Chaos, Fractals and Dynamical Systems
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Lecture No. # 08
Discrete Time Dynamical Systems

In the last class we saw that one nice way of handling or understanding the stability of closed loop orbits would be to express it in discrete time which means if there is an orbit like so then we would place a Poincare section and we would observe the successive piercing through the Poincare section. So this is the Poincare section. Now if then you look at the orbits starting from an initial condition somewhere away what will you observe? It will go through cycles like this but suppose this is a stable periodic orbit then you would see the successive piercings homing on to the equilibrium point which means on the Poincare section you will see a sequence of points that converge on to this particular point. That gives us a nice way of understanding the stability of orbits that is why this method of Poincare section is very widely used.

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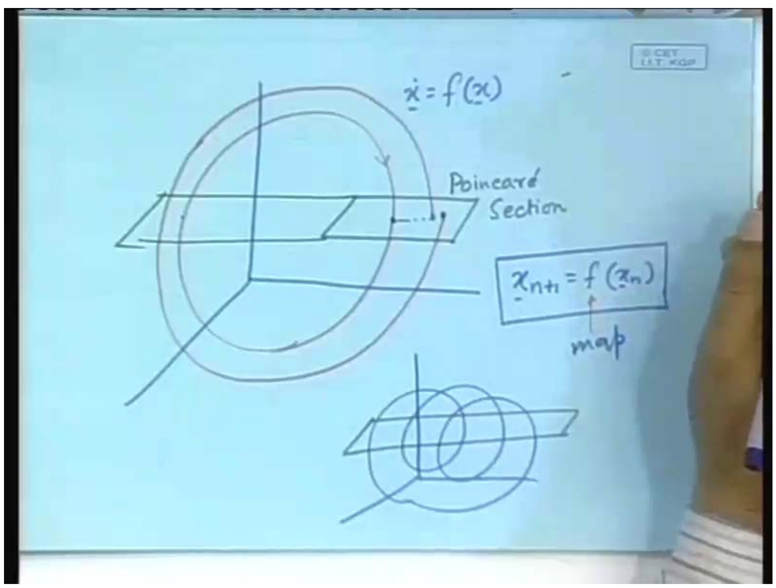

We started from equation of the form x dot is equal to f (x), these are vectors and we arrived at equations of the form x_{n+1} is equal to function of x_n these are vectors, where also understood that the description in discrete form is one dimensional less than the description in continuous domain. This original as I have depicted, the original state space was three dimensional but when we have obtained everything on the Poincare section it is as 2 d. So we have a sequence of points that ultimately would converge on to this. We are trying to analyze the character of this set of equations from now onwards. So from now onwards we will essentially concentrate on discrete time dynamical systems because mostly for convenience sake dynamical systems theories essentially concentrate on this. meaning that we are actually handling equations of this form but having reduced this form it is easier to handle and is easier to understand and easier to analyze. You might say that how to place this Poincare section.

For example you might say that it is say somewhere here, you may put it somewhere here. If it is somewhere here then obviously the orbit does not pierce this Poincare section so that would be considered an invalid placement of the Poincare section. So Poincare sections would be, the placement is in your hand. There is no hard and fast rule the only rule is that there must be a transverse intersection of the orbit of all the motions of the orbit with the Poincare section. If there were more than one for example if the orbit were something like this then somewhere you have to place so that all the loops intersect. So you need to place the Poincare section somewhere here so that all the loops are intersected so that is one condition.

Second condition is that if you extend this Poincare section here, it is after all not ending here. I have only shown it ending here just as a way of illustration but it is actually a plane that extends. So what happens here? Now this point needs to be disregarded because this is not after one full cycle of observation, so even if it pierces here you have to observe only the piercings from one direction, only this one will have to be observed, not this so that is point number 2. There are some rules of placement of the Poincare section so ultimately you will do this. You might ask for a given physical system, you coming from various backgrounds may encounter different types of physical systems, some of you may be from electrical engineering you will have a physical system that is electrical, some of you may be coming from mechanical engineering there will be physical system that is mechanical. Also there could be chemical system there could be many types of system, biological systems and each will have some kind of description in this form.

How to actually reduce it into this form? Conceptually it is easy to find but how to actually obtain this. In general these has to be obtained from here through numerical means because starting from an initial condition say here you will have to evolve or solve this set of differential equations to obtain the next piercing. Can this close it properly, till the next piercing and then only you know that this point is mapping to this point. So you have to start from an initial condition that is on the Poincare plane and then you have to evolve the whole set of differential equations till you pierce the next time, from the same direction. Now these being a set of non linear differential equations therefore you will normally not have the advantage of having a explicit solution, non linear equations will have to be solved by numerical means fourth order method or something like that which means they are starting from an initial condition on the Poincare plane, you will have to apply a numerical method to obtain this whole orbit and wait for the condition when there is another piercing. When that happens this point is stored.

So effectively this whole thing become a sub routine. A sub routine in which you start not just anywhere, you start from a point on the Poincare plane and then it calculates the whole orbit that the program to detect the next piercing also has to be embedded in that sub routine. So having done all that there is a chunk of code that whole chunk of code becomes your map. So when I say F, this F may not be exclusively given function. It can be some kind of an implicit thing whose actual expression will be in the form of a chunk of code. So when you want to evolve such a thing you will not start from any initial condition, you will start from the Poincare section and go on evolving. So what happens? The evolution goes like this that chunk of code starts with some initial condition on the Poincare plane and then the chunk of code is evolved. The final condition becomes the initial condition again for the same chunk of code, so you you put it on to a for loop and that goes on and this is called a map as I told you.

A map is obtained from a practical continuous dynamical system. Even though the map has to be obtained by that kind of a numerical procedure, often we can do a lot of analysis with it. How? Because the map often will be found to share certain characteristics with some maps whose functional forms we know and therefore by analyzing the character of those functional forms we can infer a lot of importance about many practical dynamical systems. I will come to that slowly. Let us illustrate some of the cases where you do have the advantage of being able to obtain an explicit map.

This is a system which I wrote in this form and in that we assumed that there is no time dependence in the right hand side. The right hand side is independent of time. For example look at the equations in your copies, equations of the Lorene system you will find that there is no time dependence. This is in the right hand side. Such systems are called autonomous systems.

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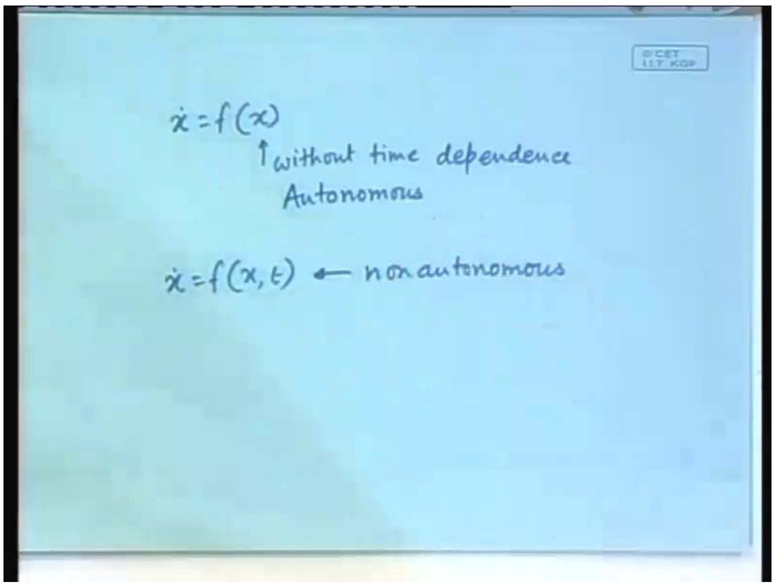

So if the equations of this form x dot is equal to f of x without time dependence then they are called autonomous system. If your x dot is equal to function of x and t then it will be called a non-autonomous system. Have you come across any example of a non-autonomous system? The force pendulum where there is a periodic forcing, you have seen that. So that is a typical example of a non autonomous system and a typical example of an autonomous is a loaded system. So you have understood that. Now while I was talking about Poincare section what I meant by this set of equation was a set of autonomous equations. So how to do this for non-autonomous system I will come to that. That will need to be understood but one thing should be clear that the asymptotically stable orbits, if it is a period one what do you see on the Poincare plane? Just one point. If it is a period two now what you see on the Poincare section? Just two points. If the system is period three? Three points. If the system is period infinity? Infinity of points. That is why I said there is a one to one correspondence between what you see in the actual state space trajectory and what you see on the Poincare plane.

Only you have to infer that way that if I see one point as a stable point then it is a periodic orbit, two points, period two orbit; three point, period three orbit and if it is infinity of points, it is a chaotic orbit.

Now let us come to the situation of the non-autonomous system. In a non-autonomous system there is some kind of a periodic forcing. A periodic forcing itself counts back to its forcing itself. If it is sinusoidal forcing, the sinusoidal variation the magnitude of the sinusoid counts back to itself after two point and therefore in that sense you might say that here is it is a 2d 2d system with a sinusoidal forcing.

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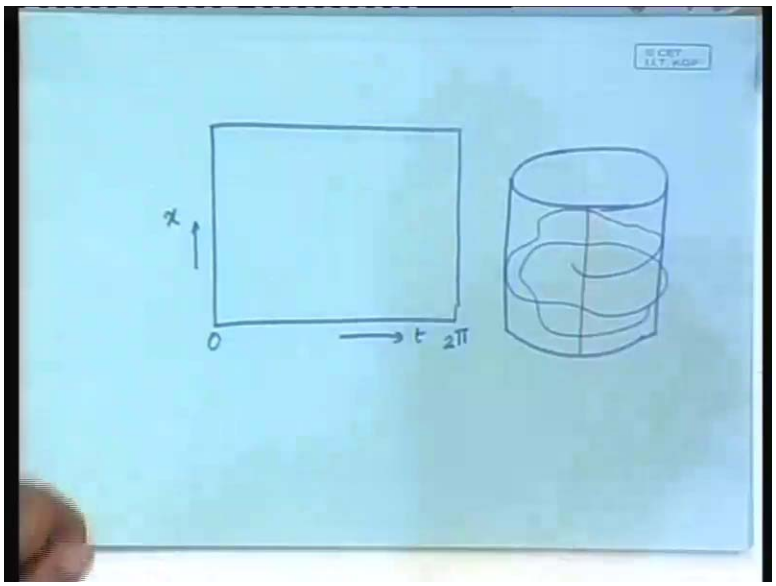

Say this is my time axis and this is say an x coordinate. In order to draw it easily I am showing just a one variable, one dimension and here is the time axis. For all practical purposes then that the time zero is the same as time 2 pi because the sinusoidal forcing counts to the same one in that sense what you can visualize this as a sheet of paper which is rolled and it goes there and becomes like so. Zero becomes equal to 2 pi which means that a non autonomous systems state space is actually on the surface of a cylinder. The state space of a non autonomous system is like the surface of the cylinder. So you would say that it is something like this. Now on this you have some dynamics going on and now I want to do something which will in effect be like placing a Poincare plane with a condition that all orbits must intersect it.

What is the most natural way of placing the orbit? Simply cut the cylinder. So the nice way of the Poincare plane would be just here which means that physically what does it mean? You have to observe it after every two pi. Physically you have to observe it after every 2 pi that’s it. I have drawn it with one state variable another time. Imagine there are N number of state variables then also this situation remains the same. If there are N number of state variables you still keep on observing every 2 pi and thereby what you get is the map but notice one thing that in case of the autonomous system, you had reduced the system dimension by one but here is the system dimension that was one, here also it remains the same.

There was a time coordinate that was actually effectively working like a variable which you have sampled and therefore the special variables, the state variables they remain the same. So there is no reduction in the number of dimension, number of state variables.

Thirdly another point needs to be taken into account. Suppose there is a non-autonomous system. How would you define its periodicity? A non-autonomous system like, how would you define its periodicity? That you are discretely observing at every twice pi and if the state variables, they come back to the same state after twice pi you say its period one. If it comes back to the same state after twice 4 pi then it would be period two and so on and so forth. The periodicity then should be counted not by the number of loops in the state space but by the number of periodicities of the forcing function that are there inside one period of the state variable. Yes there can be situations where these two can be in conflict let me illustrate.

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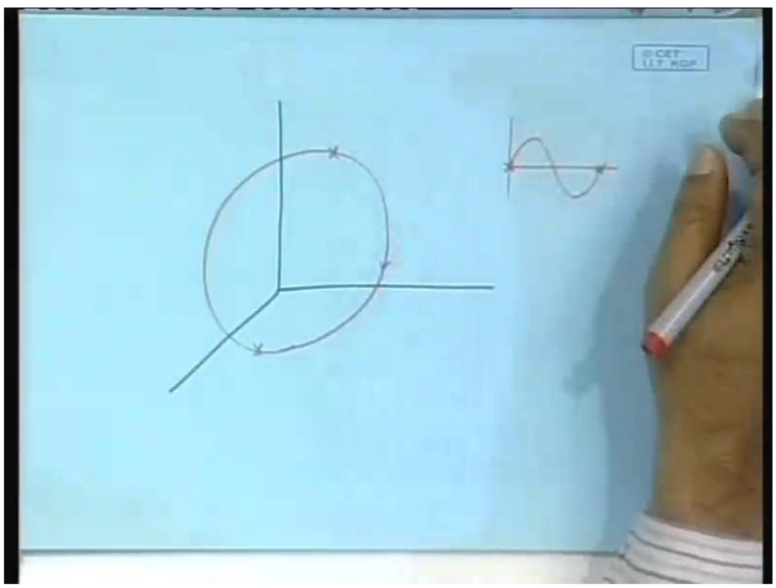

Suppose you have got a three d state space like this and suppose you have an orbit like this. What will you call it? You will call it normally a period one orbit but now suppose this fellow is occurring in a non autonomous system so that if you observe it at every twice pi of the forcing function then observe it here and here. What is the periodicity of the system then? Two because in this whole cycle there are two cycles of the external forcing function within one cycle and that is why you call it period two. So just the visual appearance does not always say the right thing. In case of non-autonomous systems therefore you have to be take some additional caution, you have to be a bit careful.

The definition of periodicity in a non autonomous system is the number of periodicities of the external forcing function within one period of the state variables but then in many systems you will find that the external perioding function that means the fact that a system is non autonomous offers some additional advantage.

Let me illustrate one case. Some of you are from electrical backgrounds this might sound familiar. There are a class of switching circuits to DC to DC converters and say let me draw one very representative DC to DC converter.

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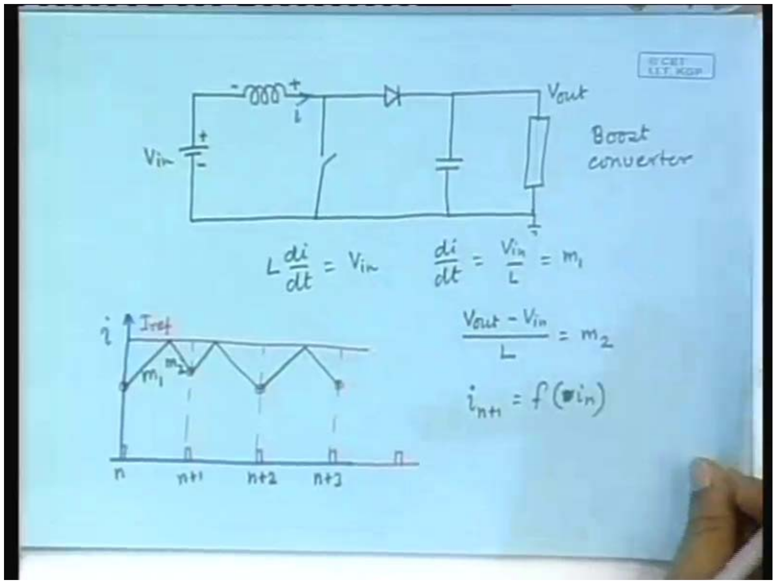

You have got a source here, you have got an inductor here and you have got a switch and here you have a diode and then a capacitor and finally the load, normally a resistive load. This is normally called a boost converter. Why boost? For those who are in electrical engineering that is trivial, for those who are not let me briefly explain. The switch is standing on and off periodically. It is a switching circuit. So it is switching turning on and off. When it’s on then you can say it is simply this circuit. A voltage source connected with the inductance what will be the behavior? It will ramp up. So it will ramp up at what slope? L di/dt is equal to input voltage, so di/dt is equal to slope V_{in} L. When it switches off then what happens? Then the whole circuit is in place and naturally this energy stored in the inductor will now go to the load. While it does so the current falls, so the current falls means now the polarity must become like this because the current is falling and the polarity of the V_{in} was like this. So you see the voltage across the inductor acts in addition to the input voltage to be applied on this capacitor. So the capacitor will see a voltage that is in excess of the input voltage and that is why it boosts the voltage. So output voltage will be larger than the input voltage that’s why it is called the boost converter.

Now often the purpose of such a circuit would be to keep the current within limit, to keep the output voltage constant so these are often done. There is a loop called the current mode control loop which I will not draw the loop itself but I will explain what it does. Suppose this is the zero value for the current and here I am drawing the current axis, current from the inductor so these current is my i. There is a reference value of the current I_{ref} and there are some clock instants that means there is a free running clock going and it is generating the clock instants. The switch turns on at every clock so that at this instant, it is a clock instant I will expect the switch to turn on and it will ramp up. When it reaches I_{ref} it is switched off, as simplest as that.

So it will again ramp down. Again at the next clock instant it will turn on and so on and so forth. That's the simple way it works, the current mode control logic essentially it is called peak current mode control it works. It doesn't allow the current, inductor current to exceed a certain value and that's how it works.

In fact you have the people who work in this field, know that this circuit goes unstable for certain conditions but let see how we can analyze it from this circuit. I have drawn it more or less like straight lines. Do you know why? Because during switch on, it is logically a straight line. If you ignore the resistance in this inductance it is fine, it is quite right to draw a straight line because of this equation di/dt equal to constant but during the switch off it is not right to do so because now it becomes say LCR circuit. Can you see? LCR so it will be a normally a damp sinusoid kind of solution but if this clock period is far smaller than the characteristic time of that LCR circuit. Then obviously this small sinusoidal segment can be approximated by a line and that is why these things are also very closely approximated by straight lines, it is not very long.

That happens if this capacitor is large enough so that the variation of the voltage here, the ripple here is kept to a minimum. So that you can assume some kind of an output voltage, you can assume this is the ground. If you assume the output voltage to be constant then you can easily see that the current through the inductor will ramp down as a straight line. What will be the down slope? If I call the up slope as m_1 and down slope as m_2 this is m_1 and down slope as is, so V_{out} minus V_{in} remember V_{out} is larger than V_{in} by L will be the down slope. Now this is a continuous time dynamical system. Here it goes following some kind of a continuous path, here it comes following some kind of a continuous path but is it an autonomous system or a non autonomous system? It is non-autonomous because even though I did not put a sinusoidal forcing function nevertheless it is non autonomous because there is a clock, some kind of an external forcing.

Wherever there is external forcing it is a non-autonomous system and in non-autonomous system therefore we have to observe it at every clock instant. So we will have to observe it here, we will have to observe it here, we will have to observe it here so on and so forth. So we are trying to express essentially when we want to observe the map is i_{n+1} where this is the n plus oneth instant say, it is the nth instant say, it is n+2 instant say, plus 3 so on and so forth is equal to some function of i_n that is what you are trying to obtain. Now can we obtain it? Let's see. I will put it here so that you can see this part on the screen.

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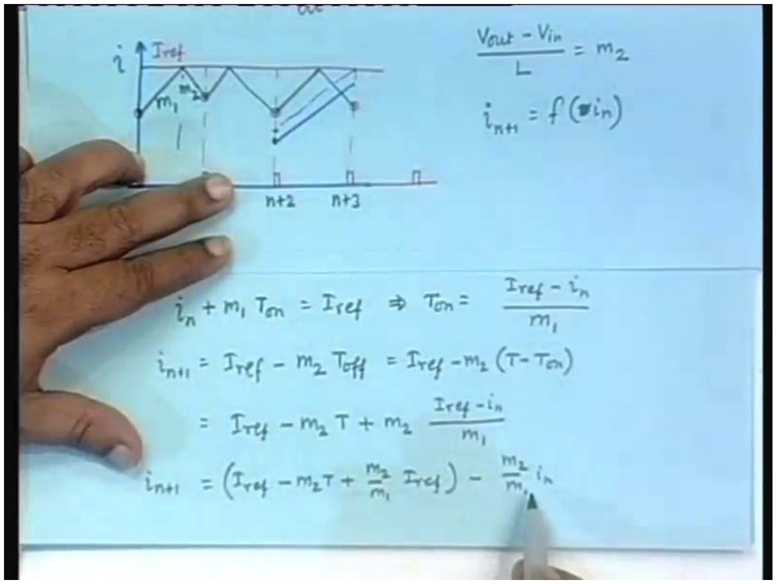

Now you start from i_n and reach $I_{reference}$. How much is the time elapsed? i_n plus m_1 T_{on}, the on period is this much is equal to I_{ref}. This yields T_{on} is equal to (Refer Slide Time: 29:00). Final value i_{n+1} is I_{ref} minus this, so I_{ref} – m_2 T_{off}, off period which is I_{ref} – m_2 T - T_{on}, T is constant because the the clock period is constant and T_{on} is this much, so we have I_{ref} – m_2 T + m_2 times this value. Can you simplify this? You will get a constant part, this is your constant part minus this is the variable part depends on i_n. So you see what is that we have achieved is i_{n+1} has been expressed as a function of i_n. This is the map, this is the discrete time dynamical system or very difficult to write or very difficult to draw at all.

There is a bit of complication here because it might so happen that one orbit may go like this say it starts from here and it goes like this. It does not reach the $I_{reference}$ before the next clock that's also possible obviously. So if the current at this clock instant is below a certain value, the certain value is given by the situation where it reaches exactly at the next clock, if it is below it will follow like this, if it is above then it will go like this and the condition for going like like this has been obtained. Can you obtain for the condition for going like this? I will again do this in the lower part here, going from here to here.

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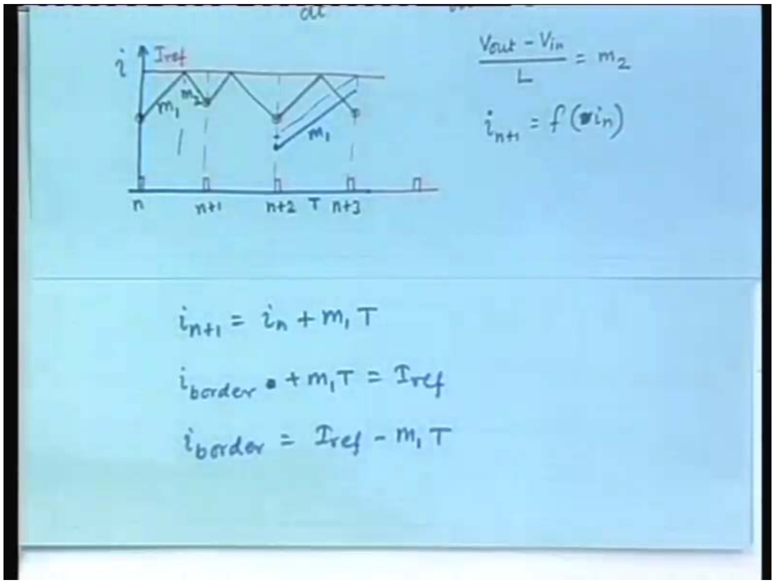

So i_{n+1} is equal to i_n + m_1 up slope times the full period, m_1 is the up slope times the full period T. So this will happen, this will be the map for this kind of evolution and the other one will be the map for this kind of evolution and this is separated by this condition. That is kind of a border line condition, critical condition. What is the critical condition? i_{border} + m_1 T is equal to I_{ref}, so i_{border} is equal to I_{ref} - m_1T. So you see all that we obtain is that here the discrete dynamical system is composed of two parts either it will be like this or it will be like that and whether or not, you have to take this function or that function depend on whether it is below or above this.

The implication and other things of this I will come later, I will preserve these pages so that we can come back to this later. So you see here was the system in which we could obtain a one dimensional description i only, we dint need v because we made the assumption that the capacitor is large so that the capacitor holder doesn’t fluctuate much. If you realize that you will get a two dimensional description but it will be slightly more complicated and we didn’t want to get into that complication today but in general it’s not very complicated one.

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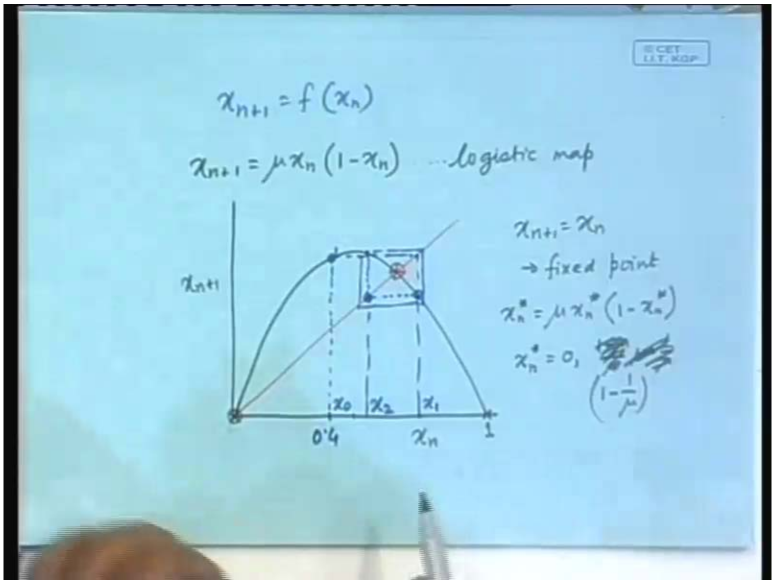

The point then is that in many dynamical systems it is possible to reduce that into one dimensional map in the form of x_{n+1} is equal to some function of x_n where x is a scalar now, 1 d means I am not no longer talking about x as a vector. It's a one dimensional thing. The major advantage of this a one dimensional map is that the graph is visualizable, the graph is drawable you can draw. For example take this function x_{n+1} is equal to (Refer Slide Time: 35:07). This is a very famous equation in fact called the logistic map. It was made famous by a man called Robert May, he wrote a paper in 1976 in nature in which he was a population dynamic system that means he worked one population dynamics.

Imagine the population of fish in a pond, the most suitable the invariant of the pond is the more will be the population of that pond it will tend to increase and the more it will increase; in the next generation it will not be able to increase further. So higher the population it will have to have a feedback effect on the population in the next generation and the fish population actually increase in the breeding times and therefore it is really discrete time dynamical system. So the discrete time dynamical systems occur in various possible ways, one is to obtain the Poincare map that I showed but there are various possible ways. For example the internet packet transfer. How does it happen? They happen always in discrete packets. So if you want to keep track of the number of packets being transferred, this is a number that goes discretely, changes discretely a discrete dynamical system.

Similarly populations are discrete dynamical system and this is the model he proposed to map a single species population, something like the population of one species of fish in a pond feeding all algae or something like that. Now the more suitable that particular pond is for that fish that means the more they feed and other environmental factors, the more will be the factor mu. So this is the parameter in the system that actually models how suitable the environment of that pond is to the particular fish and xn represents the population in the nth generation. So you can quickly do an exercise.

Do you have calculators? There are certain things you can do by hand, there are certain things you can do better by calculator. Suppose you start from any value of x_n with mu set to 0.1, for example set mu to 0.1 and start from any value of x_n say 0.6. By the way x_n is is normalized between zero and one that means x_n is equal to one represents something like a maximum possible population in that pond. So start from mu is equal to set the parameter at 0.1and x_n you start from any initial value say 0.6, see what happens. It will decay very fastly and the population will be extinct. Check out, does it not? Increase mu to 1.2, you will find that it doesn't do so, so easily. It takes more time but ultimately it does decay. Can you check?

It will be a good exercise for you to do this for different values of mu. You can do this by a calculator in fact the first people who analyze this did so by hand. They didn't have even access to calculators but you might use a calculator and check out what will happen to this population. That means first you start from 0.6 that will give you some population x_{n+1} that you put as x_n, that will give you the next generation population x_{n+2}; put that as x_n that goes on, you get a sequence. If that sequence converges to some number then you will know that will be the stable behavior.

Now you note in this case that for zero point one it converges to zero and at some larger value it converges not to zero but to some stable population. You will notice that if you increase it even further say 3.1 it does not really converge to a particular value rather in the steady state it keeps on toggling between two values. That means one here it is small number, next time it is large number again next time is small number, again next time is large number so it will toggle between two possible values. Increase it even further it goes to below 4. So all that you have seen happening in our example systems earlier are actually mimicked by such a simple equation and simple equation if you can understand the mechanism in a simple equation there is nothing better than that.

So we will do exactly that; we will try to understand why that happens using this map and that will be a rather simple loop to take. We have already seen that this is what happens in the differential equation systems that we have taken earlier. We have seen that in the Lorenz system, we have seen that in the Rosler system, we have seen that in the given pendulum that is this particular kind of sequence. Now the main advantage as I told you is that a map can be drawn, this functional form can be drawn. How? For example this one, can we infer whatever be the character of this? I will draw x_n versus x_{n+1}, can you draw? Easily because if I take x_n is equal to zero its value is zero, if I take x_n is equal to one its value is still zero. So this point will be zero, at x_n equal to zero x_{n+1} must be zero. At x_n is equal to one, x_{n+1} should be again zero and therefore there has to be a peak somewhere in between. If things are zero at two points is reasonable to expect that there will be a peak somewhere in between.

You heard of the famous theorem by Derek, in one of the parties he discovered the theorem that the distance from which a woman's face looks the best, it is always the maximum there. His logic was if she is at an infinite distance you can't see the face, is a dot. If you see the zero distance you don't see it. So in between them there must be a maxima. So that is how Derek wanted to theorize every thing but here also we can logically say that there would be a maxima.

Now the moment you draw this, a few things become immediately clear. When I say that it ultimately goes and stabilizes somewhere as a steady state behavior, what does it mean? x_n is the same as x_{n+1} is same as x_{n+2}, is the same as x_{n+3} and so on and so forth. How can you find that out? That is the condition when x_{n+1} is equal to x_n that is the fixed point, so that is the fixed one of this map. You can easily find out the fixed point here, can you find it? Yes so you will put x_n^*. The fixed points generally denoted with a star is equal to mu x_n^* $(1-x_n^*)$. Obviously the solution would be one at mu xn is equal to zero. So the solution would be x_n^* is equal to zero. This I didn't calculate, let me calculate first one (Refer Slide Time: 45:56). It will be one minus you can write it this way. Now a nice way of representing this is simply to draw the 45 degree line.

Any point on this 45 degree line will have this property and therefore where it intersects the graph of the map this point will be the fixed point. So this is one way of calculating, again this will also be a fixed point. When I said that take a calculator and start from an initial condition say x_n is equal to 0.6 and keep on iterating. What you will do? You will put a 0.6 here, 0.6 here calculate this put it here, again put it in the left hand side that mean you have gone iterating it. That iteration can then be done graphically. How? Let's say here is my 0.5 and say I start from say 0.4. If my initial condition x_n is 0.4 where will the x_{n+1} be? I have to go up vertically and cut here so I will go vertically up and reach this point.

In the next iteration what will I do? I will put that value in the right hand side and again recalculate the left hand side. So I have to somehow bring this value to here and then recalculate and how do I bring this value to here? Simple, you go in the horizontal direction, meet here and this will be that value. So if this is my x_0, this will be my x_1 and then where will this one map? It will go to this point, again you have to bring this value to the x axis and go on calculating. How will you do that? Come horizontally to the 45 degree line and this point is what x_2 is. How will you go on further? Again go up to the graph of the map, come here again you go this way. Again you go this way, again you go this way, again you go this way so on and so forth. So what actually are you doing? You are iterating the map, what I asked you to do by pressing a calculator. You can also do it by computer but a nice visualisable way of doing that is this way. Practice this because this is a very important tool in our hands to understand the dynamics of systems. We can easily see whether or not an orbit will converge or not or will it diverge, you can easily see. How?

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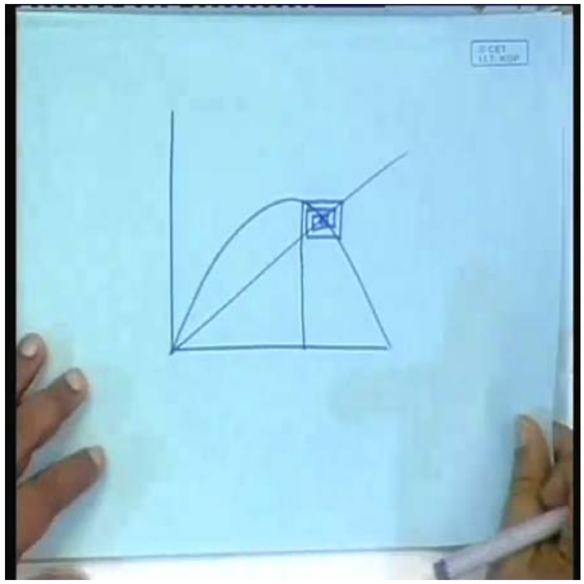

For example, if you have the graph of the map is like this and you have got the 45 degree length like, so I have drawn it badly, not 45 degree nevertheless don't mind. Suppose you start from this initial condition that means this initial condition. How will it go and it will converge. Just by doing this we don't really pressing a calculator, I could see that it is converging but can you see what will be the condition for convergence? That is slope here has to be less than 1+2, when you are very close to this fixed point you can more or less approximate by a straight line.

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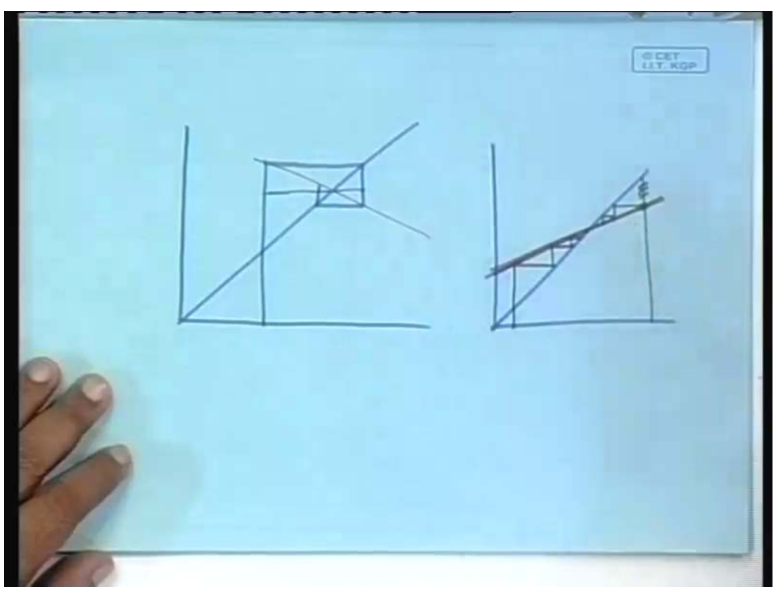

Then you say that I have here is the 45 degree line, I have the graph of the map like so, enlarged then if I start from a point here I will go up like this and you will go like this and if it lands closer to the fixed point then I know that I am at homing on to it. I will go again if this distance is also closer than this distance I know I am homing on to it. Now just apply simple trigonometry to check this condition that it will converge, if this lines slope is less than one, check. You will essentially need to make sure that this distance is larger than this distance but I will leave it to you. It is too simple for me to do, it's trivial.

By the way if it is like this, then if this is the graph of the map then if you start from a point somewhere here, some initial condition somewhere here, you will go to the graph of the map; you will go to the 45 degree line; again you go to the graph of the map, will again go to horizontal to the 45 degree line. Again go to the graph; again go to the 45 degree line and so on and so forth. Ultimately you can see that it is converging.

It is easier to check that in this condition then in this condition so you can easily check that. Starting from here how will it go? It will go to the graph of the map and stops here, go to the 45 degree line, again to the graph of the map, to the 45 degree line, to the graph of the map so it is. So you see here the slope of the map was less than plus 45 degrees and here the slope of the map was less than or greater than minus 45 degrees. The modulus has to be less than unity then only it will be stable.

So just trace back your thoughts, we decided that the whole orbit, its stability can be understood by studying only the character of the fixed point. If it is stable then my whole orbit will be stable. Now you see we are homing on to a condition by which we can check the stability of the fixed point from the map, provided the map is one d. I will come to the higher dimensional cases later. One dimension is easier to understand, so we did it for 1 d first. Just before we break off, notice two things one is this type that it homes on to the fixed point from one side. It starts from this side, it remains this side, starts from that side, it remains that side. While if it is like this it goes on once to the left then to the right then to the left then to the right, then to the left. This character we will again comment on in the next class. That's all for today.