Dynamics of Physical Systems
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Lecture - 20
Numerical Solution of Differential Equations

You have, been mainly dealing with how to obtain differential equations. Now, once we have learnt how to obtain the differential equations then we will do something with it, obviously we need to solve them, now what we have obtained, so far can be expressed in this form there are a few variables, the state variables.

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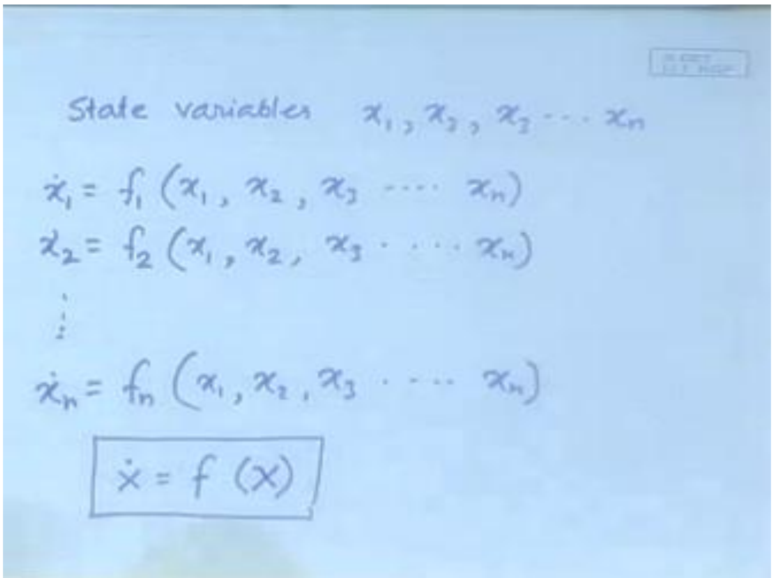

Normally, these are defined as the minimum number of coordinates that you need in order to specify the dynamical status of a system uniquely, so these are the state variables and you have also understood that normally, the positions and the momenta of the masses would form the state variables or electrical circuits. The charge across the capacitors, which are equivalent to the position variables and the current to the inductors, which are equivalent to the momentum variables are proportional to these would be the state variables.

So, there would be a momentum associated with an inductor, there will be a position associated with a capacitor and these would be the state variables. So, you have got the state variables say these are x 1, x 2, x 3 and all that and then what we have done, so far amongst to saying that we have obtained x 1 dot is equal to some function of x 1, x 2, x 3 to xn. Similarly, x 2 dot is equal another function of x 1, x 2, x 3 to xn, so on and so forth. xn dot is equal to, so that what we have done so far.

In brief, if you express that in a compact matrix form you will say X, capital X representing this whole vector dot is equal to some function of capital X, so this, what we have obtained, so far and then the next task is to solve it. Now, if the right hand side all these are linear; that means, this is a linear combination of x 1, x 2, x 3 and xn and all that in that case the right hand side is expressible in nice matrix form. And remember whenever you did obtain that kind of equation with a right hand side in a linear form, you should always express in the matrix from, in the answers.

Now, when that is true then the solution of this equation would be obtainable easily in an analytical manner, we will deal with that. But mostly you have also notice that this side is not expressible in a linear form in that case the numerical technique is all that we have in hand. But, notice that here even though you may have attended some classes on the numerical techniques. I will briefly only in one class cover the essentials of the numerical techniques, so that the whole thing is complete.

Now, what these define is something like this suppose, after we have defined this, the next conceptual step is to define a space in which the x 1, x 2, x 3 and all that these are the coordinates, thus would this takes place. So, since we have constant to draw and think on a two dimensional paper.

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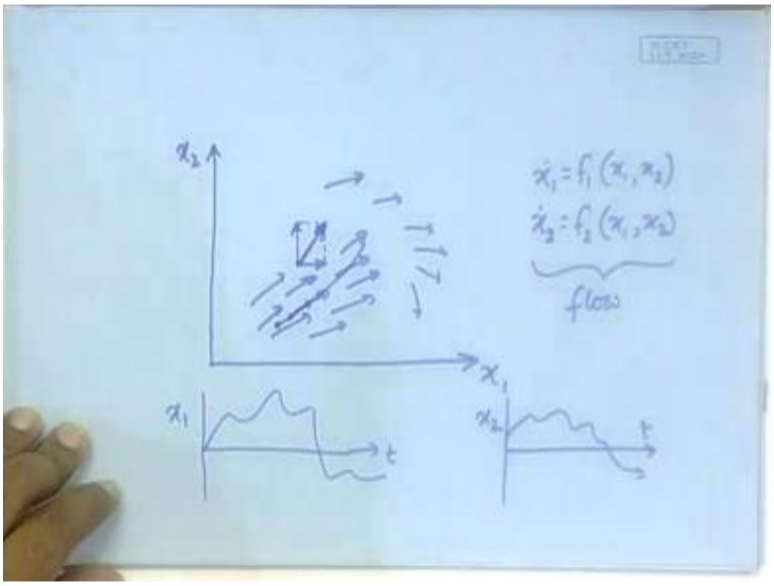

So, I will use a two dimensional state space to illustrate what I mean, you have say x 1 and x 2, what these ((Refer Time: 05:10)) differential equations actually specify is that say if my point is here which means, that now that system has a specific value of x 1 and a specific value of x 2 in that case ((Refer Time: 05:23)). What would this say, it says that x 1 dot is a specific function of x 1 and x 2, which means, I will write it, here x 1 dot is equal to f specific function of x 1 and x 2 which means that if x 1 and x 2 are known.

Then x 1 dot is known, x 1 dot means how would x 1 change in the next instant, so that is sought of something like this. Similarly, the equation x 2 dot is equal to another function of x 1 and x 2 will give this, not only in the direction, because x 1 dot means, how does x 1 change with time, it could be this way or that way and it is magnitude be given by the length of this vector, so from this you could infer that in the next instant, it would move in this direction in the state space.

So, what effectively the differential equation has given is this vector in the state space, similarly if the point is here it would give another vector in the state space meaning that if the point is here it would move in that direction. So, notice is then that this state of differential equations, actually define a vector at every point in a state space and say there is a set of vectors like this, and this is not difficult to see that if you have a initial function say here it would more or less follow the stream of this vectors.

So, anything that we try to think up regarding the solution of the differential equations either numerically or analytically should follow this conceptual, if you that it would follow the vector. It is like a flow, it is like a flow that is going and if you drop a particle somewhere in the flow, it moves with a flow, so that what we are trying to find out, that is why state of differential equations is also called a flow in literature, you will find that these what flow and a set of differential equations are being used interchangeably, the conceptual issue is this that a set of differential equations actually define a flow.

So, now you have a initial condition and the solution of differential equation would essentially means how it moves with the flow for a given time, now how to do that, so in that case you would get some kind of a movement in the state space. But you could say no, no I am interested in mainly how it changes with time say how x 1 changes with time and how x 2 changes with time, in that case you will get a wave form something like this, say this is your time and this your x 1.

So, for every value of time you get a value of x 1 that is a solution of the differential equation. Similarly, you have another solution for say time this x 2, these put together; that means, if you forget, now about the time rather you are more interested in how the point state moves in the state space then you would plot x 1 verses x 2 and that is what I have drawn here. So, now we will deal with this issue analytically later, but first let us find out how to obtain the numerical solution.

You might say fine, what I will do is at every point I will find out, I will find out the direction of the vector which you can do the movement you have it and then I will take a step from here to here along that direction. I come to this point at this point, again I evaluate the right hand side and I again draw the vector and so on and so forth, I drop. Yes that is the simple thing that, we can do and that is what is actually done in many cases, the only problem is that in that case how big a step will take.

And there is; obviously, the underline assumption that while you take the step from here to here the vector actually does not change, so you have tactically made that assumption that in that step length, the vector does not change much, but it could. So, this method would incur some error and we need to do some approximation in order to do better and that is what all the numerical techniques are all about. So, in concept what are you doing, we are taking the steps trying make sure that we are along the vector as close as possible.

So, let us start with considering a one dimensional differential equation that will be easier to start with and then we can continue to high dimensions.

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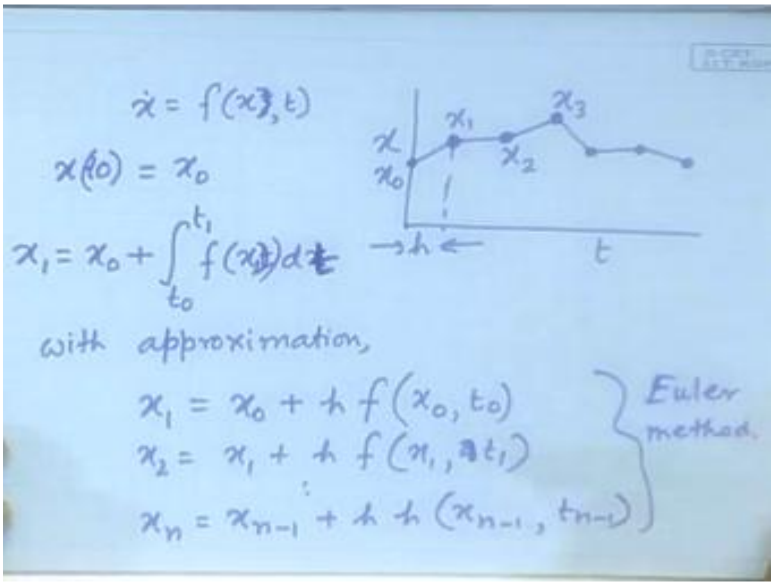

So we are considering… Say x dot is equal to some function of x only, very simple and what we are trying to do is to plot against time, the value of x which would be actually continuous line, but we will not be able to obtain the continuous line, because we are taking. ((Refer Time: 11:13)) final steps. And therefore, we will say that at the 0 time it was say here and then we this will tell us how does x change with time.

And therefore, along that we take the next step say at this point, we come here then and at this point again we calculate the right hand side take the next step say we come here, take the next step we come here, take the next step we come here, say this way you can proceed. But actually we are calculating in steps and this step length say is your h. So, if you let the initial value say x at 0 is equal to say x naught then a x, x 1 is what x 1 is this one, this is x 0, this is x 1 this is x 2, x 3 and all that.

So, x 1 is equal to x naught plus ideally you would like to do the integral, integral of starting from t 0 to t 1 f x this is what you want to do. But then this is not know and that is why you needed to take a step that is why you needed to make an assumption or approximation and the approximation was that this slop that was calculated at this point remains the same till this point. So, with approximation then we would say that x 1 is equal to x naught plus h f of, it is actually I should say also dependent on time.

So, it is not only dependent on x, but it also depended on time, so f of this is x naught and t naught, so at t naught whatever was the x naught f is calculated on the basis of that, times h the time is what you have to add to the original value. Similarly, x 2 is equal to x 1 plus h f x 1 t 1 and then you continue this, you have x n is equal to x n minus 1 plus h times f x n minus 1 t n minus 1.

Student: ((Refer Time: 14:28))

Yes you are right, this will be t; obviously, not differentiate this x t is depended on t and then you integrate over t, so this is how you calculate the different steps and this method with the first approximation is called the Euler method, this is the Euler method.

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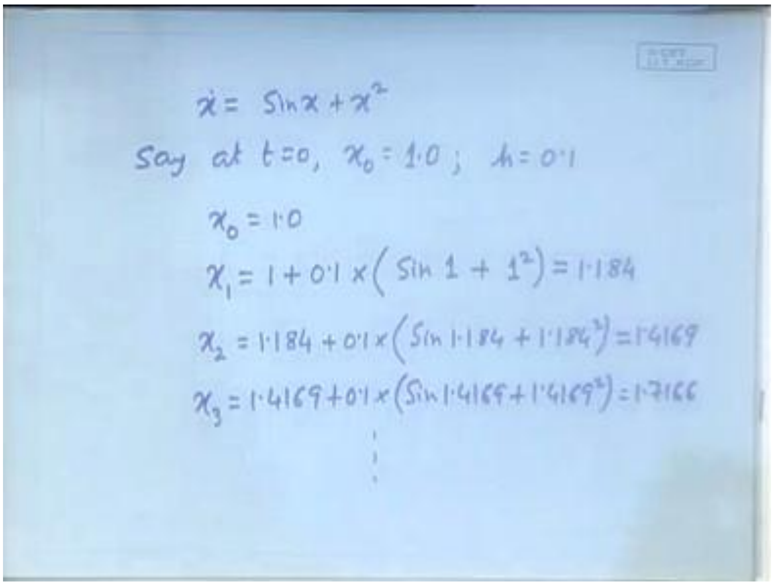

Let us illustrate this with the help of one simple example say your x dot is equal to sin x plus x square, say at t is equal to 0 x naught is equal to 1.0 and say h is equal to 0.1, then how would you calculate the first step, start x naught is equal to 1.0 we have known, x 1 is equal to how do you calculate that x naught plus, so 1 plus ((Refer Time: 16:10)) h times f calculated at the initial condition. h is 0.1 times this thing sin 1 plus 1 square that comes out to be, so you have got the numerical value 1.184.

x 2 is equal to this plus 1.184 plus 0.1 into sin 1.184 plus 1.184 square is equal to 1.4169 and so on and so forth you can continue. x 3, I will write x 3 because we will compare with the other methods also 1.4169 plus 0.1 times sin 1.4169 plus 1.4169 square is equal to 1.7166 and so on and so forth, we are taking, ((Refer Time: 17:45)) steps exactly like this, so this is how you would calculate by the Euler method.

Now, it is clear that in the Euler method at every step the errors will build up and naturally when you are calculating x 2 you are putting the result of the x 1 in here. And all that and therefore, if there is an error, it will build up in the next step, it will build up in the next up and therefore, there is a possibility that will that will blow off. In order to avoid that you need to make the h as small as possible, the movement you make very small h the competition becomes very slow, so we need to do something better. In the step of the assumption, we can say that say we are now…

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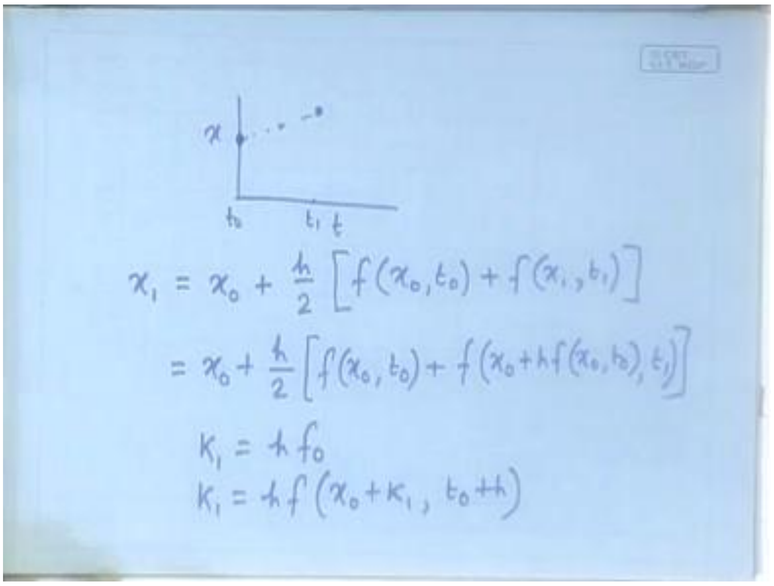

This is t, this is x, we are also here and in somewhere suppose, you know that at x 1 you are here what way I will come with that later, so this is t 1 time, this is t naught time then the next approximation is that what was our ((Refer Time: 19:01)) Initial guess, it was the slope remains the same from this point to this point. Now, this slope will still we assume that it remains the same, but will sort of improve upon the value of the slope, what can you say will say will not take the value here, rather we will take the value somewhere in between.

So, in that case our solution at x 1 becomes, x 1 will become x naught plus this is a total h, this is h by 2, h by 2 times then we have to take the average it is f of x naught t naught this one plus f of x 1 t 1, so what I have taken the slope at this point and this slope and this point we have taken divided by 2, so it is a average of the two. So, we have taken the average slope, but still the question remains how do you know this, because you have not yet, so you do not known this.

Yes, then the nice thing is that you have the earlier. ((Refer Time: 20:20)) Calculation based on the Euler method, so which you can substitute here that was approximate, so this will be approximate all right, but ultimately this will be a little better approximation then what it was earlier, so you substitute it here. So, we have x naught plus h by 2 it would be f of x naught t naught plus this whole thing f of x 1 was x naught plus h f x naught t naught comma t 1.

Normally, when we will write this we try to abbreviate, so that the right hand side can be expressed in a more compact form then we say, K 1 is equal to h f naught, f naught is f calculated at x naught. And K 1 is h f of x naught plus K 1 coma t naught plus h, so if you define these then we can simply write on in terms of that.

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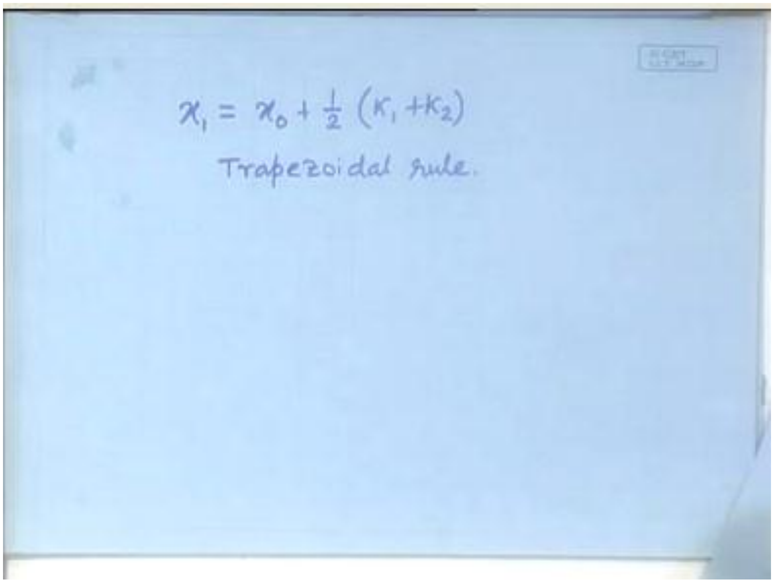

X 1 is equal to x naught plus half K 1 plus K 2 ((Refer Time: 22:10)), so this term is K 1 this term is K 2, so we have x naught plus half K 1 plus K 2, so then these is the next type of better approximation. So, what have we done, we first say that we will take the, we will improve upon the slope that you take constant between this point and this point and in order to improve upon we say that earlier we are taken this slope to be constant at up to this point. Now, it take we will say no, no the slope here is constant from this point to this point; obviously, better approximation.

And then in order to do that we wrote it this way, the slope is then this slope this plus this by two, so you have a better approximation, then we substitute it from the Euler method. Whatever was this value and then we wrote it separately this way this part is this, this part is this, K 2 is equal to fine, so ultimately you have this, now this is called the trapezoidal rule, trapezoidal means whatever. ((Refer Time: 23:35))Done is essentially say that it is a trapezoid and then we have taken the middle point.

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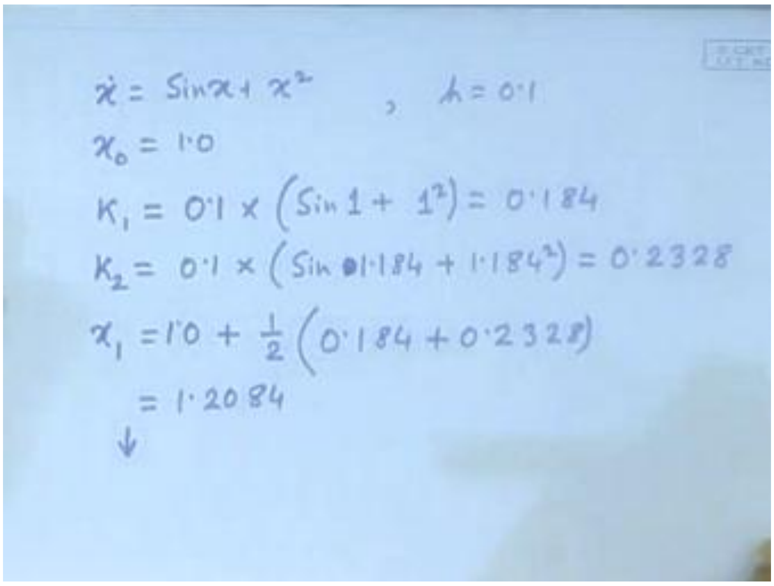

Let us illustrate the same problem that we did earlier on the basis of the trapezoidal rule, our problem was X dot is equal to sin x plus x square and our x naught was 1.0, h was 0.1. We now calculate with the help of the trapezoidal rule what will we say. We will say that when calculating x 1.

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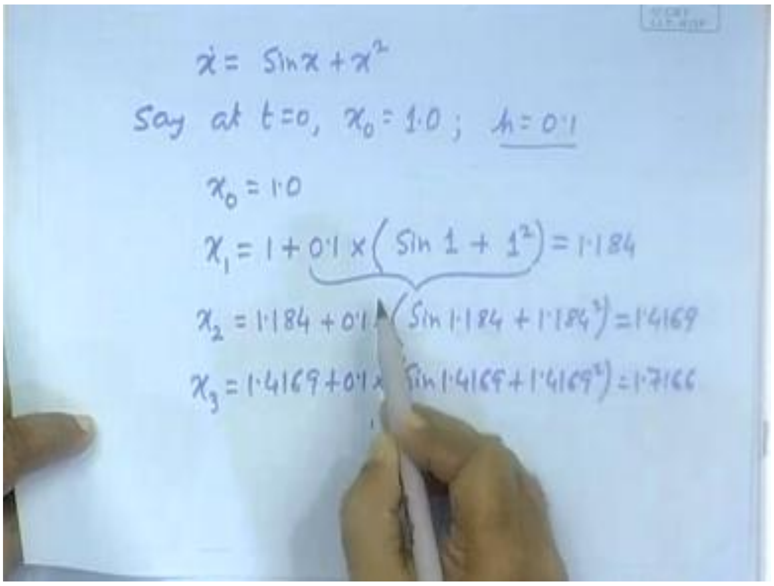

We need to calculate K 1 and the K 1 is exactly the same as whatever we have calculated earlier, so no this part, this part not 1 plus where is it where did I keep it ((Refer Time: 24:58)), h times f naught, f naught is only this part, so it will be, it will be 0.1 times sin 1 plus 1 square is equal to 0.184., K 2 is this substituted here 0.1 times sin 1.184 plus 1.184 square is equal to this again we have calculated it is 0.2328. What I have calculated this, we can substitute ((Refer Time: 26:04)) in this equation to obtain x 1.

X 1 is then x naught was 1.0 plus half K 1 plus K 2 is 0.184 plus 0.2328 is equal to 1.2084. Where is the calculation last time yes, see in the Euler method at the end of x 1 we had calculated 1.184 ((Refer Time: 26:54)), while by using the trapezoidal rule we have found x 1 is equal to 1.2084, which is a better approximation than the earliest step. Similarly, you are x 2 in order to calculate x 2 go to next step then we calculated it again.

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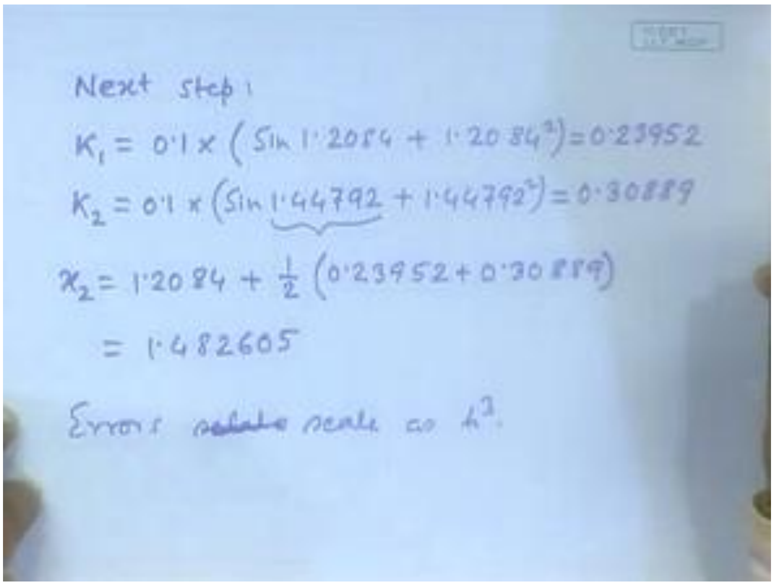

Next step, again K 1 is equal to 0.1 times, now these has to substituted it is sin 1.2084 plus 1.2084 square is equal to 0.23952, K 2 how to calculate K 2, how to calculate K 2 yes

Student: ((Refer Time: 28:10))

(Refer Time: 28:09) So, here to calculate the f at the next instant, so it is 0.1 times sin it would be 1.44792 plus 1.44792 square is equal to 0.30889, once you have done this you can calculate x 2 is equal x 1, 1.2084 plus now half of K 1 plus K 2 0.23952 plus zero 0.30889 is equal to 1.482605. Now, you notice that how much is it better than the last calculation ((Refer Time: 29:25)). This was calculated on the basis of Euler method 1.4169.

Where in this case x 2 is equal to 1.482605; obviously, that is a better approximation, but still is an approximation. Now, it, so happens that in this method you have still errors, but the errors scale as h cube, so you have learn the trapezoidal rule also.

Student: ((Refer Time: 30:11))

Where did we get it from, yes x 1 plus K 1, so if you do that you get this, now you notice that we are going into successive stages of better and better approximation. The first was the Euler method, which is also called a first order approximation. Second was the trapezoidal rule that is called a second order approximation and then you can easily imagine that we can make third order, fourth order, fifth order approximations? Because what was the logic in the trapezoidal rule we said that this point. ((Refer Time: 31:11))

We will be approximate by approximated by the first order approximation, in the next stage you can say no no, no this point will be approximated by the second order approximation and therefore, the midpoint again will become a better approximation. A fourth order approximation would be this point will be calculated by a third order approximation and then this slope will be approximated in the fourth order. There by, you can go on increasing the accuracy of the approximation and this whole logic were laid out by two people Runge and Kutta, so the whole logic is called the Runge-Kutta method.

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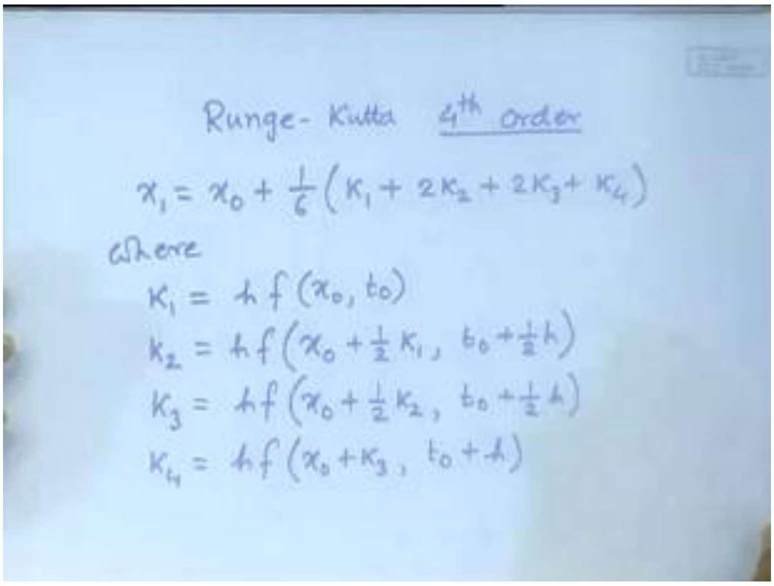

Therefore would be pronounce as a hu, so is not Katta is Runge and Kutta that is why the Euler method is also called the first order Runge-Kutta method. The trapezoidal rule is also called the second order Runge-Kutta method and naturally you can easily imagine how the third order Runge-Kutta method would look, fourth order Runge-Kutta method would look, the fifth order Runge-Kutta method would look and so on and so forth.

And if fact all these are used, but once you understand the conceptual clarity with conceptual clarity how the progress in the order of approximation, you can the whole thing falls in place, out of all the various things what is most commonly use is a fourth order approximation. So, without giving the details which you can find in any numerical analysis text book, we let us just write down the steps in the fourth order Runge-Kutta method.

In that case the x 1 will be x naught plus something that will be one sixth of K 1 plus 2 K 2 plus twice K 3 plus K 4 ,where K 1 is equal to the first one; obviously, h f x naught t naught. K 2 will be h f x naught plus half k 1, t naught plus half h, K 1 is known so you can substitute it here, h is know so you can substitute here and you can evaluate the function therefore, K 2 will be known.

K 3 will be h f x naught plus half K 2 t naught plus half h and K 4 will be h f of x naught plus K 3 and t naught plus h. So, once you have calculated this, this you can substitute it here and you can get the first step, then the second step is all these exactly the same only in place of x naught you substitute x 1 and so on and so forth. So, let us just illustrate one of these calculations, we still keep these in the hand, so that we can compare, still we are going with the same equation.

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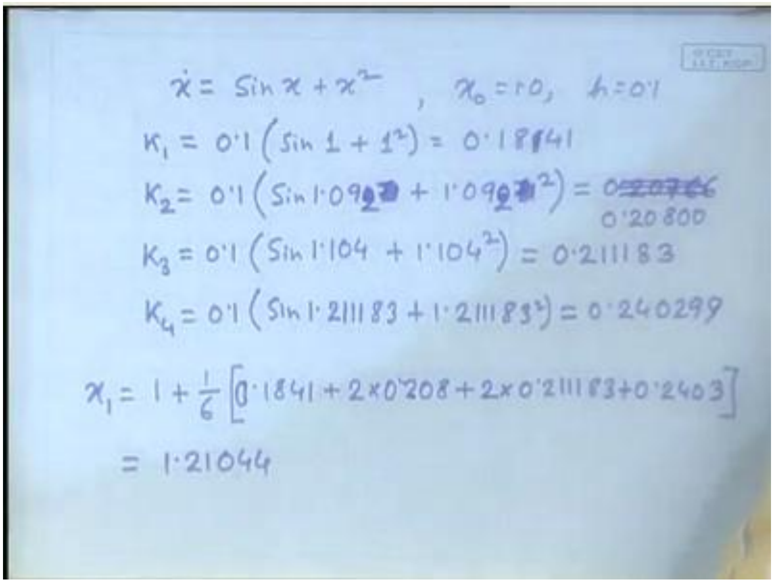

X dot is equal to sin x plus x square, we know that our x naught is equal to 1.0, h is equal to 0.1, so we first calculate the K 1, K 1 is what we already did. So, that would be 0.1, we have done it, just it will be right, so that we can write in the same style, sin 1 plus 1 square is equal to 0.18, I will write it to higher decimal places 1 4 1, then your K 2 will be 0.1. ((Refer Time: 36:24)) h f of x naught plus half K 1, so x naught plus calculated at that particular point. It would be sin, now just calculate, what is x naught was 1.0 plus half of this, it would be 1.09071 plus 1.09071 square is equal to 0.20766.

Student: ((Refer Time: 37:15))

Now, you have K 3 which will be ((Refer Time: 37:38)). H times f of x naught plus half K 2, K 2 is now known substitute it here, you get what 0.1 times sin of what x naught plus half of this.

Student: ((Refer Time: 38:00))

It will, this one is calculated on the basis of with one now, can you just check with a calculator, weather the mistake was here or what ((Refer Time: 38:23)). Here because I had done the calculation once which one of the steps ((Refer Time: 38:29)), may have been mistaken or I may have written it wrongly ((Refer Time: 38:34)). Whereas this is correct, or this is correct.

Student: ((Refer Time: 38:45))

So, what is the result of the this step then…

Student: ((Refer Time: 39:00))

Can, you tell me what is the right hand side 0.20800. So, now K 3 will be calculated with sin 1 plus half of this, how much is that 1.104 plus 1.104 square is equal to should be 0.21 something right how other come to be 0.211183. Yes and your K 4 is 0.1 times.

((Refer Time: 40:24))

K 4 is x naught plus K 3, so we can directly add this sin 1.211183 plus 1.211183 square is equal to…

Student: ((Refer Time: 40:44))

0.240299, fine, so we have calculate the case now your x 1 will be x naught 1 plus one sixth of K 1 which is 1.1841 plus twice into 0.208 plus.

Student: ((Refer Time: 41:40))

Which one, 0.1841 plus 2 into 0.208 plus 2 into 0.211183 plus K 4 is 0.2403, anyway you are able to calculate, so let us not drag with it 1.21044, you would notice that calculation is definitely ((Refer Time: 42:37)). Better than the first approximation that we did it is also, ((Refer Time: 42:42)) better than the second approximation, where is it x 1 is here second approximation that we did. Now, it is ((Refer Time: 42:51)) rather simple to write down a program to proceed this way, so you should all have to do that; that means, that will be an assignment. But let us first understand how to extend these two higher dimensions, we have done it in one d. If it is a two dimensional system exactly the same logic box, but independently in the two direction, so it will be…

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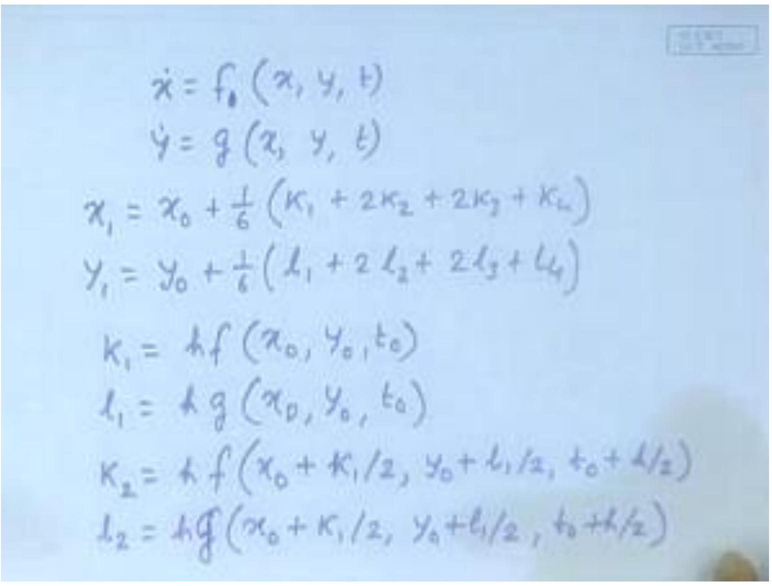

Suppose we have the equation given as x dot is equal to some function write into easier x y and time, y dot is equal to another function of x y and time. In that case your x 1 will be x naught plus one sixth of K 1 plus 2 K 2 plus 2 K 3 plus K 4. And y 1 will be exactly in the similar way plus one sixth of another function will say l 1 plus twice l 2 plus 2 l 3 plus l 4 and these will have to be calculated exactly the same way for example, K 1 will be what was it h f. Now, this only thing is that it has to be calculate x naught y naught and t naught, l 1 is h g of x naught y naught t naught.

Then is next step K 2 is h f of x naught plus K 1 by 2, this has to be also written in terms of the y, y naught plus l 1 by 2 t naught plus h by 2, so the conceptual change is that while you are changing this the y has to also be to change to the next step. Similarly, l 2 is equal to hg x naught plus K 1 by 2 y naught plus l 1 by 2 t naught plus h by 2. Next step, next step, next step, goes in the exactly the same way. The result of all these is that if you write down the solution programs and studied with some initial condition.

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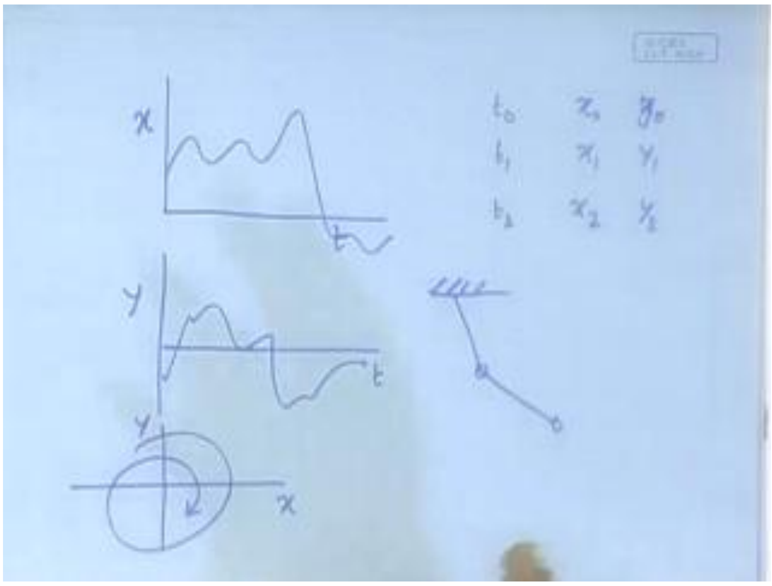

Then you would be able to get a result file that will be the t naught and the x naught, t 1 x 1 not only that y naught y 1, t 2 x 2 y 2 and so on and so forth. Three columns if it is a two dimensional system, you get three columns, so as the time progresses how does x change, how does y change and then you would able to plot it like t verses x, some kind of a wave form, y verses t again some kind of a wave form, not only that also x verses y some kind of a not a waveform, but it is movement in the state space.

So, your assignment will be to write the solution program in fourth order Runge-Kutta method not the others, not necessary because they are relatively less accurate than the fourth order, forth order is accurate to the extent of h to the power of five. So, it is more accurate and then you will have to solve any one of the problems that I give you in the tutorials for example, you have written down the code for the written down the solution of the inverted pendulum.

We have done it for the double pendulum the one that was like this, any one you take and obtain the solution starting from some initial condition, here the initial condition would be some initial position and the momentum of these two masses. You have written down the solutions, already as part of the tutorial and for the part of this assignment you will have to obtain the solution. Let us for the sake of uniformity do not take just any, but do this for this system, we have all done it as a part of your assignment earlier. Now, you obtain the solution starting from some initial condition. Now, the additional necessity that one does is normally, if you write the differential equations and then try to solve it one sets the h.

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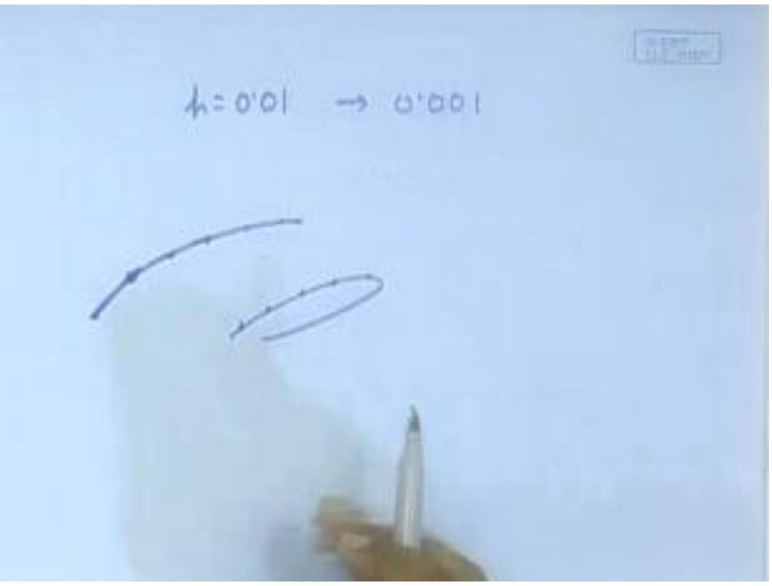

Something a small value something like this, but then this h may not be sufficient where for example, imagine that you are taking the step length like this and you are calculating. So, if it is going like this then the amount of approximation that would be involved in going from here to here to here to here. Even if that approximation is accurate to the h to the power of five extent that is reasonably good, but suppose it say some like this and you are taking the step like this, then what is the solution has sharp turns; obviously, this will incur error still.

The other problem is that in order to avoid entering that error you might say that no, no, no I will then change it to 0.001, you will be better off, but then the program will be ten times slower. That is why one is to strike some kind of a compromise and the compromise is often not available a priory, you have calculated it you have got an orbit and you never know if it is correct, so that is a typical problem in solving the differential equations.

Now, given a specific problem, you should be able to see whether you are likely to encounter this kind of situations, When are you likely to encounter that is what I refer to in the last class as stiff systems, stiff systems are where there are completely different time scales within the system, something is oscillating say at megahertz frequency and something else is oscillating say hertz frequency then you have trouble, because something is really oscillating very fast, and therefore your program should be tuned to that frequency.

That means the h step will be, so small as to be as to be able to see those oscillations also, but if you said the h larger then you would be completely missing those oscillations. If you try to capture, that oscillations the program will horribly slow and you will not able to proceed much. It may. So, happen the some system is more or less smoothly proceeding, so you that you are happy, but at some point it takes a turn, while it smoothly proceeding like this, like this you never know that is going to encounter a turn.

So, something has to be done, something as to built in to the built in program, so that you can you can you can find out that yes I have encountering a problem, so I need to reduce my step size and that is why most of the model programs for the Runge-kutta solution incorporate. Some kind of a variable step size routine, the variable step size routine is where depending on the character of the solution you vary the step size, where it is going smoothly in a more or less linear fashion, you make it large.

So, that it calculates faster, but where ever is something like this you make it small. How do know, the nice way to know is that we have already talked about the second order, third order, fourth order, fifth order Runge-Kutta methods. So, supposing you have started here and you are you are trying to find out whether before the next step there is going to be term, where will that be manifested, it will be manifested in the calculator value of x 1, if you calculate by the Euler method, by the trapezoidal rule, by the Runge-Kutta method, values will be widely different.

But, If it is more or less smooth or linear then there will be more as the same, so what one does is normally, you set a tolerance and then calculate it for the with the third order Runge-Kutta method, as well as fourth order Runge-Kutta method. If the difference is beyond a tolerance limit then you make the h say half, because that is it detected no there is a, there is a turn, so you make it half and then do the same thing once again when it goes below the tolerance limit use the value of h and calculate.

But it has to be again dynamically done, so that before the next step, if you find that no, no it is below the tolerance limit increase it, till it still remains below the tolerance limit, so that when it goes like this you should be able to calculate with a larger time step. In your program incorporate that also, because otherwise for practical systems your program will not be able to execute properly, include that also.

Your notice if you look at the MAT LABS code, MAT LABS solution of differential equations are given in two types of ode's, two ode solutions one is called ode23 or ode34 and ode45. What is it mean it calculates it by the suppose ode45 it means that it calculates by ode4 Runge-Kutta fourth order method. The next step also calculated by the Runge-Kutta fifth order method, the next step the compares with there is a tolerance and then does the variable time scale algorithm, you have to incorporate the same thing. So, that is how you would write the program as the assignment to be done over the next week end, fine; that is all for today.