And y t is c 1 minus write it as minus 2 minus twice c 1 e to the power twice t plus minus c 2 e to the power minus 3 t. So, these are the individual equation, but we would prefer to write it as the vector equation like this, because of some conceptual clarity I will come to that, so where do we get c 1 c 2 from, from the initial condition.

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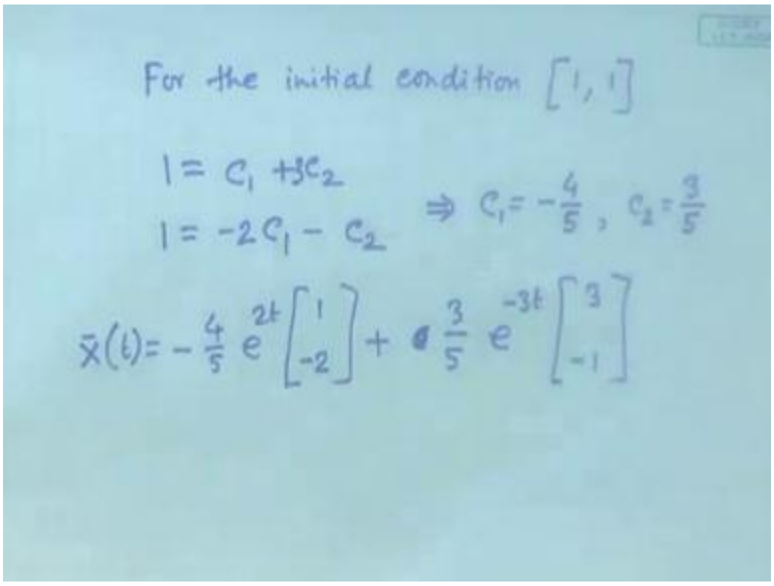

So, just calculate that once, for the initial condition and calculate the values of c 1 and c 2, what will you write these two initial condition mean t 0, so x and y at 0 is 1 1. So, you will say 1 is equal to c 1 e to the power twice 0, this is 1, so c 1 plus c 2 thrice c 2 and 1 is equal to minus 2 c 1 minus c 2. Two equations two on solve it and get it, what you have this will give c 1 is equal to what just do that minus 4 by 5. And c 2 is which means the actual equation if you have this as starting point would be simply this ((Refer Time: 21:26)).

So, it will be X vector of t equal to c 1 is minus 4 by 5 e to the power twice t 1 minus 2 plus c 2 is 3 by 5 e to the power minus thrice t 3 and minus 1 that will the solution. Obtaining such solution is not every difficult thing, but I am trying to divert is that that gives a very nice geometric way of looking like this, what is happening.

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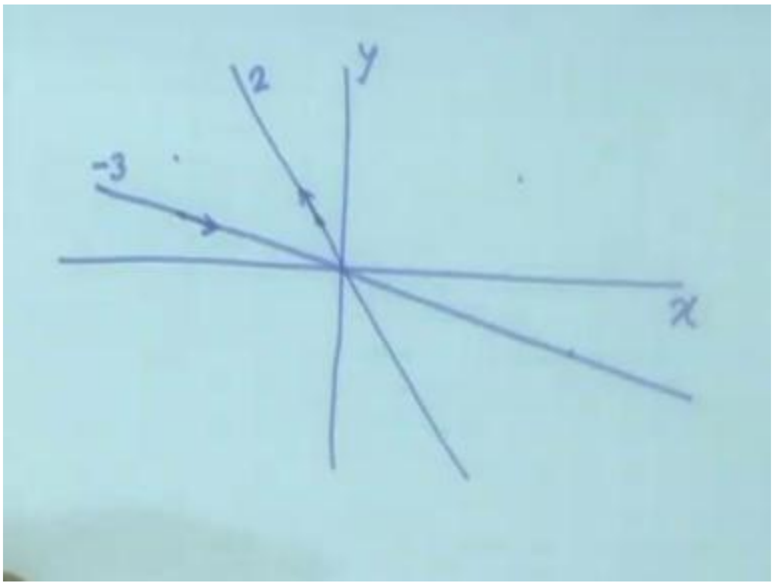

You have the state space like this with x and y are the coordinates, now where is this vector 1 minus 2, so it should be something like this that the Eigen direction physically. This is 3 minus 1, so go to 3 and slide bit here, so it would be I will draw different, so for the Eigen vector that is 1 minus 2 your associated Eigen value is 2. So, with this you have 2 and this you have minus 3.

Now, then this one any initial condition along this, say like this will increase as e to the power twice t it will increase. While any initial condition on this will go as e to the power minus 3 t it will decrease exponentially, so anything here will go out, anything will go here in, so is that clear. In general I will come to the case is where the other points are also to marked for example, if starts from this initial condition, that initial condition I will come a little later. But, let us understanding the situation where both the Eigen directions have negative Eigen values. Because, this is the case where one Eigen value as positive another negative, but let us consider sequence with negative Eigen value.

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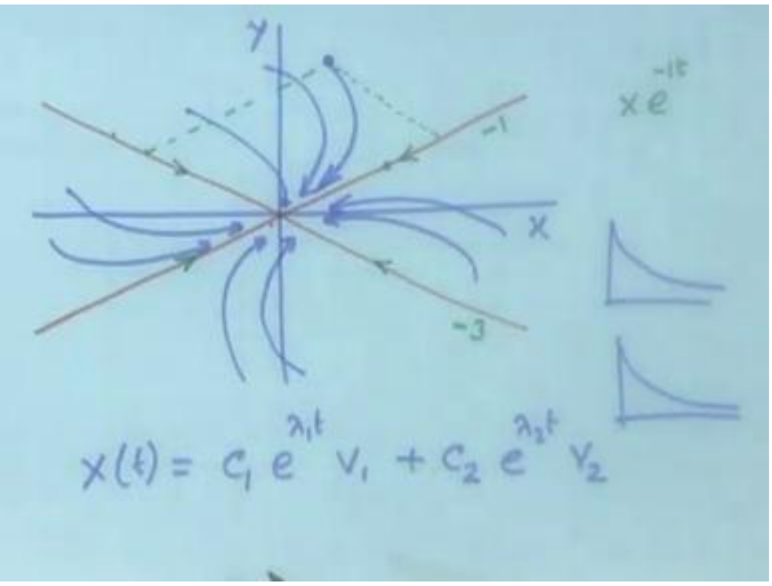

Then, suppose you have an Eigen vector like this, another Eigen vector like that and suppose the associated Eigen values here are say minus 1 and say minus 3. Try to figure out, how the behavior will be after all the behavior will be something like this, only thing is that here in case of 2 it will be minus 1. My point is then the equation would be of this form, which means that if you ignore this it will be an exponentially decaying value, if you ignore this it will be exponentially decaying value.

So, starts from here it will exponentially decaying, starts from here it will exponentially decaying. So, I put arrows like this, make it clear why because, if you start from an initial condition say here, it is evolution will be govern by initial condition times say as X, so X times e to the power minus 1 t. This minus 1 t is exponentially decaying function, therefore this magnitude will go down and down and down and finally, it will converge that is why I put the arrow X.

Now, suppose you are start an initial condition somewhere here, what will it is behavior to try to figure out on the basis of this, this kind of equation. So, the equation would be of this form X of t is c 1 e to the power lambda 1 t V 1 plus c 2 e to the power lambda 2 t V 2, these two are the Eigen vectors. Notice that the ultimate solution is a linear combination of these two solutions, which means that effectively these two solutions involve independently and the ultimate solution is just the addition of them, which means if you drop such parallels, then you will find that the initial deviation from the origin along this Eigen direction is this much, and that will evolve by this, initial deviation along the other Eigen direction is this much it will evolve of that, this as a relatively larger Eigen value minus 1 and this as a relatively smaller Eigen value minus 2. So, which one will decay faster, this term will decay faster, this term along that.

So, it will decay faster along that and relatively slower along that, what does it mean what will be the evolution like. It will be like this, did I really solve equation, no from logic we could infer that the behavior would be like this. So, depending on the relative magnitudes of the Eigen values, if there both negatives, then it will converge. But, it will converge always after converging first on along where after converging first two, one of the Eigen direction.

It will not just converge just like this no, it will first converge on one of the Eigen vectors and then, along that it will finally, converge on the fix point, equilibrium point. Ultimately if you draw individually this and that you will find both will be like this, you will actually like to see the evolutions along the x and y, they will both will like that. So, just by looking at this you do not see, that it is actually converting on to the Eigen direction and then, meeting the fix point, but it will always do so.

If you start from another initial condition say here, it will what will do how, this will have these as the initial deviation, which one will decay faster this one will decay faster. So, as a result it will convergence on to convergence on to, converge on to this first and then it will start from here it will converge like this, start from here it will converge, start from here. So, you can see that without ever doing anything else I am drawing the vector field, just by working out logically that this will have the behavior like this, this is the vector field. So, the vector field around a the equilibrium point, if the two Eigen values happened to minus 1 and minus 3 it should be something like this. And this I did not work out simply by computer, but this can be work out by logically.

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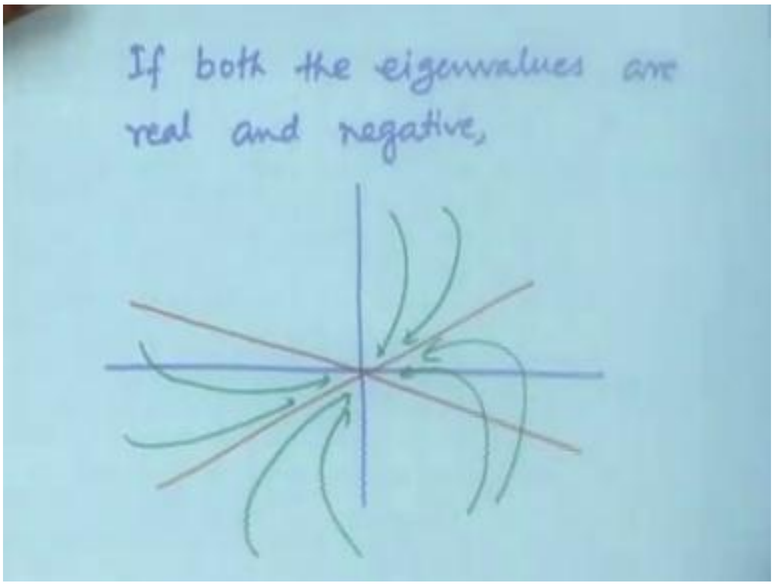

So, you see if both the Eigen values are real and negative, then the behavior will be always like this. If you have the two Eigen directions drawn, then it would be no, I of course, this is Eigen directions it is wrong.

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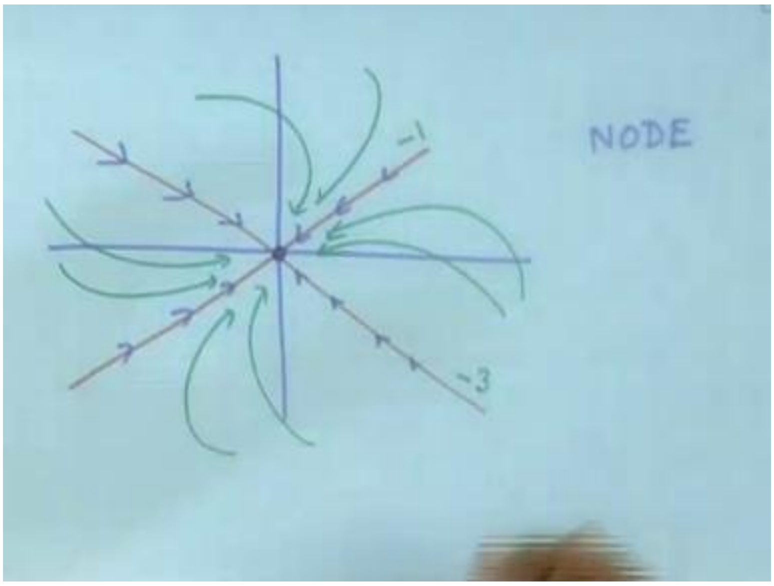

It will like that, provided this as a Eigen value minus 1 this as minus 3 or this will larger than this, what if both the Eigen values are positive and real. It will be just the opposite only the arrow heads will be in the other side, only the arrow heads in the other side, so if probably you have notice that I said that this is wrong, why did I say it is wrong, because we will never cross an Eigen directions. So, that is one of the important points. The vector fields will never cross the Eigen directions, vector fields will always in this direction it will be just like this. In this direction it will be just like that, what you effectively done is to imagine a coordinate transformation, where we are call this as a new x coordinate, that as the new y coordinate, then it is simple.

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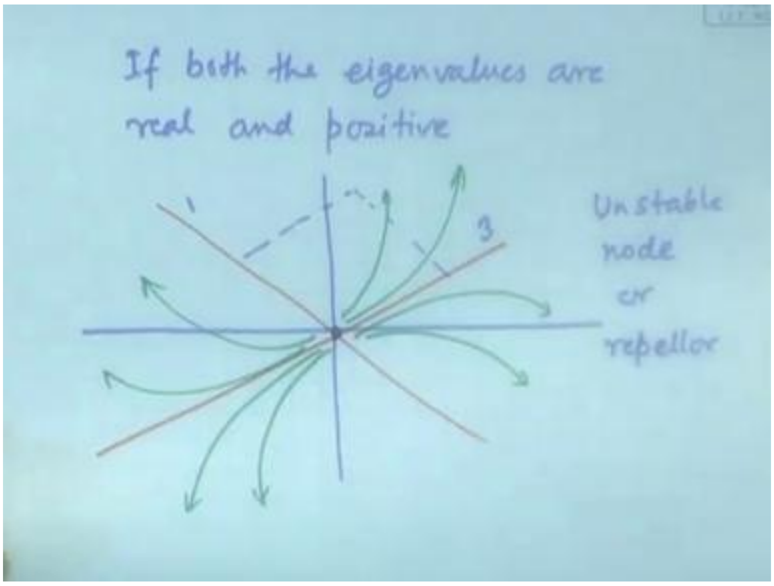

When if both the Eigen values are real and positive, then it just the opposite it will be, so if you start an initial condition very close to the equilibrium point it will go out. If you have an initial condition exactly at the equilibrium point it will stay there. Slide part of then it will first start off along an Eigen vector and then, go away in other direction, it will always start off along parallel to the initial to one of the Eigen vectors. And then, it will go away.

Student: ((Refer Time: 34:09))

Suppose, this is that is the good question, out of this which one has a greater Eigen value, the way I have drawn it imagine you have taken a point, it has two projections. And in this case this one is increasing faster, that is why it is fast going this way, which means this as this particular direction has, so say it is a 3 and this is a 1. I need not really write down the equations for this can be inferred from logic. Now, let us come to the case that we started with Eigen values 1 and minus 3. This had to be understood a little later that is why I first came to this and then talked about this.

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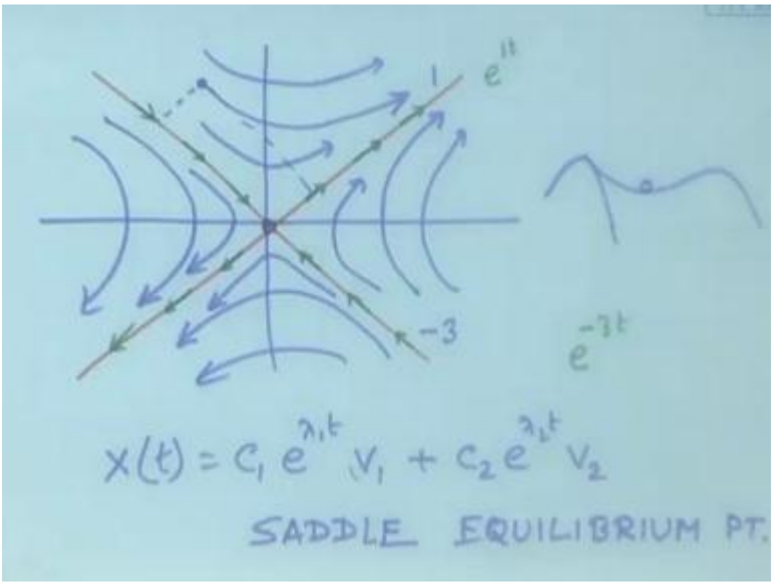

Now, you have say two Eigen directions like this and say this value as Eigen value 1 and this value as Eigen value minus 3, then what will be the area. Notice that any initial condition along that will do what it will go, so start from here it will go I will have to draw the arrows like this. I will have to draw the arrows like that, along this anything will evolve as e to the power minus 3 t, so along this it will be e to the power minus 3 t along that it will be e to the power 1 t.

So, it will be decaying, so I will have put arrows like that, start from initial condition somewhere else, say here what will be the behavior like. Again imagine in terms of the two projections and how the projections evolve, because ultimately your X of t is c 1 e to the power lambda 1 t V 1 plus c 2 e to the power lambda 2 t V 2, so the projections. This V 1 or this V 1 and the V 2 are the components along the two Eigen directions considering separately, because there actually evolve in separately.

So, this is the component along this Eigen direction which decays, and this is the component along this Eigen direction that increases. So, it is behavior would be, start from any initial condition here what will you do, again along this it will decay along that it will grow. So, it will be start from initial condition say here, along this it will decay, along that it will grow and initial condition here it will go like this. So, do you see the structure of the vectors and if you want to visualize the vector field draw a few more of this lines, it will be easier then to visualize, that is how.

So, any initial condition its component along this directions will reduce, along this directions will grow. It is somewhat like the behavior if you release a marble on say a saddle, then saddle means, the saddle on the top of the horse is something like this. If you release a marble somewhere here it will stay there, but if you starts from somewhere here it will roll back, if you have give a slide part of this way it will roll rolled down. So, the behavior of anything placed on a saddle, it is similar to here can you see.

This part of vision is similar to this part of vision along this line, and this part of vision is similar to this part of vision along that line, that is why such equilibrium points are called saddles. So, this is called saddles, an saddle equilibrium point, the situations like this are called nodes. Situations that means, this equilibrium point is called a node, where the vector field lines converge, this is where vector field lines are convergence, so it will be called a stable node.

In this case the vector field lines are immerging from there and going outwards, it will also be called a node, but it would be unstable node. It is also called a repelled, because any initial condition here is repelled away, so this is the stable node. Or it also called attractor, because any initial condition attracted to this point, any initial condition will be attracted this point. So far so good there is no problem we have understood the cases where there are real Eigen values. Now, the troubles on case, the complex Eigen values we have to understand that also, what to do with the complex Eigen values.

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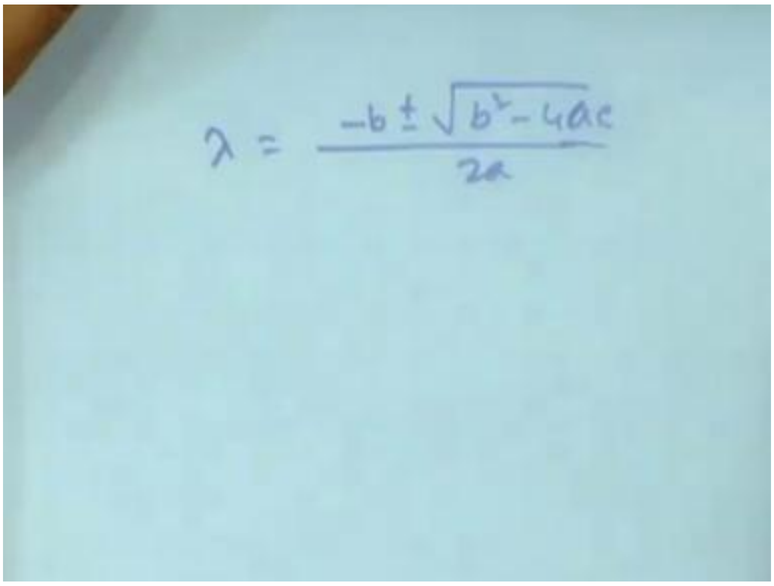

Obviously the Eigen values are obtained from a quadratic equation, a quadratic equation solution. So, that will be lambda is equal to some kind of minus b plus minus root over b square minus 4 a c by twice a, this term depending on the b square minus 4 a c could as will be negative or positive. If it is negative we will a complex number, but notice that it is not just a complex number, but always complex conjugate do you know why. Can it yield just one completes the loop why not, at least not this, but say high dimensional system, where this will not be a quadratic say cubic quintet whatever.

Student: ((Refer Time: 42:29))

Yes, if the polynomial coefficients are real, then there is a theorem law proof very long back provide the D Alembert. That if the polynomial coefficients are real, then it will either yield real Eigen values or complex conjugate Eigen values, it can never be a single complex value. So, it is always be complex conjugate, if there complex conjugate, then if see what will our logic. Our logic is that we will try to identify one direction in which we will place an initial condition will obtain a possible solution.

Another direction along which I will place another initial condition, I will get a possible solution, but now we are trouble because these two it is if one is p plus j q the other is p minus j q. Which means that they do not have any independent information, the information there is contain in one, the same information contain in the other fellow, which is a conjugate. So, they cannot give two independent directions, that is one trouble, but will overcome this trouble, that is not a big problem as such.

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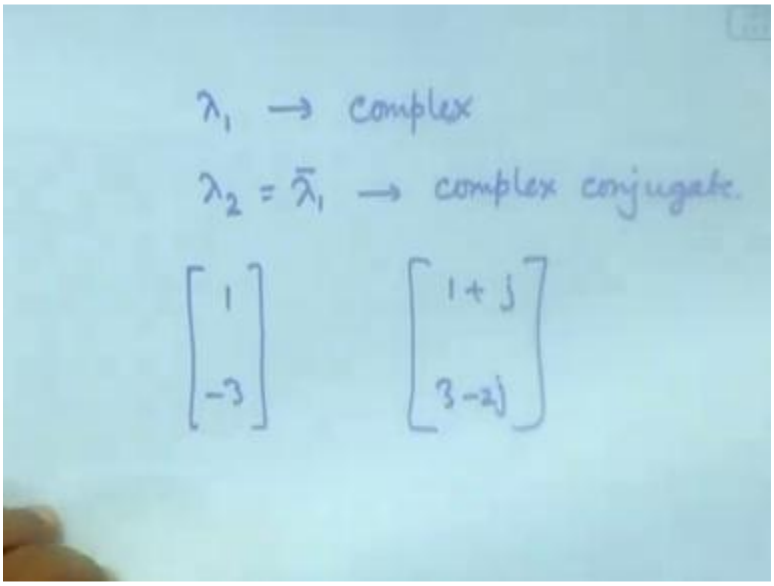

So, you have say an Eigen value that is lambda 1 which is complex, they will of course, be another lambda 2, which is lambda 1 bar complex conjugate. Now, the next stage was that we have to identify the Eigen vector, and then place an initial condition on that Eigen vector. Now, what the held in the Eigen vector is this case, that is obviously, a question because, if we had a complex number the Eigen vector will also be a complex Eigen vector.

What do you mean by complex Eigen vector, normally we were writing the Eigen vector like 1 minus 3, these are real vector. While a complex vector might be 1 plus j 3 minus j something like this, that is all. A vector in which the individual elements are complex numbers, but still the question remains in a real space, real valued space I can draw the direction of the Eigen vector. What is this obviously, this is not any real direction in the real value takes place. So, how do visualize them wait patience, because the way to visualize would be that still this equation is valid.

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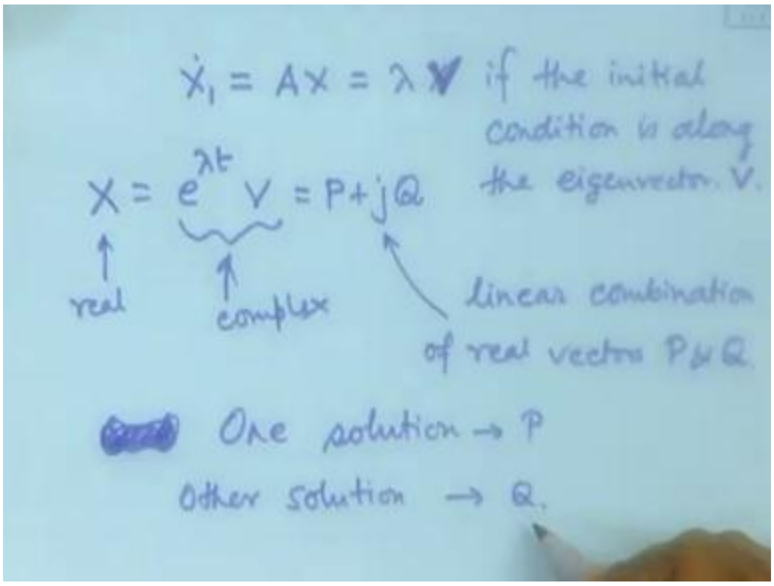

X 1 dot is equal to A X was equation that is equal to lambda X, if the initial condition is along the Eigen vector. What is the meaning of the term if the initial condition is along the Eigen vector obviously, it is not a physical vector, but still I will clarify that… But, if that is true then the solution is X is equal to times the Eigen vector V. Along the Eigen vector that say V, so if the initial condition is along the Eigen vector V, then this is true, this is then V lambda V.

X dot is equal to lambda V and then, solution is this, let us figure out this solution lambda is a complex number V is a complex thing, so this solution is also a complex number. What do you mean by a complex number as the solution, now this left hand side is definitely real. Why because, it is the solution of this equation X is a real thing, A is a real thing, X dot is a real thing, so this has to be real.

This tells us that the state goes from here to here and there therefore, there must be a real number, so this is real, but this is a complex. So, there is a problem no, that is not a big problem because this can always be expressed as P plus j Q, in that we expressed their notice this is a linear combination of a vector P, real vector P and the real vector Q. So, this is a linear combination of real vectors P and Q. So, X a solution is a linear combination of the real vector P and the real vector Q, which means the P and Q must individually solutions.

So, one solution P, other solution Q, the moment you have to real solution take an always constructed a any solution out of this combination. Let us see, so what was the logic again let us recapitalize or reiterate the logic that I was proposing. What I am saying is that if you have a complex valued Eigen value and complex value it is complex conjugate, the Eigen vectors will also be complex conjugate. There is no point using both the Eigen values why, because they carry no new information.

So, just take one of them, say lambda 1 obtain the corresponding Eigen vector, how will you come to obtain corresponding complex Eigen vector, I will come to I will show you, not problem. Obtain the corresponding Eigen vector, the moment you have obtain the corresponding Eigen vector you can write in equation like this, because the vector, the Eigen value. And therefore, you know the solution that solution will consist of the solution will be a complex valued solution.

That can be broken into a real part and imaginary part with j multiplied, so this term is a real valued thing take them separately. They will definitely be two separate linearly independent solutions and then, constant the final solution out of these two. Logic wise at least I know everybody is somewhat initially, some kind of discomfiture with this kind of situation, because I can visualize the vector. Real things are nice, because the vector I got a component here, it decays, it grows what is this. Yes, but at least there is flow in the logic, go it the logic let us see, if there is somebody who is not convince with this logic, I can take it to another logic also.

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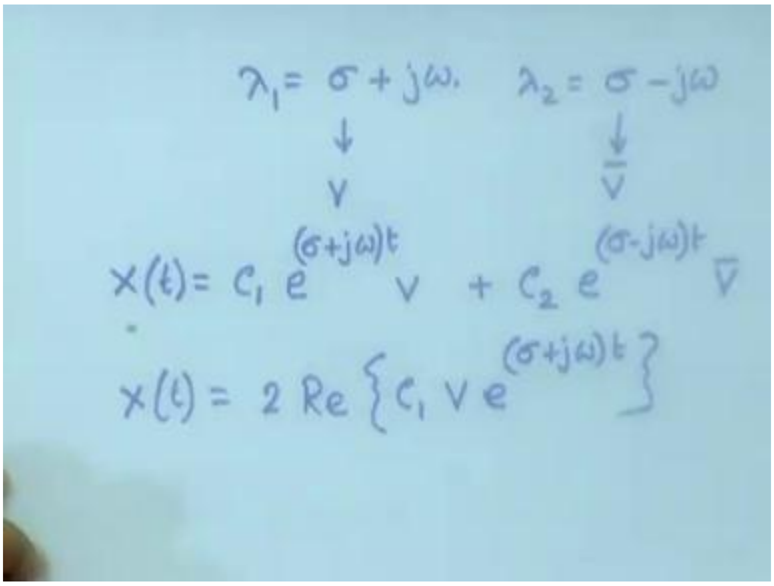

For example, suppose you have two solutions, two Eigen values which are say lambda 1 and lambda 2. Lambda 1 is equal to say sigma plus j omega, lambda 1, and then lambda 2 must be sigma minus j omega. For this the Eigen vector is a V, then the for this the Eigen vector must be V bar, which is the complex conjugate part of it, which is the complex conjugate part of it.

So, you have V and V bar and lambda and lambda 1 lambda 2, so the final solution is definitely X of t will be c 1, c 1 then e to the power sigma plus j omega times V plus t plus c 2 e to the power sigma minus j omega t V bar. C 1 and c 2, these things again have to be obtain from the initial conditions, that is what you have to done, but then in this kind of a setting the c 1 and c 2 will also the complex.

C 1 and c 2 will also the complex not only that, these two terms will also be complex conjugate. So, if you actually do it, do the algebra you will find the these two term will always be complex conjugate. So, ultimately what you have is X of t will become what I mean to say is that, it is breakable into two parts, one with the real part of this plus the real part of that, another with the imaginary part of this and the imaginary part of that.

It is breakable into two components, they will be twice the real part of c 1 V e to the power sigma plus j omega t. So, that will be the value here and they can be another part constructed out of, so this already construct this. So, you have this component which is, which are the real part and imaginary part, this can be separated out into a complete real part plus j into complete imaginary part and these two individually will become the solutions, that is the logic.

So, in the earlier logic we where will not considering both the Eigen values and Eigen vectors, but if do consider then also it least to the same result. In the next day we will illustrate this with an example, with an example where we will actually derive the solution, will you continue with the in next class.