Chaos, Fractals and Dynamical Systems
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Lecture No. # 09
The Logistic Map and Period Doubling

In the last class we have seen that any continuous time dynamical system can be discretized and the method of discretization was that if it is an autonomous system means there is no external periodic function then we would place a Poincare plane, any plane in the state space that is called the Poincare plane and you observe the piercing through the Poincare plane. What would you do if it is non-autonomous? That means there is some kind of external forcing, you would have to observe it in synchronizing with that external forcing function and that defines the periodicity. We also understood that if this system is periodic period one, limit cycle then on the Poincare plane you will see one point as a steady state behavior, if it period two it is see 2 points, period three 3 points and so on and so forth and for periodic orbit you will see infinity of points.

The question then is that would be infinity of points be scattered just any way. You would imagine that there is a Poincare plane on which there will be a scatter of points but then would the scatter of points have any shapes, structure, organization or it is just some points. It turns out that always the scatter of points have some structure that is what tells us that the whole thing is not completely unruly or unorganized. There is a very deterministic phenomenon going under it which is organizing that kind of structure. Now whenever you place a Poincare plane and you see things happening on that. Essentially you are seeing a picture, some kind of a portrait of the system and that is often called phase portrait. A phase portrait is nothing but the picture of the orbit in general that refers to chaotic orbit as it appears on the Poincare plane. In some books you will also find this same word, same phase used to mean the continuous time orbit in the state space but often that is called a phase plane orbit.

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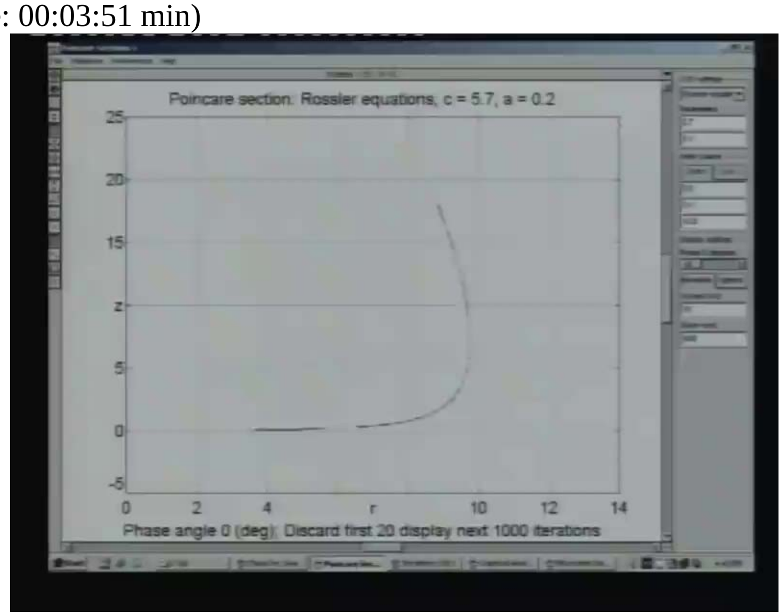

Now let us see some phase plane orbit for example this computer application is extremely slow. This is the phase portrait for the Rossler equation for which you have already seen the continuous time orbit. The phase portrait has a very distinct and definite structure. It has infinity of points but it has a very well organized structure. So it not just any arbitrary shape. For any system for example if you do the same exercise for the Lorenz system there also you will see a very definite structure. In that case it takes the form something like this. His question is what do you mean by a definite structure? This is not just any way scattered around the whole thing. The points fail into a particular shape, even though there are infinity of points the infinity of points are in a particular shape and you can almost write to in equation for that. So that is what is meant by a definite structure system. This is the phase portrait of the Rossler attractor.

Now what do we do next? In the Rossler system as you have seen earlier that as you are changing the parameter, the behavior goes from a periodic orbit, period one orbit to a period two orbit to a period four orbit. I would like to understand that might happens for any system, any practical system and we might like to understand how the behavior of the system changes as you change a parameter. Now as I told you that if you look at the phase portrait then when it is period one it is just one point, when it is period two that is 2 points, when it is period three, three points and now there are infinity of points.

The point is that as you are changing a parameter, all these are happening. So you would like to have a complete picture of the events that go on as you change a parameter because in a practical system there would be some parameters and you would like to know in which parameter range, what kind of stable system behavior pertains. There is a nice way of doing that, invented by the people non-linear dynamics. There is a nice of plotting the behavior as a parameter is changed that is called the bifurcation diagram where what we essentially do is now let’s look at this.

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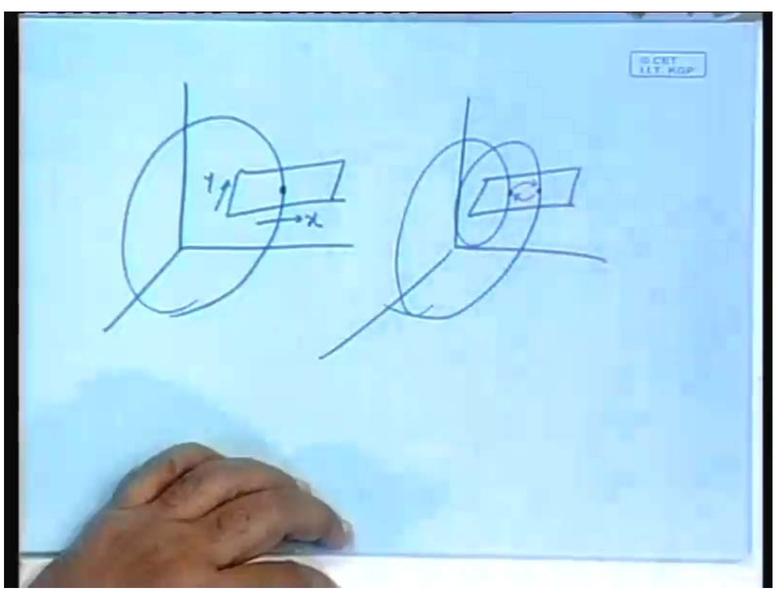

Let’s look at the paper. If you have a system where the behavior is some kind of a periodic orbit and you have placed a Poincare sections somewhere here and you are observing this point. As you are changing a parameter, this point, this behavior changes to say something like this so you are observing these two points. Now on the Poincare plane there would be one x coordinate another y coordinate.

Suppose you observe one of them and you keep on keeping track of the subsequent values of this x on the Poincare plane. Then what will happen in this case? You will see all the points having the same value of x. If you make a data set containing some hundred subsequent points of x, you will find all of them having the same value. In this case you will find 50 of them having this value, 50 of them having that value and they toggle in the sense that this point goes to this point and this point goes to this point. So you see a pair of points (Refer Slide Time: 08:30).

The way to draw the bifurcation diagram is for a particular parameter value eliminate the transient, allow the system behavior to settle on to its asymptotically stable behavior and then plot some hundred such discrete points. These are the discrete points so plot it. If the parameter is plotted in the x axis and those points are plotted on the y axis. If the parameter value is such that all the points fall on same location and you will see only one point. If the parameter value is such like this period two orbit that the two values of x would toggle between two things then you will see two points and so on and so forth. As you change the parameter then you can study the system behavior and this is often done by the help of a Poincare maps and I will just show you.

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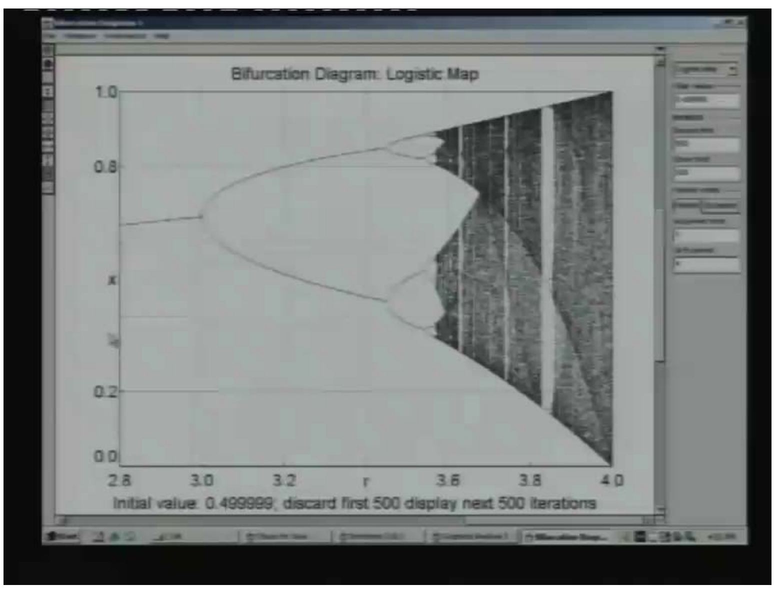

If you observe this then this is a bifurcation diagram of the logistic map that I brought forth in the last class. Logistic map means the one that you saw in the last class, mu x_n (1-x_n). Initially the behavior would be period one and all that I will come to that but here looking at this it would be immediately clear to anybody that in this range up to this parameter value, the behavior was period one.

What does it mean? It means that at this point where my cursor is, at that point actually there were large number of points but all the points fail on same location. That is why you see just one point here, another point here, one point here but here for this parameter value here, the r is being varied, the parameter varied there are two points and at this point there are 4 points and at this points there are 8 points. So by looking at this anybody would be able to immediately say that from this parameter value to that parameter value the behavior was period two, from this parameter value to that parameter value the behavior was period four and so on and so forth. This gives you a sort of panoramic view of this stability status. For some range the period one behavior is stable, for other range period two behavior is stable, for another range period four behavior is stable so on and so forth. By the way I will come back to this later.

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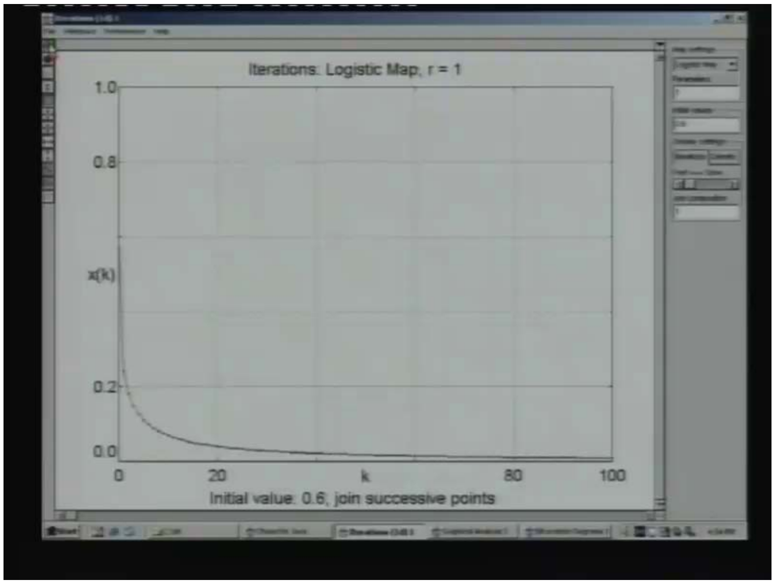

Let us see what we saw in the last class as the behavior of the logistic map. Here we said r is equal to 0.2. Let us change the parameter to say one, r is mu. What I wrote as mu here they are using the r, so it is same thing. If you change it to one then also you starting from something like 0.6 and it is decking to zero. That is what you expected by doing the calculation.

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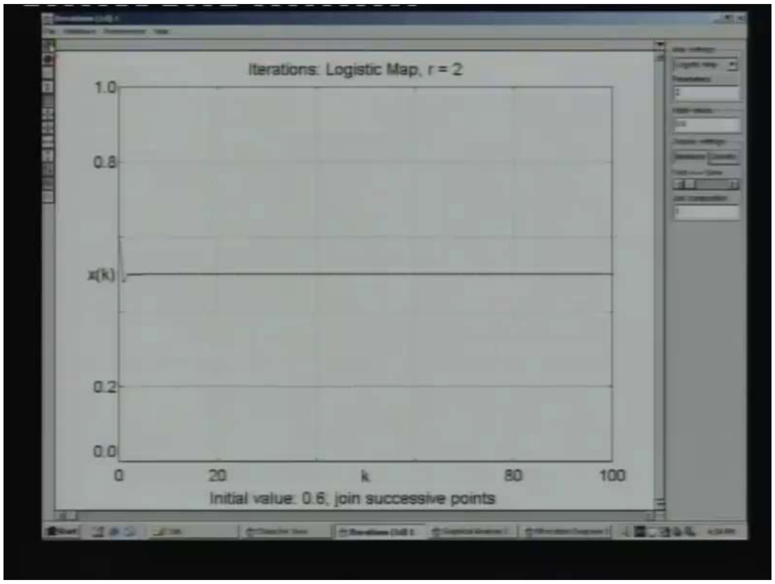

If you change it to two it is not decking to zero, it is ultimately stabilizing at some value. Can you see that? Change it to 2.5, it is stabilizing to another value.

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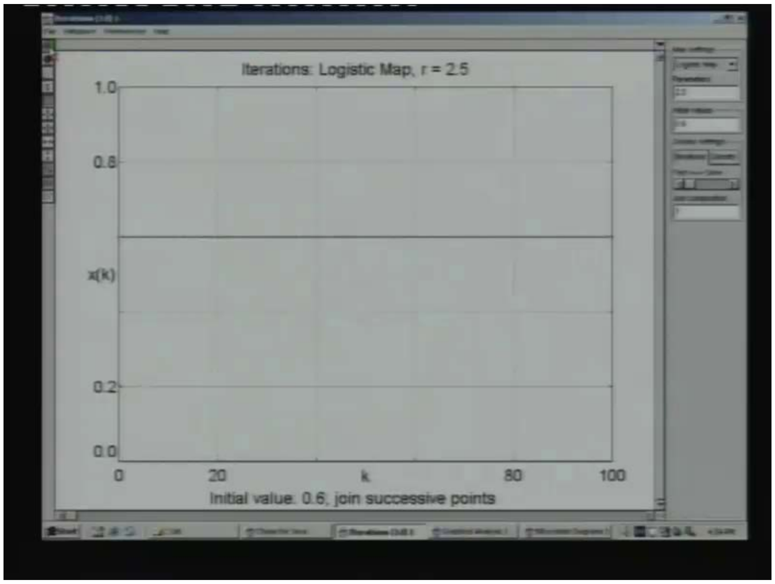

2.9 it is stabilizing to another value. See there is some kind of oscillation but nevertheless it is coming down to a stable value.

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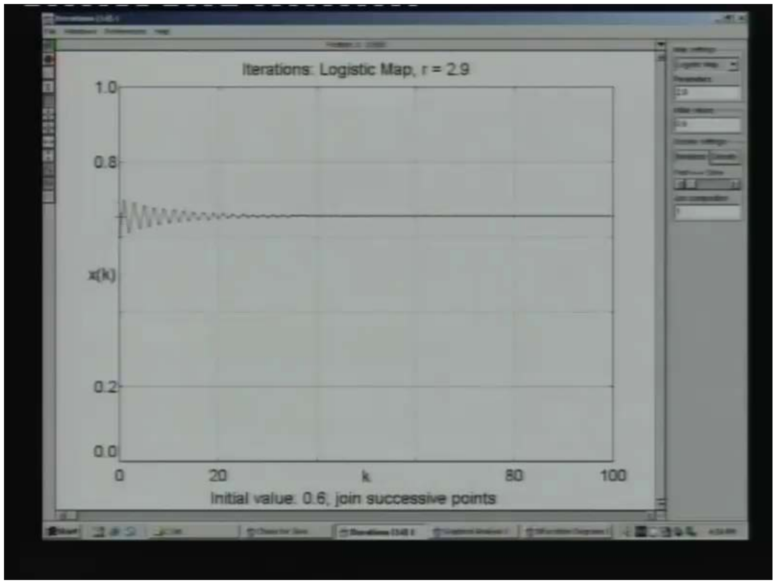

Now let us change it to 3.1. So the last day you did the calculation. I asked you to do that and you saw that at a certain parameter value which was you calculated it to 3 and at that parameter value, the period one orbit becomes unstable and I will show why it happens. When you saw in the bifurcation diagram, let the period one behavior is changing to a period two behavior. Naturally it raises the question why and now we are trying to answer that question. Again if I can answer what is happening here, then it is not difficult to see that a similar thing is happening here. Again a similar thing is happening here. Do you notice that what is happening essentially is the repetition of the same kind of events?

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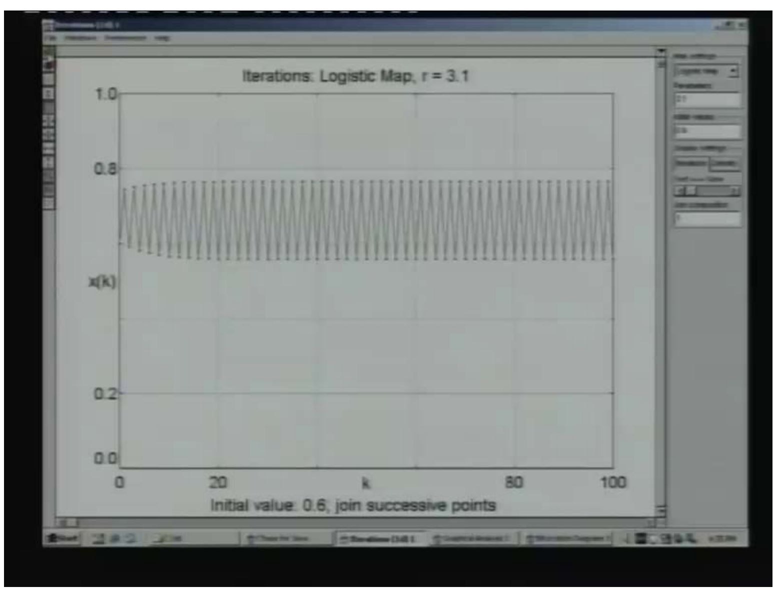

So it essentially demands the explanation of one them then you like extrapolate that same explanation to find out what is happening in the rest.

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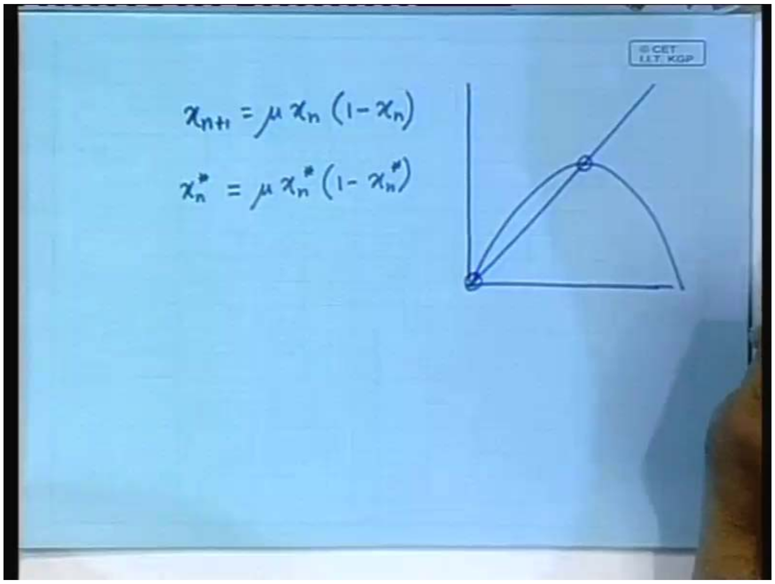

So let us start with the equation that we wrote last time x_{n+1} is equal to mu x_n 1- x_n that is where we started. What was the method of analysis? We say that the x_n^* which is the fixed point is given by; the left hand and right hand side are equal, so x_n^* (1- x_n^*) solve it, you get the location of the points. You have already seen that there are two positions, there are two equilibrium points and we had decided that 45 degree line is an important thing that we need to watch out for and it has a curve something like this. There are two points at which they intersect so one is at zero, the another is at mu -1 by mu so that is a point. The issue is then why did this orbit become unstable at a certain point? This was the orbit that where observing here, when we are observing this point we are observing it at this one and not this one. Why not this one?

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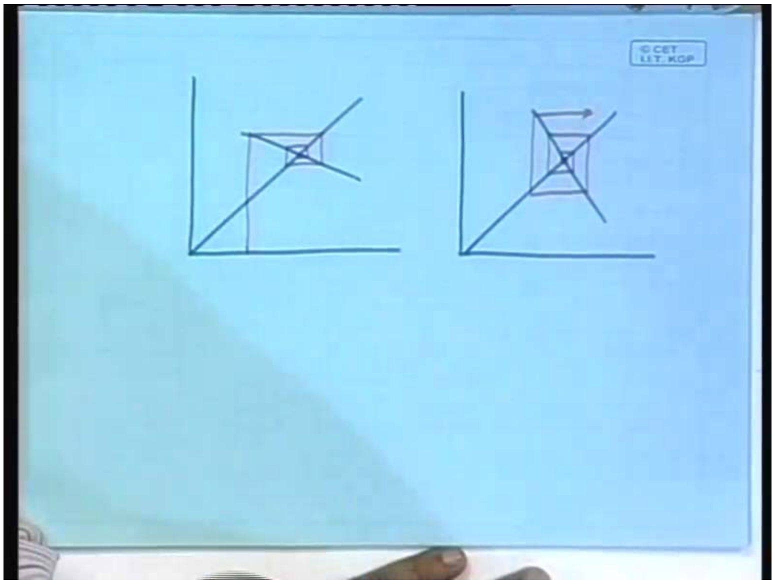

For that we had already seen, again recall what we said in the last class. If say this is the 45 degree line and the graph of the map is like this having a slope less than unity. Then if you start from here, remember the graphical analysis that I talked about, it will go like this. You have to go the 45 line, you have to come back here, had to come back here so on and so forth it converges. If the slope is at that value is greater than one then you start from a point, you see it diverges. So we had concluded that this stability of the equilibrium point depends on slope of the derivative of the map at the equilibrium point at the fixed point. So naturally whether or not a particular fixed point will be stable depends on the slope there.

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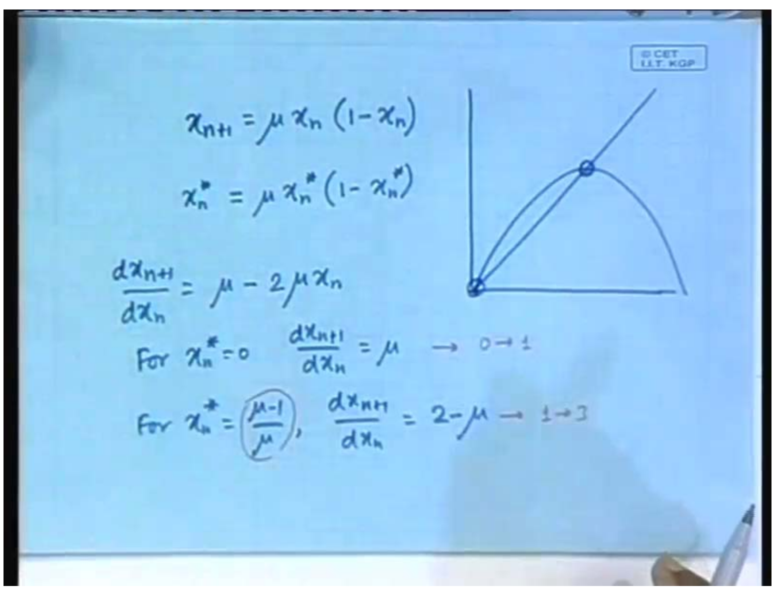

Now here you can see the slope is larger than unity and therefore this point even though that is a fixed point, the orbit will not stay there. Here this slope is less than unity so the orbit will stay there. So we have the map express like this and we are interested in the derivative dx_{n+1} by dx_n is mu minus twice mu, if you take the derivative you get this. So if you talk about the equilibrium point or fixed point at the origin, at the zero position then you put zero here as a result the derivative will be just mu which means that's so long as mu is less than one this point will be stable. So as long as this is less than one this point will be stable that is exactly what we observe when we did this less than one say 0.9, zero is stable. Things are converging to zero so zero point is stabled then.

What about the other one? Now do this exercise. Would this position of the equilibrium point of this fixed point here and evaluate this. So for x_n^* is zero it is… and for x_n^* is equal to dx_{n+1} by dx_n is 2, if you substitute this here. At what value of mu then would you expect this particular equilibrium point to become stable? Here mu minus this, 1 to 3 it will be stable. So this one would be stable up to one and this one would be stable up to 3. Let's change the range of the bifurcation diagram.

What is it? Where is the parameter? So we have decided that this fellow will be stable from zero to one and this fellow will be stable from 1 to 3. At these three point, when the parameter value becomes 3 I expect this fellow to be unstable and that is exactly what is happening here. At 3 I know this fellow will become unstable. If it is unstable then obviously if I keep on iterating this map, the iteration will not go there and that is the exactly why you do not see this line extending after that. That is why you do not see this line extending after that part. This calculation tells you that this point is any way there, even when you set mu is say 3.5, this value is there.

You can calculate this value which means that this value is there, this point is there but now it is unstable. So try to understand the distinction between things not existing and things existing but is unstable. In both cases you don't see them but only theory allows you to understand that this fellow is there and there would be effects of that, it will have its own dynamical limits I will come to that later but try to understand then it become unstable still the period one orbit is there but it is unstable.

Now let us see what happens when this fellow becomes unstable means when you have changed the parameter to a value greater than 3. What do you expect? In order to see what you expect, we definitely know that the period one orbit has become unstable, we either expect the orbit to diverge to infinity or would you expect some other orbit to become stable. So we start by checking if period two is stable, period four is stable and all that, so let's see if period two is stable. How do you find out if period two is stable? On the bifurcation diagram of course you can but that is only simulation, it is not theory so let's try to do it theoretically.

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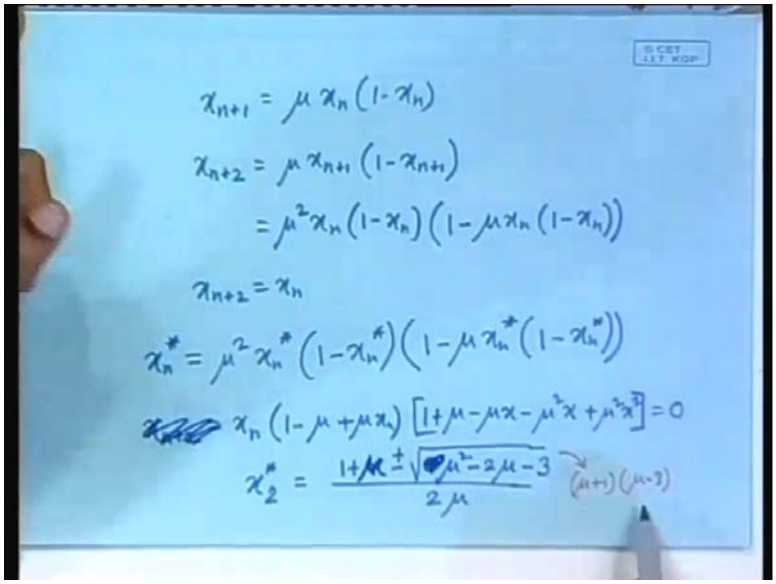

You have x_{n+1} is equal to mu x_n (1- x_n), therefore x_{n+2} is mu x_{n+1} (1- x_{n+1}). Substitute this you get mu square x_n (1-x_n) for this one and for this (1- mu x_n (1- x_n)). This is a fourth order polynomial equation and in order to find out where are the equilibrium points or fixed points of the period two orbit. What will have to do? You have to say x_{n+2} is equal to x_n. Why? Because if you are trying to check for the period one orbit what will you do? You said that in order to check that I have to set x_{n+1} is equal to x_n. So if you are now trying to check if period two orbit is stable then what will have to do set x_{n+2} is equal to x_n so you will have to set this. So you arrive at an equation something like this, x_n^* will have to substitute it here is equal to mu square x_n^* (1- x_n^*) (1- mu). It is a fourth order equation in x_n and you have to solve it.

Fourth order equation is difficult. We have never learnt how to solve a fourth order equation. Notice the argument. A fixed point is the fixed point of period one orbit that must also be a fixed point for the period two orbit. Why? Because if the point comes back to the same location every iterate then obviously it will come back to the same location to the second iterate also which means that earlier we had found two locations for the period one fixed points. They must also be roots of this equation. So you already know two roots, you only need to factorize the rest. So you have to take out those two roots one is already there, take it out, zero and take out the other one means the one that is at mu -1 by mu, extract that and you have only a quadratic which we will need to solve.

If it is clear, still this needs to be expressed in those two factor terms. Can you do that? It is not difficult though but I can write it. So this is x_{n+2} is equal to x_n. I am writing from here x_n (1- mu + mu x_n). This is the second root, I have already written these two roots and I have to find out what is to be multiplied here so that I get this. So if you want to do this, you can do but I can write what is there plus mu to make the calculation shorter but in the exams you will have to do it yourself.

So we have factored it into this root which you already know and these are the two new roots that have appeared. Notice the line of argument. We have written the expression for x_{n+2} in terms of x_n. Now we say that in order to find the location of the fixed point, I have to equate the left hand side and right hand side. [Student conversation: you will have to put the left hand side equal to the right hand side. I will have to put zero (Refer Slide Time: 00:29:30).] So now after you have substituted this, we have taken out those roots which are also the roots of the period one equation. The logic was that if there is a root or a fixed point of the period one solution then it must also be a root of the period two solution.

Therefore these are already known roots which we are not looking for, we are looking for the additional ones which are here. So if you now do that you get x^*, we have to give a separate name, x_2^* then equal to 1+ mu plus minus root over mu square minus twice mu minus 3 divided by twice mu. So these two are the new roots, these two are the roots of the period two orbit. In fact that is exactly what you see here, these are the two positions.

Now let us further notice that the one that is inside the square root can be factored. So this can be factored as mu +1 and mu-3. So if you can factor this also notice that this fellow inside must be positive because we are looking for real root. If you are looking for real root this fellow must be positive and therefore what is the conclusion? You get real root, real solutions only when mu is greater than 3. So between 1 and 3 this fellow is positive but this fellow is negative and therefore here you get a negative number. We are assuming mu is not negative. We are assuming that it is varying between zero and upwards because you see as I told you that this equation came from the modeling of populations and mu represented how good is the environment for that population and obvious reasons that cannot be negative number.

If it is very bad it is zero. So we have earlier concluded that period one orbit is stable up to 3 and it becomes unstable at 3. We now conclude that the period two orbit starts to exist after 3, not before that. It doesn’t exist before that, it starts to exist. That is why I wanted you to distinguish between existence and stability. Here it starts to exist at three we have concluded that. Is it stable? Naturally we need to prove that question.

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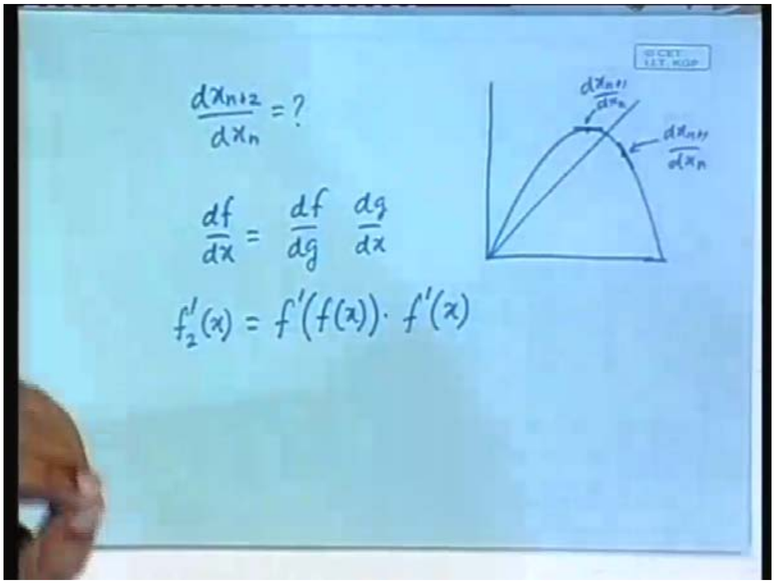

Stability is again prove by as I told you, you have to write down the dx_{n+2} in terms of and since you have the expression you can always do differentiation. You can do that. While you do that let me do it in round about manner. I will say you know that if you have a function of x then you can write df by dx is equal to df by dg dg by dx, you can write that. Now if you say that this is g, the f is x_{n+2}, g is x_{n+1} then you can write it in two separate parts. So what do you have? Ultimately how does it get written as? You have the second derivative of x this one, it becomes the derivative of… times the derivative of… (Refer Slide Time: 35:51).

So what does it mean? It means that if I have like this and I have one point here, another point here, these two solutions. Then all I need to do is to have the slope here and have the slope here. The slope ultimately of this one is nothing but the product of these two slopes. So ultimately if we want to calculate the total slope of this one then you could do the hardware writing down the whole expression in the derivative form, put the locations and then extract the number. We can do that but it's easier since have the expression for the derivative of first equation. So here this one is dx_{n+1} by dx_n and here also it is dx_{n+1} by dx_n. Now simply multiply these two that what it says. So in order to calculate the stability of the period two orbit all we need to do in a simpler way of calculation, simply calculate the first derivative at these two points and the points are given by this.

Now let us see what actually is happening. This is the way of calculation. From now onwards, in general we will not do it in the long way because you can easily understand the moment it goes into a third iterate or fourth iterate it will be going out of hand. You won't be able to do that but you will still able to do it this way because if it is fourth iterate, all you need to do is find out the four points and find out these slopes, multiply them. There is the stability of the fourth iterate. So in order to go forward in our discussion we will have to take this root.

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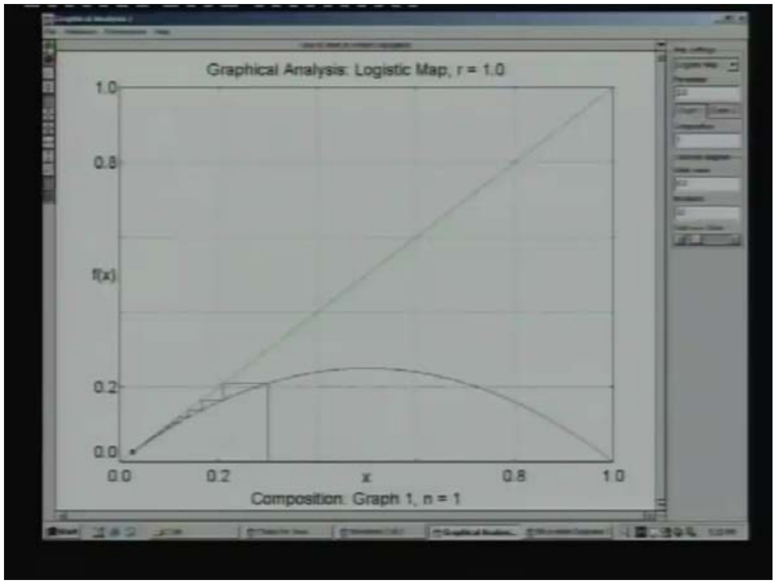

Now you see let us do the graphical analysis here. This is the graph of the map and this is the graph of the map for the value of mu as one. Now let us change the value of mu to 2.

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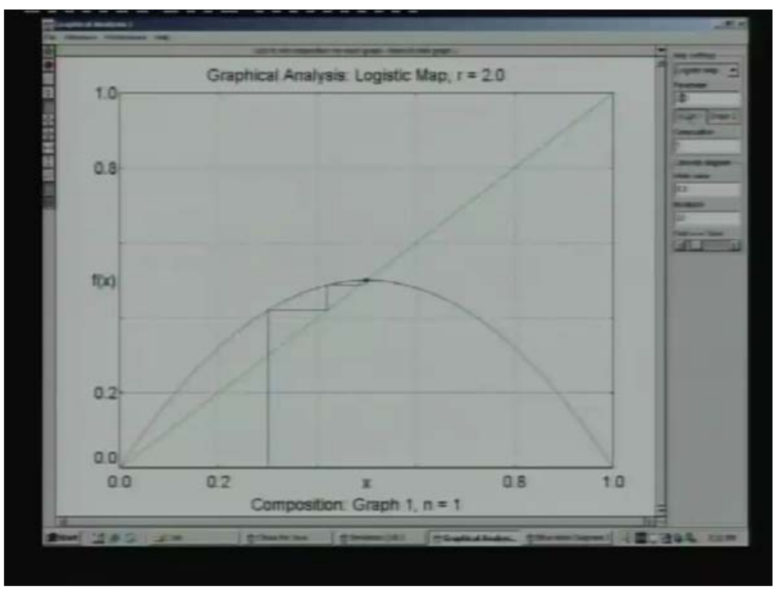

Its slope has now become greater than unity and therefore this point is now unstable and this is now stable point. As just you can see, the iterate started from here 0.3, it went up like this, went to the 45 degree line and it went up like this, went to the 45 degree line and this way it converges on to this point. Let us increase it further to 2.5 or 2.6 something like that.

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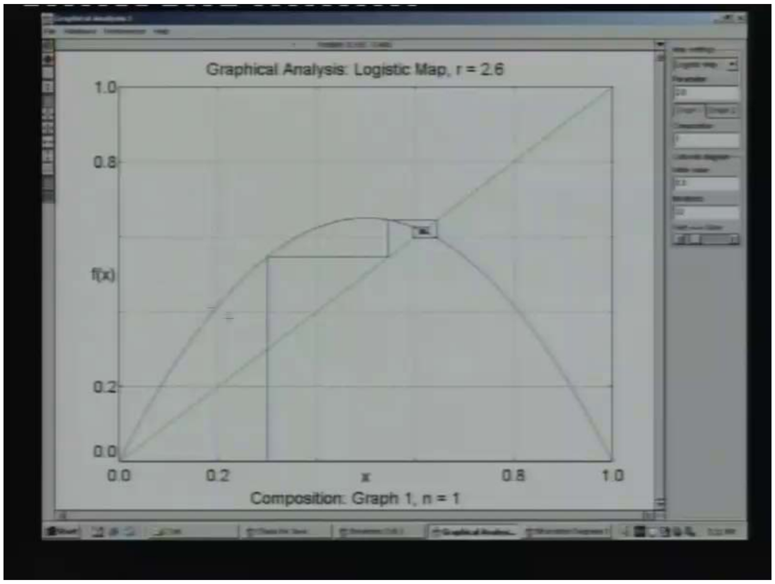

Still it does the same thing. If you zoom this part then also you can see same thing is happening.

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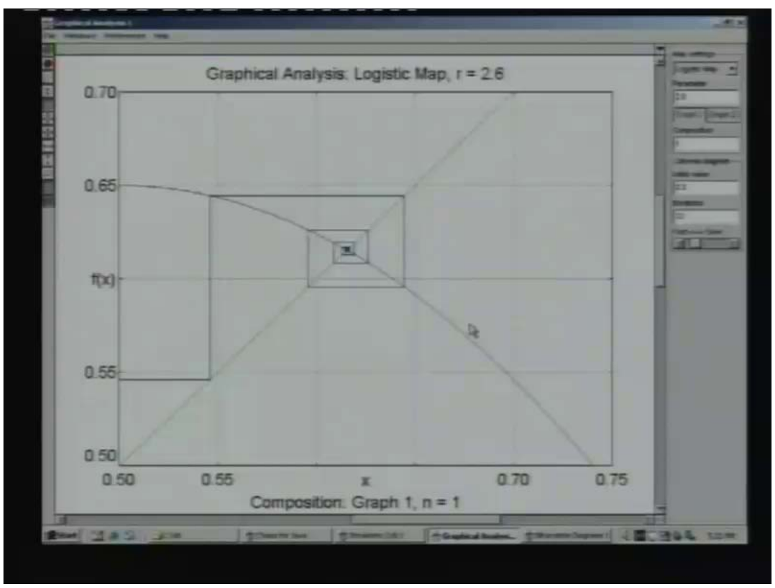

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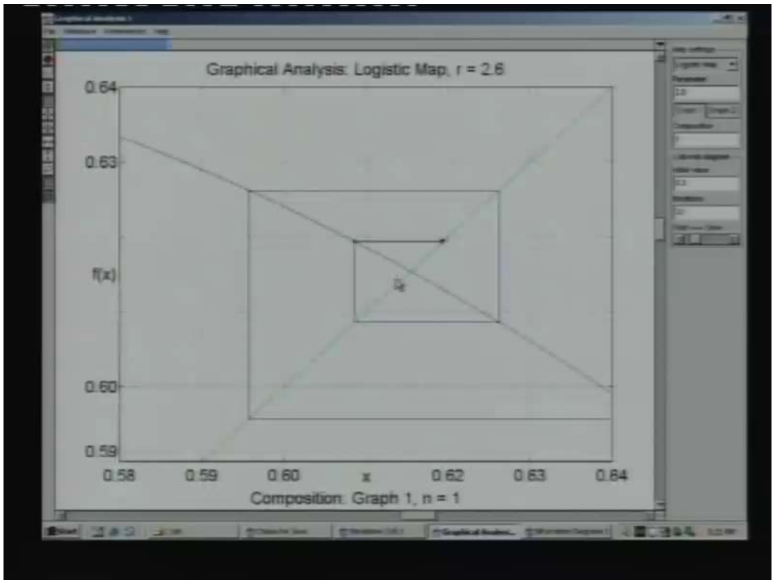

If you zoom it further, you see it becomes almost a straight line and the argument that you gave now can be seen in action. It is going like this and it is finally converging because the slope is less than unit. Now let us increase to three, let us do the calculation.

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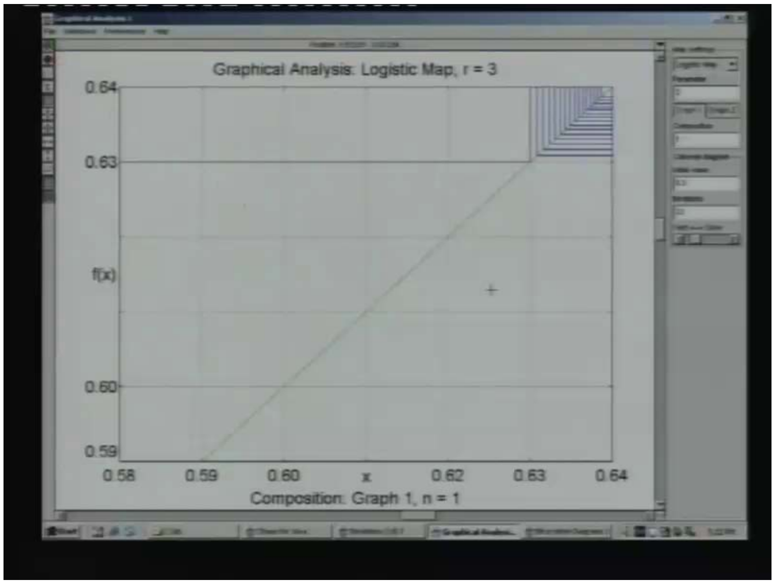

It is easier for me to just restart it all over again.

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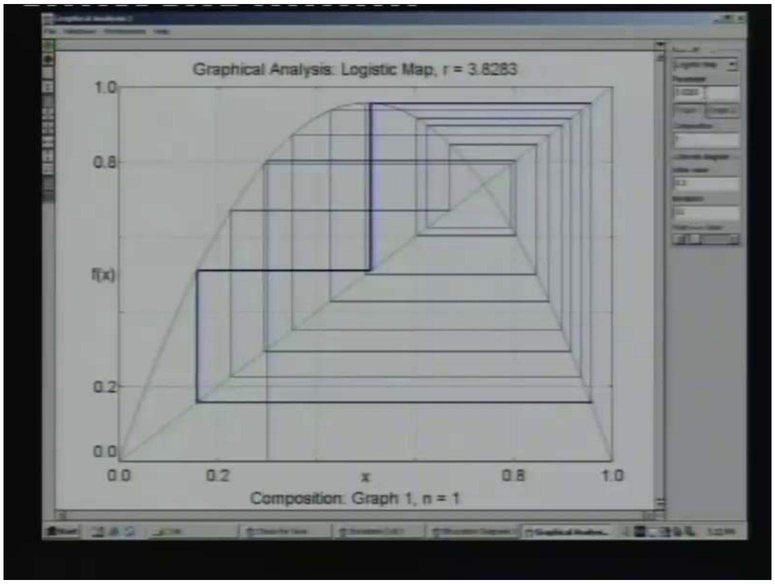

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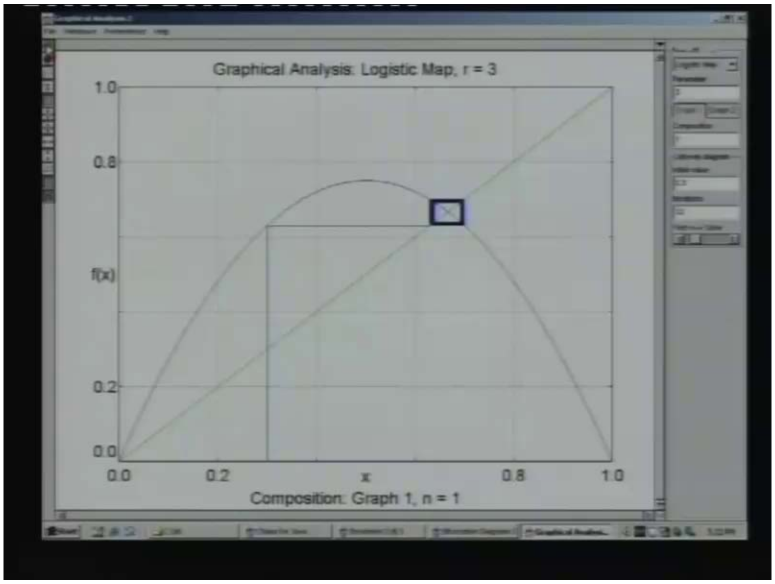

So we were in 3. Notice what is happening if I start from a point somewhere here, it goes like this and then it is going on and on. Let me increase the number of iterations to hundred, you see ultimately coming very close to this point but the rate of convergence is very slow. Let's change it to 3.4 then you can see it very clearly.

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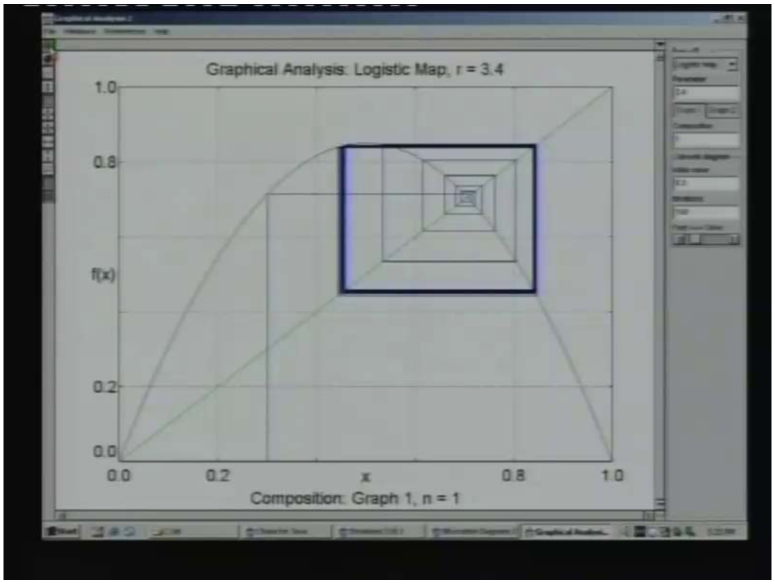

What is happening? After having done this it has again diverged out and ultimately it has come to this point and next to this point and next to this point. On the x scale you will find this point is mapping to this point and thus mapping back to this point and thus mapping back to this point and that is exactly what you see like this, period two.

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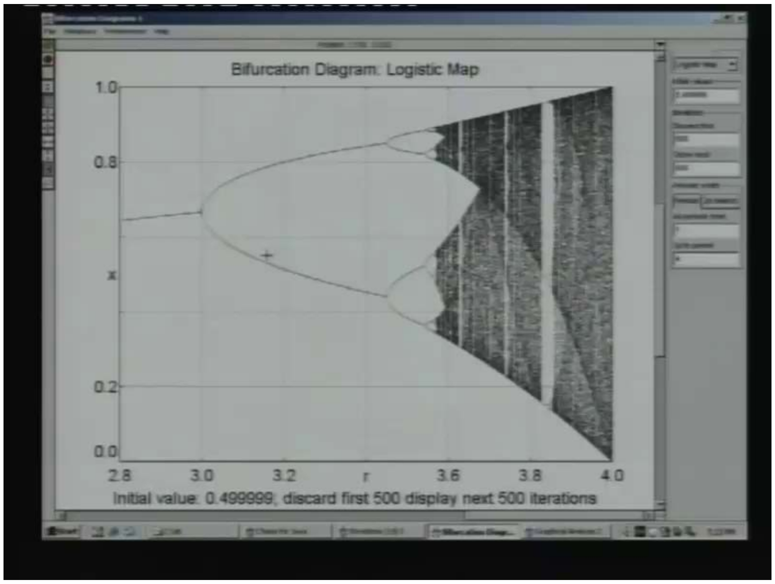

So what is the mechanism of the generation of the period two from period one? It is essentially that the period one orbit become unstable its slope, the derivative becomes… (Refer Slide Time: 42:13). Exactly at this point what is the value of the slope of the period one orbit? Yes, it was minus one, not just plus or minus one it's just minus one. Beyond that it becomes less than minus one and that is why the slope is greater than unity and at the same value of the parameter, the period two orbit starts to exist that is what you see and the period two orbit then is stable. If you do this calculation simply, these are the two points, put them into the derivative equation, calculate the slope, multiply them. You will find them that's less than one, it is stable. There is another way to see that this is stable. Let us take a value close to this.

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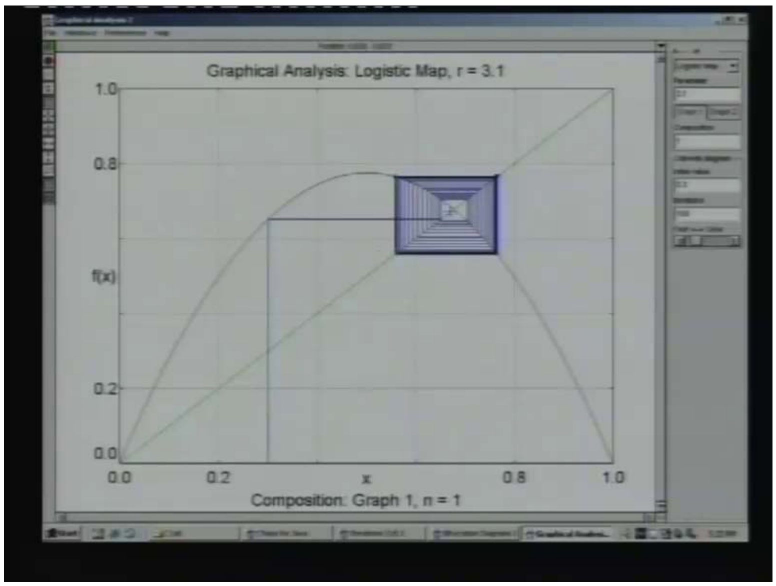

So it has gone here and finally it has diverged out and it has converged to this two points. So what you need to calculate is this slope and that slope. Now try to understand a simple logic. The logic is that here the slope is changing fast, here the slope is changing slow. At this point the slope is less than minus one, this point has become unstable. So now in order to calculate the slope of the period two orbit what will have to do? You will have to multiply this slope and that slope. So just keep on imagining and notice where I am placing my curser. If you go this point, it will map to this point but the two slopes are almost the same. Again this slope multiplied with this slope, again that slope multiplied with this slope you would see that this slope is not changing much but this slope is reducing very fast.

Therefore they must come a parameter value when this slope times this slope becomes equal to or less than one and therefore that must be stable. Again here the slope is negative, here the slope is negative multiply them what you have? A positive slope, so you have a positive slope that is less than one. Now let us see the behavior in this second one.

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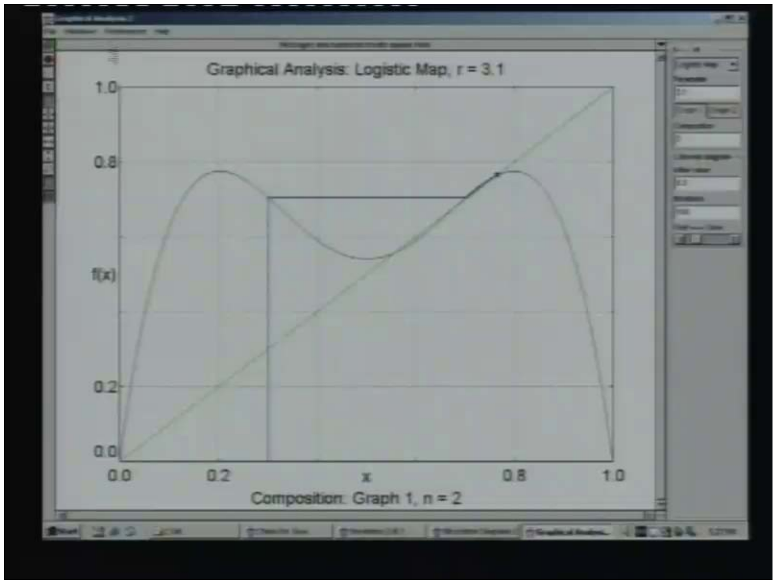

What I have drawn here is the x_{n+2} as a function of x_n. I have already written that here is x_{n+2} as a function of x_n equation, this equation can also be plotted and that's what I have done. The moment you do that you notice at that parameter value the period one was like this, same way but the period two is a two humped map with 1, 1, 2, 3, 4 intersections with the 45 degree line, therefore four fixed points. Out of that two were also the fixed points of the period one orbit. What are they? this and that (Refer Slide Time: 45:47) zero will continue to be but notice and remember that this is also a fixed point of the period one orbit that has now obtained a slope that is greater than one. Can you see that? Let us draw it for a little bigger value so that you can clearly see that.

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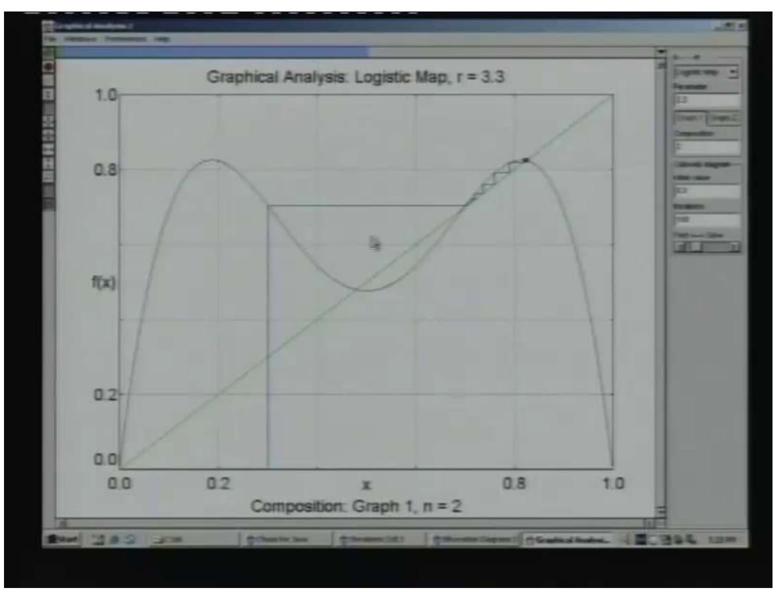

This fellow is unstable now but the fixed point is there but it's unstable now. So if you start from a initial condition very close to that, as you can see here it diverges out but that's cannot diverge to infinity, it converges to this point. So here you have in the period two orbit two new equilibria, new fixed points are born, one is here another here. Remember they are mirror image of each other, in the sense that if you start at this point next it will go there and if you start from there, it will come here but this one what I plotted here is a period two orbits. That means it is x_{n+2}, 2 as the function of x_n. So this fixed point in x_{n+2}, in this particular system will map to the same point. See if you draw a first composition in this stage, first composition means xn plus one as a function of xn.

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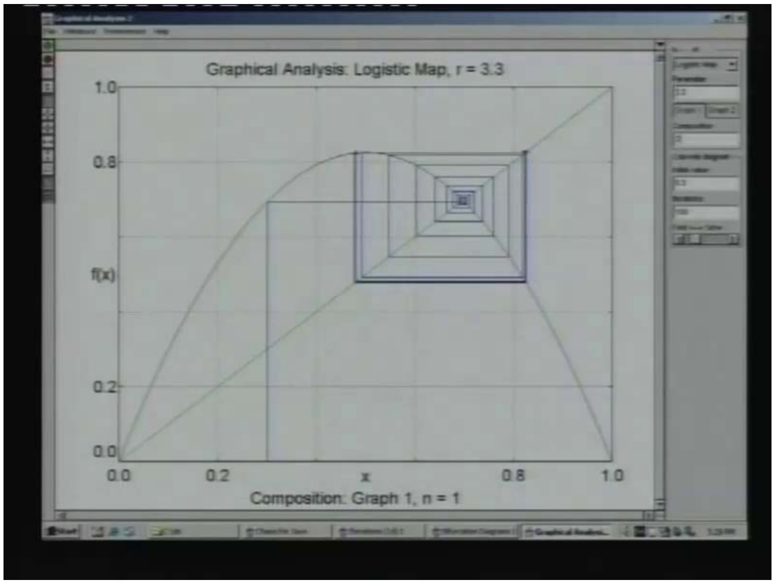

See this is like this, this point mapping to this point, this point again mapping to this point but now if you plot it as x_{n+2} as a function of x_n, then this point will keep on mapping here. If you start it here, it will keep one mapping here these are the two new fixed points. Now since this slope is a composition here, composition means notice what this right hand side is saying, this one is the derivative the slope calculated at, let me plot the one. This one is calculated here and that one is calculated here, their product we are talking about (Refer Slide Time: 48:20). We are talking about these two now. Look at the computer screen, it is a product of this slope and that slope. So what it could do analytically, you can also reason out.

It must be that way that means the period two must become stable not anything else because there must be a particular slope here, the way the slope is reducing fast. There must be a point at which the product becomes less than one. So now you got this little time. So we will say that this is the reason why you have only this kind of a phenomenon happening. Student conversation: (49:29:00). His question is that at this point will the slope be always minus one?

We have analyzed the situation where it is minus one and whether or not it is always minus one, we will prove this question later because in order to generalize we have to take out many possible situation and then analyze but notice one thing that if in the graphical analyzes we have here the slope minus one and as a result of that the period two orbit as become stable. Now try to reason with your selves, the other way for the periodic orbit to become unstable is to have the slope plus one.

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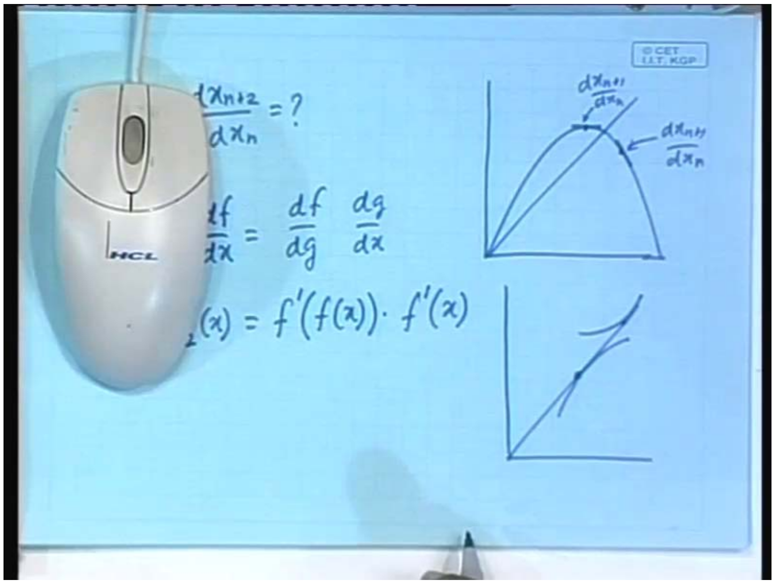

If it is plus one then if you look at the thing that I am drawing here, it should be something like this. The fixed point at that point slope is plus one or fixed point at the slope is like this. It means something like that. What that results in, i come to that later but that does not result in a period doubling.

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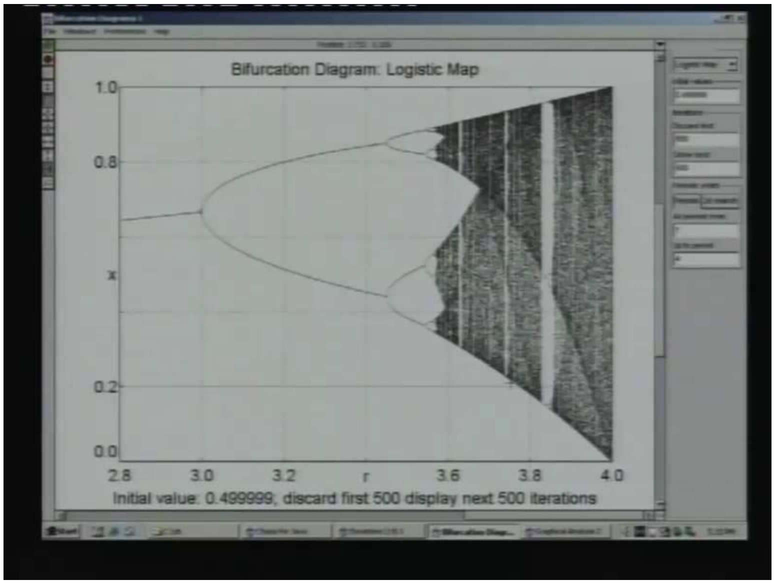

So what you have seen here is a period doubling phenomenon. Now remember when we are talking about this, we have obstructed one stage from the actual thing was a complete orbit. We are placed a Poincare section and then said that here is what I am seeing and in this Poincare a section we have seen that it period doubles means that it's periodic orbit becomes like this and that is exactly what you have seen.

Exactly what you are seen earlier when we are considering the differential equations, their orbit, the limit cycles that is exactly what happens. We have seen multiple examples in which period one become period two, it become period four so that is a period doubling cascade. Here we have what is known as a period doubling cascade. What is known as a period doubling cascade? So can you argue out what is happening here? This is as you can see close to, this is 3.4, 3.5 something like that.

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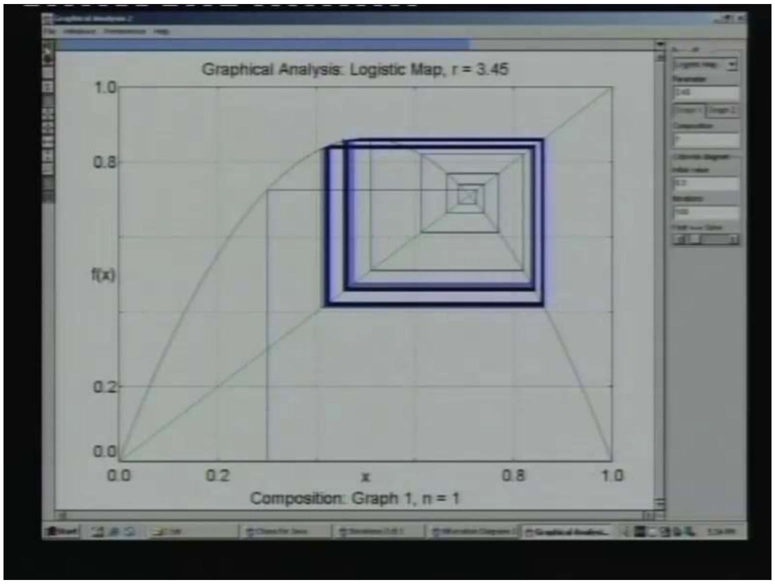

Let us just put it 3.45 and see what is happening. See it has become period four, so let us see what has happened to the second iterate of the map.

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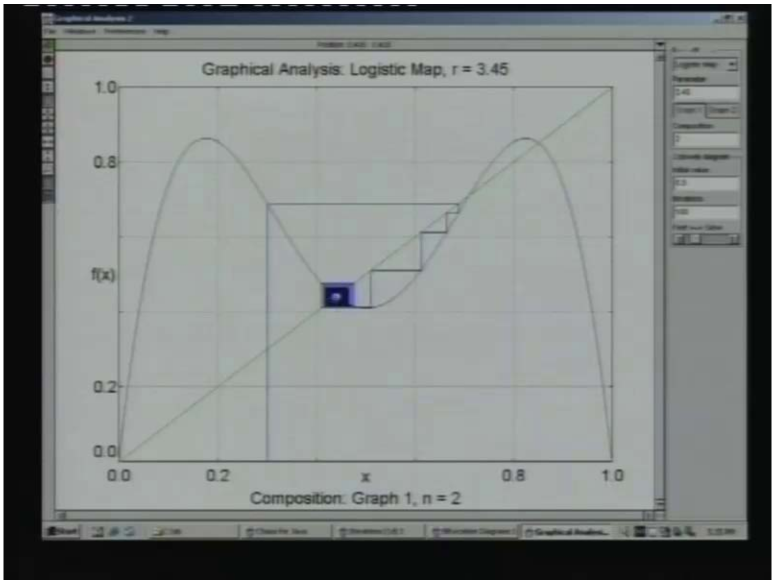

Now this point and that point, they are also losing stability. So if you make it 3.5 it will be clearer.

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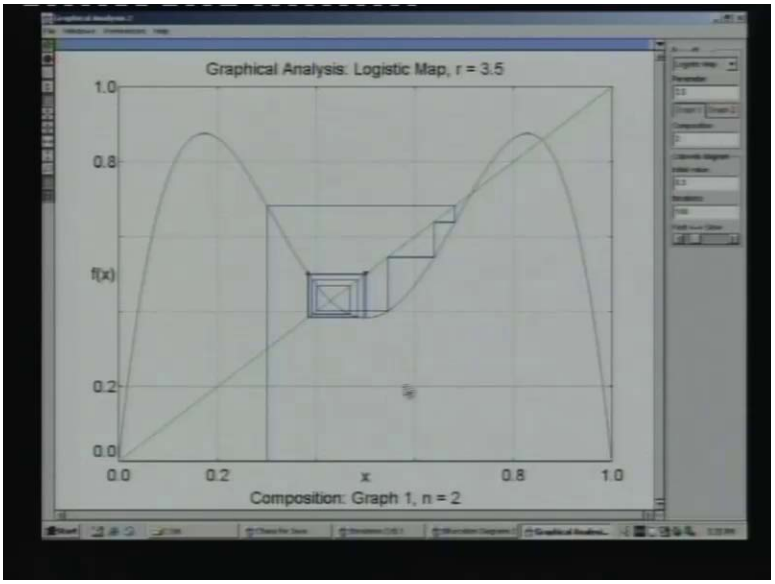

So this point earlier was stable, this point was earlier was stable (Refer Slide Time: 53:18). They have also lost stability.

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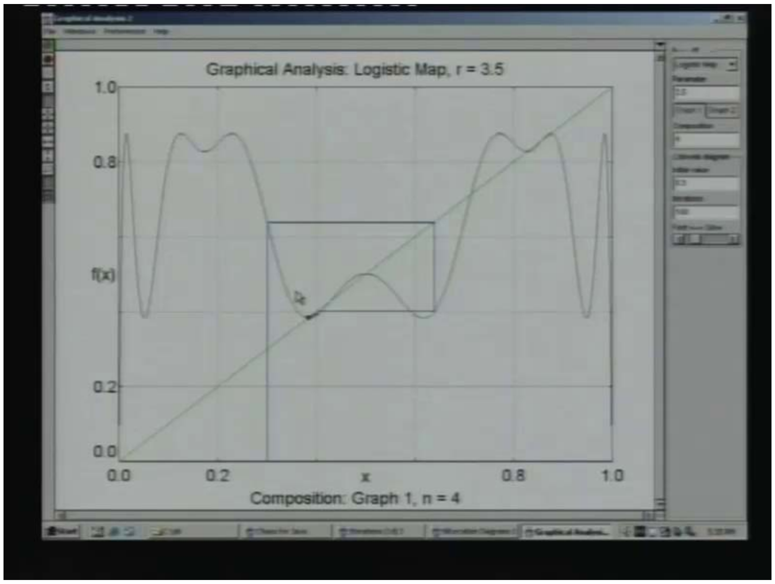

If you now draw the fourth composition x_{n+4} as a function of x_n, now you will see this is a graph. It will be ugly to write it down nevertheless you can always do that by mat lab or something like that. You can always get it plotted. If you plot get it plotted you can find that. Now out of this how many intersections will be there? With the 45 degree line how many fixed points will be there? Notice 1, 2, 3, 4, 5, 6, 7, 8, there will be 8 intersections.

Out of that two will be also the fixed points of the period one orbit. Which are they? 1, 2, 2 will be also the fix points of the period two orbit. Which are they? Here and here (Refer Slide Time: 54:20) but now around this again two fixed points have appeared, around this again two fixed points are appeared. That is why it goes from period two to period four and it's not difficult to see that further the same logic will apply. So you will have an evolution from period 4 to period 8, period 8 to period 16 and so on and so forth. That is why you have period doubling cascade which is very common in nature, in engineering in various types of fields, this kind of a change of this asymptotic stable orbit as you change the parameter is very common. Is the reasoning beyond the period doubling cascade understood? We will come to the issues of the universality and other things in the next class.

Thank you