Chaos, Fractals and Dynamical Systems
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Lecture No. # 05
The Lorenz Equation – II

In the last class we had concluded that for the set of equation that I have given you is called the Lorentz equation, the origin will be the only equilibrium point before the value of r =1. after that, as the parameter r is changed through the value of one, you have that particular equilibrium point becoming unstable while two other equilibrium points then appear and becomes stable. as you change r further, then you will see that the real parts of the Eigenvalues of those two equilibrium points which were having complex conjugate Eigenvalues, i.e., having inwards spiraling orbits, there the real part would slowly go towards zero and finally at some parameter value it will cross zero which means that they will also become unstable. So we ran into situation where the the fellow at origin was throughout unstable after r =1 and the other two also became unstable. So let us see how the orbits look like all through these sequences of events as the parameter is changing. I will use a readymade program.

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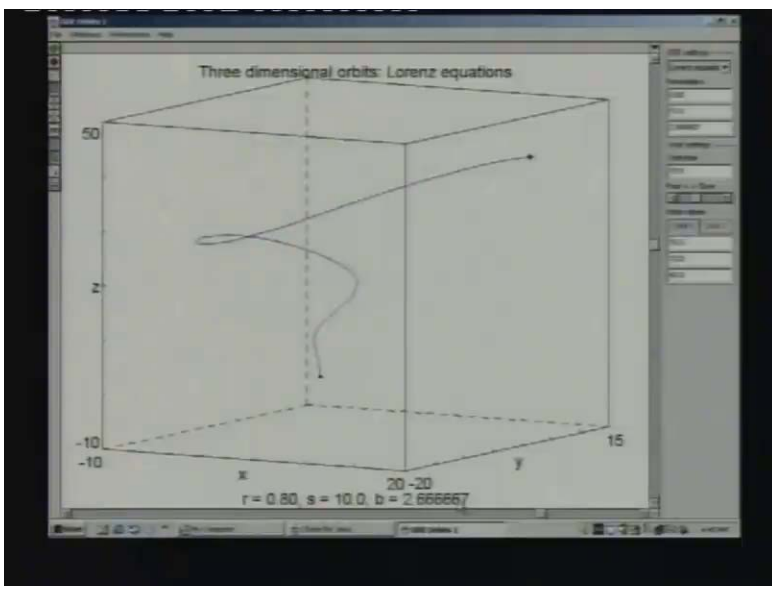

You can easily do this by Matlab coding. That is exactly why I asked to do it. So first let us start with a parameter value something r = 0.8. Here it is‘s’ but we have written sigma and its b is 8/3 which is 2.667. So if I now start the orbit simulation, you will see it starts from this point and it goes to zero. Start from any other initial condition it will also go to zero and you can see that it is going to zero through one of the stable Eigenvectors. Now let us increase the parameter beyond one while from theory, we expect that the origin fellow will become unstable and something else will become stable.

Let’s see if that happens. Let us increase it to say 1.8. Let us increase the time. You can see here it was coming here (Refer Slide Time: 04:00) but it is going away. If you increase the parameter further, say 10 now it will be clear how it is going away.

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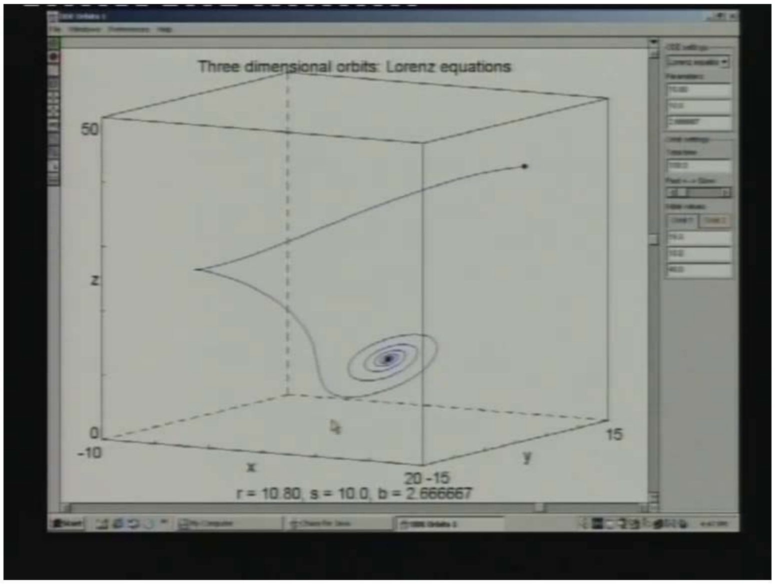

Now here there was one fellow that has become unstable but now you can see clear spiraling orbit spiraling inwards towards this and do you see that this spiraling orbit is in a plane? It is this plane that I was talking about. How do you obtain this plane? As I told you have to obtain the you can easily obtain the location of this equilibrium point at that equilibrium point you have to evaluate the Jacobian matrix obtain the eigenvalues now there will be one real eigenvalue and two complex conjugate values. You can by by looking at it you know that the real eigenvalue is directed something like this (Refer Slide Time” 05:17) and the complex conjugate eigenvalue associated with the Eigen plane and you can see that Eigen plane. That Eigen plane is to be obtained by taking the real part of that eigenvector and the imaginary part eigenvector both real vectors. The plane passing through these two vectors is the plane here. So you can analytically calculate this plane. But nevertheless the point is that here we have starting point and it is going through all sorts of things and finally it is moving on to the equilibrium point. But you have to notice from analysis that there should be two of them. So let us try to do that. Its starting point was 15, 10 and 40. Let us start from -15, 10 and 40. It was a symmetrically placed location. So let us recalculate it.

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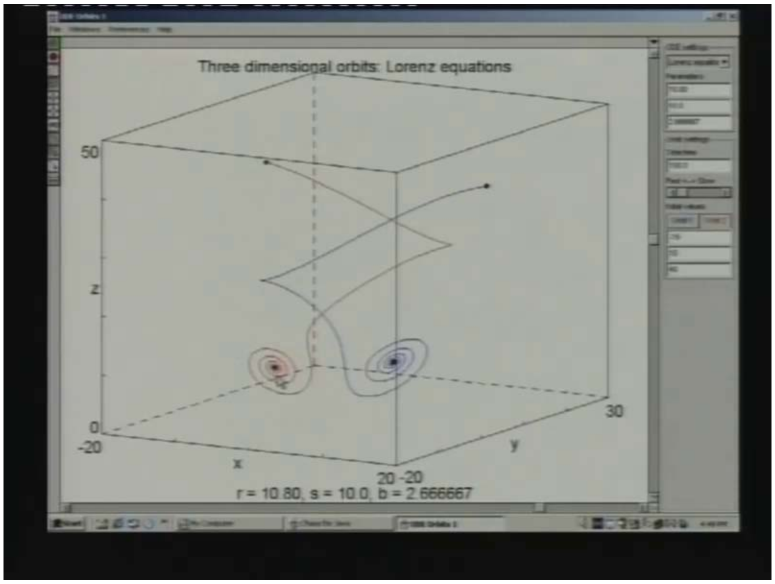

So this starts from the from the right-hand side and it goes and converges onto this one. This fellow starts from here and it goes and converges onto this one. There are the two equilibria. The other fellow is unstable and these two equilibrium are now forming a plane and along the line which is orthogonal to it. There is the stable eigenvector and these planes are also stable Eigen planes at this parameter value. Let us increase further. What do you expect? As you increase the parameter further, its real component will become closer to zero. Suppose I don't go to the positive side but it becomes closer to zero. What do you anticipate to happen here physically? What does the real part give? Real part gives e to the power sigma t. it will come in very slowly. The rate at which it is coming in will slow down and therefore it will have very long-going spiral. Let's increase it to say 20 and see what happens.

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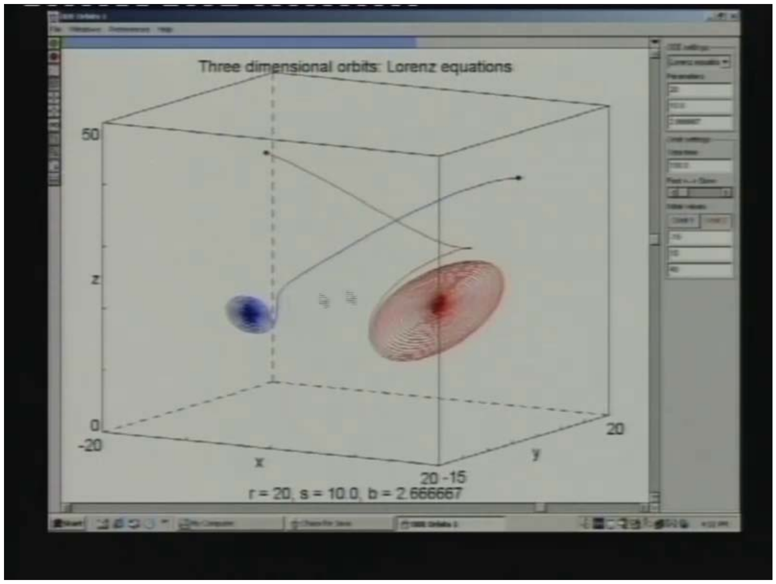

See if you run it slowly, notice how it is going. Still it is coming into the equilibrium. So one from this initial condition, another from that initial condition. But you can clearly see the planes. now let us increase the parameter more and run it again.

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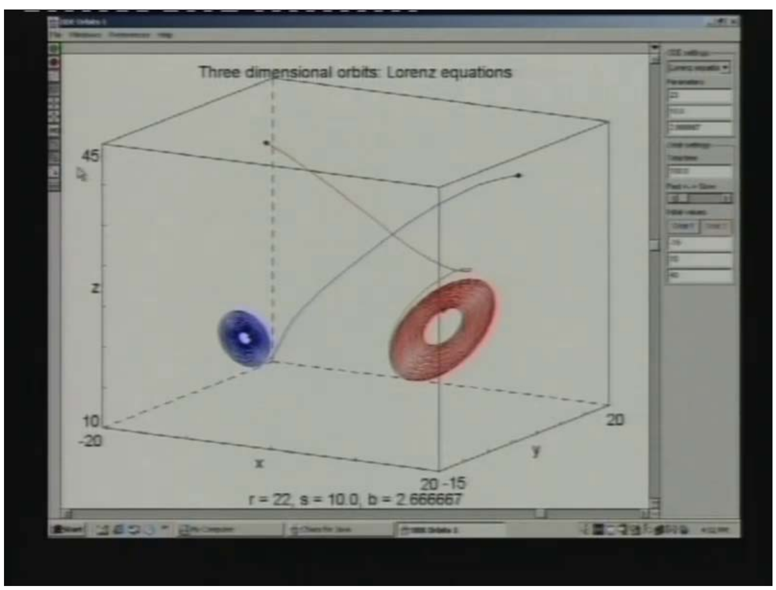

When we run it now it will become even slower in going down. The rate is slowing down. Now increase it further. Now I am slowly increasing it. It’s 23.

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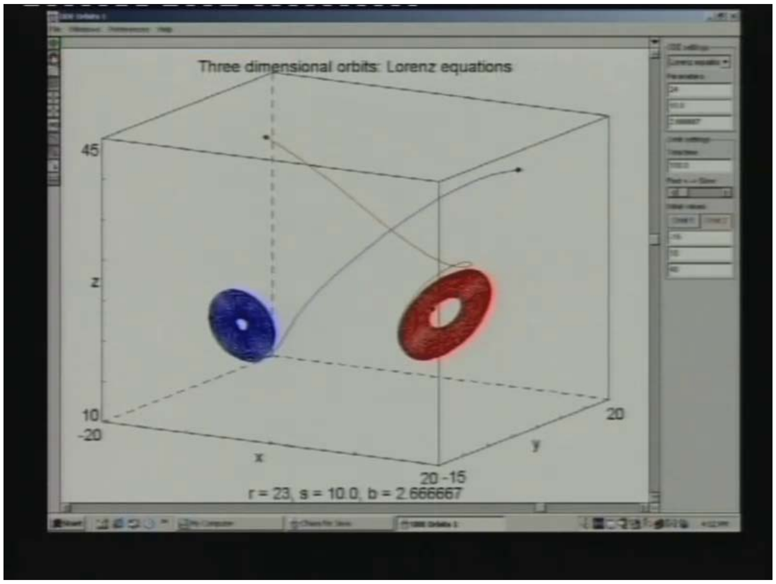

We can anticipate that at 24, something will change I am changing it here.

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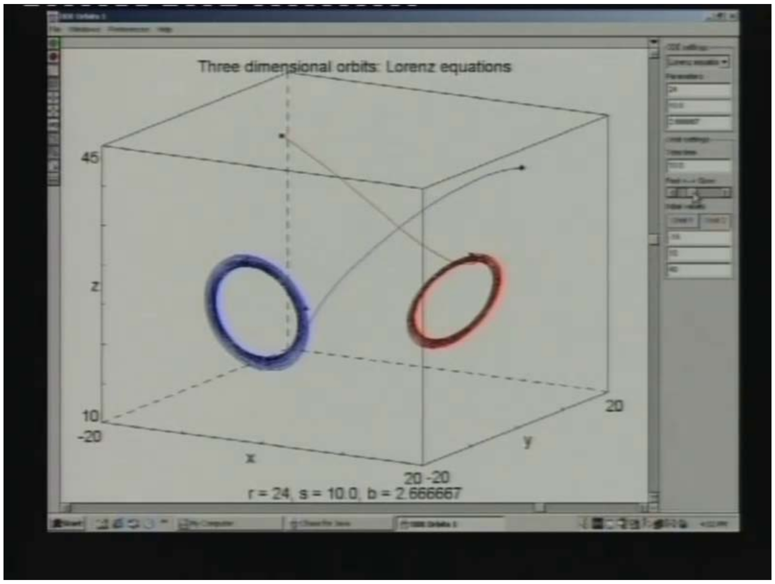

They become unstable. They are going out. Let us execute it little slowly so that you can see the evolution. They are going out. If they are going out, the natural question is: where do they go? They are spiraling out. If you stop at time 10 seconds, then they will stop here. This fellow has gone only this much and that fellow has gone only that much. But if you can anticipate that if you running for a longer time then this plane will intersect with that plane somewhere. So this orbit will run into that orbit and that orbit will run into this orbit. Now if I increase the time of computation to longer time, they are going out as anticipated from theory.

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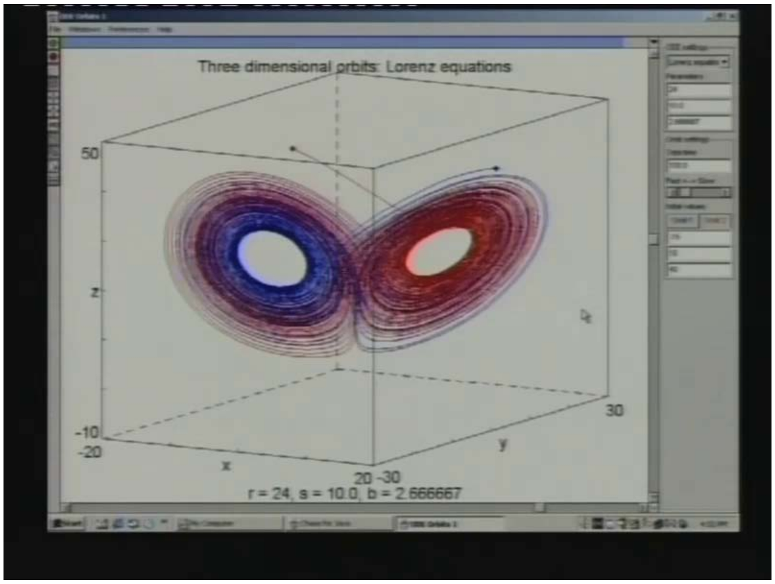

Then something that cannot normally be anticipated from theory happens. Do you see their movement? This is called chaos. So it is not a very difficult thing to understand. It is reasonably simple. Now while this fellow goes on, let me tell you how this was first discovered. It was actually model of the weather. There are a large number of people who try to model the atmospheric circulations and from there, they tried to predict what will be the behavior or they also try to predict the weather. For example you may have heard that the weather prediction business in India requires a super computer. Do you know why? why does it need a super computer because at different places, people measure the air pressure, humidity, temperature and things like that and all that makes a grid and finally there has to be a computation and the computation obviously takes into account simple hydrodynamic equation. Heat produces convection. Next things go up as it goes up, it has to come down because it cannot indefinitely go up. There has to be circulating current. So around the early 60's, 60 to 63, when the computers were very primitive, the kind of computer that I was talking about. In MIT, there was a group working on that the person whose name associated with the set of equation his name is Edward Lorenz. He simplified the set of equations into this set of three equations. Normally it will be enormously complicated stuff. But that can be simplified.

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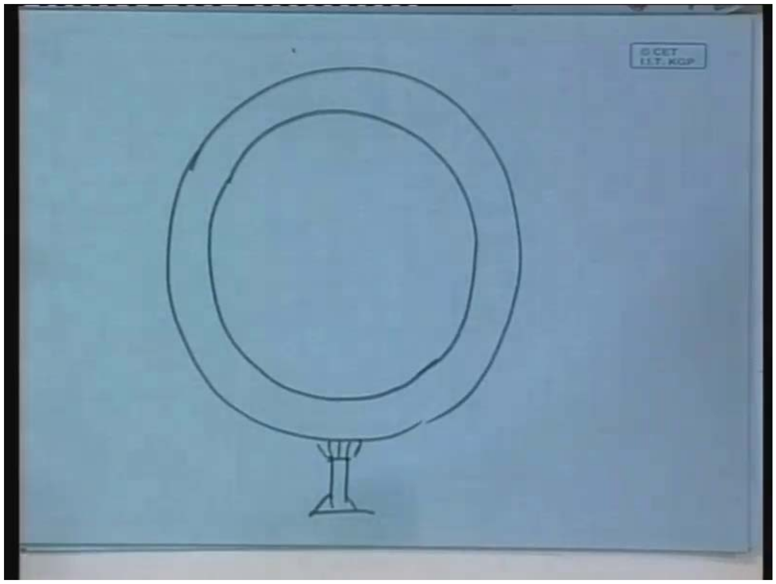

Just imagine the whole thing is simplified into this situation that you consider a ring containing some kind of a fluid and you have say, a Bunsen burner here and you heat it up here and suppose you have got some kind of cooling device by which you cool the upper part. So what will happen? It will tend to go up and this fellow will tend to go down. But it cannot really go up in both the directions. It cannot go down in both the directions. So otherwise it will tend to take this route or that route and these are represented by these two different directions of rotations that you see on the computer. so such a simple thing where the at atmospheric circulation has been reduced to this simple annular ring and heating in one spot and cooling in another spot that can easily be modeled and he showed that it can be modeled into these three simple set of equations and he was trying to understand how this simple set is.

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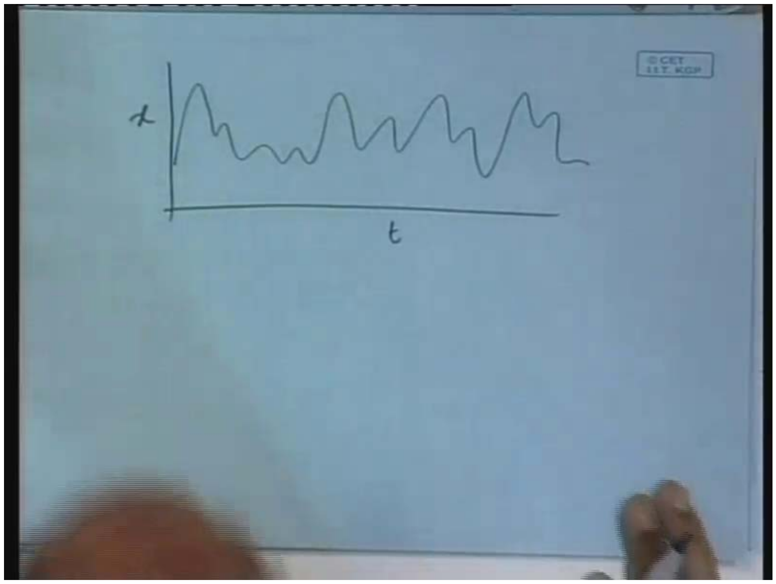

Now as for any system, for example, many of you are electrical engineers, some of you are may be mechanical engineers, for any given system you will write down the set of differential equations and you solve them and ultimately you get an evolution of something like this. Some kind of an evolution against time. He also got an evolution against time. This will also give an evolution against time. There was nothing very holy about it. But then one day he noticed something clear because I will as I told you that those days the computation procedure was not as simple as you have today. So the computers were slow. the programs were slow and you had to give those brick size collection of cards and finally that goes into the computer and then the next day, the print outs come out in the form of a thick collection of printouts. Now supposing you do some computation for some time and then you want to start the computation all over again, what will you do? The graph will not be available to him. It was the numbers that were printed out. So the numbers at the last column would be entered into the computer and that would go on again. If you go for lunch, you enter another card with those initial conditions and let it run again and then after the whole day's work, you go home, note down the final value and then next day you come start all over again. So that is how it ran. So one day he did the same thing but he noticed while he went for lunch, he entered the final value and after he came back from lunch, he entered the initial value as the final value of the last computation and it went on. It so happened that the same computation had been done the last day while it has been done by ignoring the lunch time. That means he went on. So he had at the end of the day two computations starting from the same initial conditions but one which had been interrupted at the lunch and one which had not been interrupted at the lunch. But there should not be any difference because final value was entered by hand. But two days after he noticed that for some time, the two wave forms were almost the same and but after some time, it was completely different. That was the starting point of what we today understand as sensitive dependence on initial condition. Let us try to understand here.

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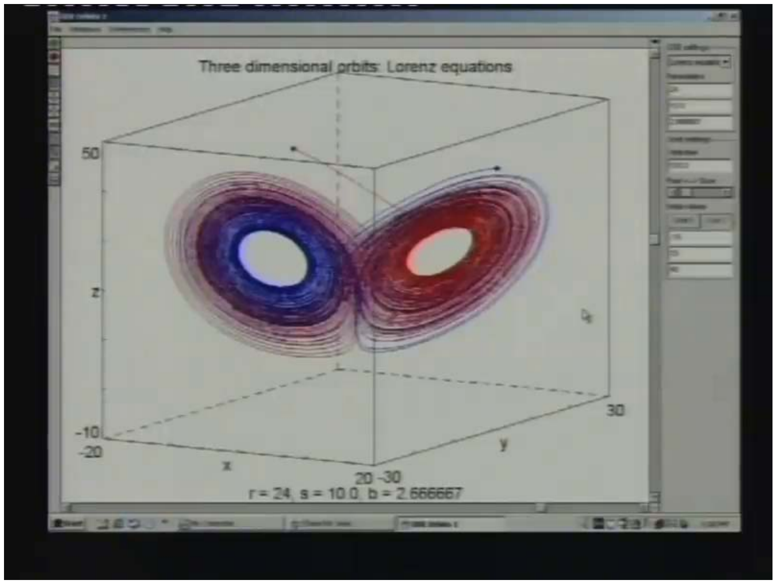

Suppose the two initial conditions for this system, let us make it 26 say, and let us make the two initial conditions, the blue one and the red one very close to each other. Probably you cannot read this but the starting point is 15, 10, 40 for the blue one and the red one, I will make it 15.00001. They are sufficiently different and close. So we will restart the whole business.

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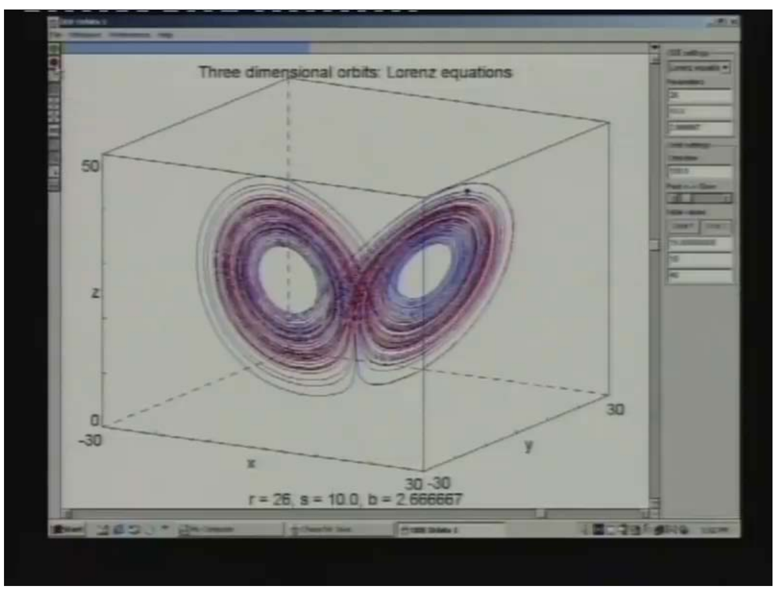

Now they are going together. That's why you can see them. Now they have started going in two different directions. You must say that 0.0001 is not sufficiently close to each other. Let me stop it and run all over again.

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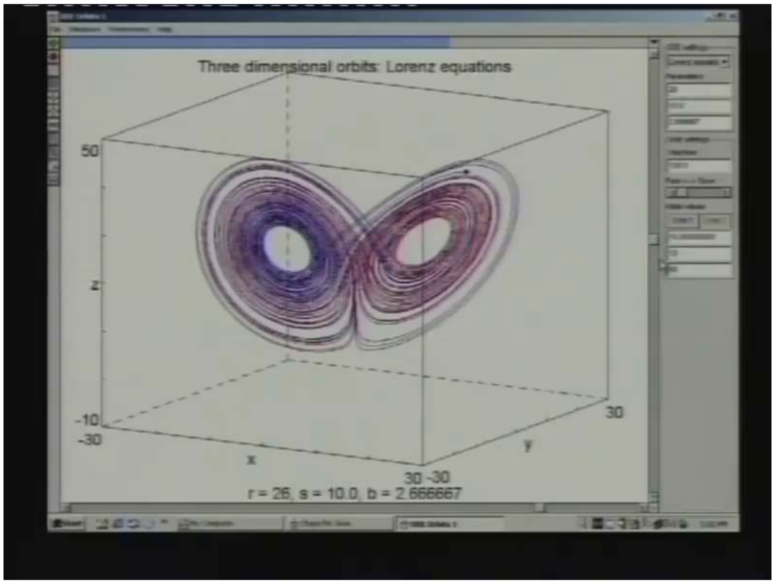

They are moving together. That is why you can see only one. You might notice that though they are evolving independently and therefore they go in two different directions leading to different predictions about the behavior of the system or the state of the system yet more or less, their behavior overall is highly predictable. All remain within this range. They move in those two Eigen planes and so they are not completely unorderly in the common parlance of the term. So what is happening?

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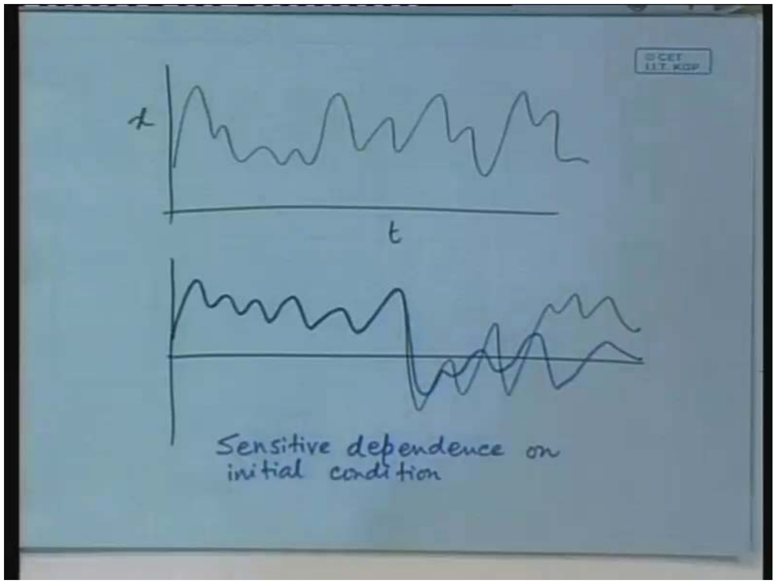

You have started from arbitrarily close points and one evolution would be showing something like this and other one would be starting at almost the same point and it would go on evolving almost the same way if you plot them in time. Finally at some point some deviation will start popping up and then after some time you will find that they evolve in completely different way

this phenomenon is one of the hallmarks of the case it is called sensitive dependence on initial condition and in fact, however fast these two initial conditions may go, it will only increase the time for which they more or less follow each other. If you make it 10^{-12}, that kind of close to each other, then it will only go together for a longer length of time but after sometime they will be in going separate directions. There exists no separation which will make the two evolutions the same. What is the physical import of this? The physical import was to very long time sink in the scientific community and since you are learning this subject some 25 years after this event, we are at a better place to give some idea about what is happening but it took a very long time for people to really realize what is happening here. See the point is that throughout human history, we have done this business of starting from initial condition, trying to predict what will be the behavior of anything later. The whole business of classical mechanical essentially this. What Newton did was essentially this. What all the physicists after him did was essentially this. Trying to observe the objects in the sky like mars, the moon; take the reading of the initial condition, write down the system equations and solve it. That is how they finally were able to predict where these boundaries will go and they were successful and you see in the engineering community will do the same thing. When we try to design a circuit, when we design a system when we design a mechanical apparatus, then we do the simulation, that means we write down the differential equations we send it up with an initial condition and we see how it behaves. if you can predict that it will go from here to here in the state space, it will really does go from here to here. What creates this confidence? It is this. Now look at it from a different point of view.

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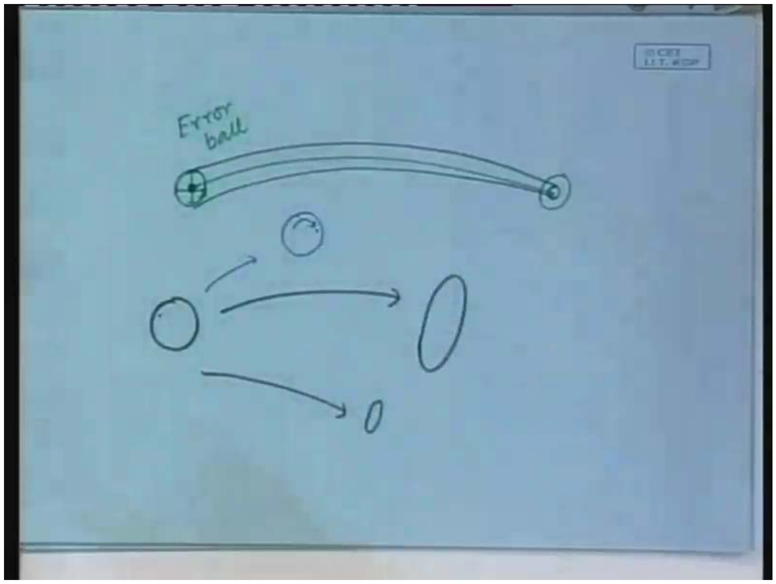

Suppose you have measured the initial condition to be here in state space, for all practical purposes, is it really a point? No, because supposing you are measuring the position and the momentum of a pendulum, can you measure to the accuracy so that you can state that it is a point? You can never do that. In fact, astronomical observations on the basis of which most of the predictions are made are even more inaccurate. So they will always be some inaccuracy in the specification of the position and the momentum, the variables in the state space and therefore you cannot really say that this is a point. All you can say is that I have measured it to this value but I know how much my error bounds are. Normally a proper scientific measurement will always specify that this is my error bound, this is the accuracy of my instrument and therefore I know that I have measured this value but I know that it could be erroneous to this extent but not more. One would say that here there is an error bar along this axis this is an error bar. So for all possible scientific statements, one would say that the initial condition is somewhere in this error ball. so when you actually do the computation, try to predict what will be the state of the system say one hour later, what are you doing? You are actually evolving from this initial condition to another state. But you can also apply this to all the initial conditions within this error ball. If you do that then the error ball will change its shape when it reaches here suppose it has become this much which means that it has evolved somewhat like this. A bigger error ball has become a smaller error ball. Then even though the initial estimate of the initial condition was inaccurate, you can say with reasonable confidence that the final prediction is accurate. So, initial error actually is shrinking as we are going on the prediction process. The character of the vector field is such that your final prediction still becomes quite accurate and it becomes somewhat insensitive to the error in the initial measurement. Till the 1960's, people had encountered or studied this kind of system in the mean. What is the character of the system that the error ball containing the initial conditions that shrinks as the system evolves. Now you might again make a measurement here and then the measurement here would have the same kind of the accuracy as here. So you will get an error ball again like this. So you noticed that having measured the initial condition earlier and having predicted is more accurate than the measurement now.

If that is true, then our predictions are accurate reasonably accurate. In fact our predictions are accurate. How successful is the Newtonian program. This is the result of the Newtonian program. His program was that I will write down the differential equations, I will set out the basic rules that formulate the differential equations and finally those equations have to be solved and finally you get them. See in 1995, there was a total solar eclipse. About a couple of years before that, the scientist had calculated using this program using the Newtonian program. He predicted that the totality line will pass through this particular zone and the totality belt will be this big. If you stand at the particular point of time from this time to that time, you will be able to see the total solar eclipse based on this program. Having observed the position of the sun, moon and earth earlier and having predicted by writing down the system equations, solving the system equations, they had predicted. How accurate that where we stood at a point, we watched our clock we had to synchronize it with the Greenwich clock but nevertheless we kept the watch. It started exactly at the time. not only that, in order to make that prediction in future even more accurate, this scientist will have to observe it properly. So professor J. V. Narlikar had a program in which he stationed school students along the fringe of the totality belt asking them to note at what time it started and about what time it ended. That information were again put together at one place to correct the prediction for the next point. This is how the Newtonian program works. It works with that kind of accuracy. Now we know that is because of this particular property that you can do the predictions. You can shoot and you can finally find out where it will lie because of this property.

Now why is this happening? This is because now we can easily understand in terms of the character of the system here. If you are observing at that point you can at this point locally linearize the system equations. You can obtain Eigenvalues and if the Eigenvalues were real and negative, then we expect any deviation from here along these direction will shrink and along the other direction as well. Therefore this ball which may be have started as a circle will take the shape of an ellipse. It will take the shape of an ellipse but the major and the minor axis both have been shrunk because the Eigenvalues were negative. So as it evolves, the circle evolves into an ellipse but in both the directions, it would have shrunk. So this render is not really correct. It will be something like this. So this is one related to one Eigen direction and the the minor axis also related to another Eigen direction and both directions shrink. So if the system is stable, then you have this property. If the system is unstable, in engineering you do come across unstable systems. Don’t you? They their Eigenvalues are in the right hand side. If the eigenvalues are in the right hand side, it will expand but it will expand to infinity. So it is as if the initial condition, the error ball of the initial condition that is expanding continuously and finally going to infinity. That is what an unstable system is. So these are the two types of systems that we had encountered so far. The system that we see around us which are stable have this property that in all the directions, they shrink and if they are unstable, they expand and go to infinity. We will learn how to keep instability within bounds and how to stabilize a system that is a whole gamut of control theory. Just one concept if you have the starting a setup initial condition as the error ball; a circle then it will take the shape of an ellipse if the Eigenvalues are real. If they are complex conjugate with a negative real part, the orbits will be spiral but inward spiral as a result of which these set of initial conditions will shrink. But it will rotate. It still remains a circle but the initial condition that was say here will rotate and remain a circle. But it will rotate and will come somewhere here. Are the concepts understood? See I am not really solving an equation now. I was so far trying to convey the ideas about how we can relate dynamics with the

Eigenvalues or Eigenvectors and at the end of the day; you should be able to visualize that. A situation with real Eigenvalues will mean that a ball of initial condition with shrink but will take the shape of an ellipse. A ball of a complex conjugate eigenvalue it means that the ball of initial condition will rotate as it goes on it will rotate but depending on the real part, it will either shrink or expand. There are two possibilities.

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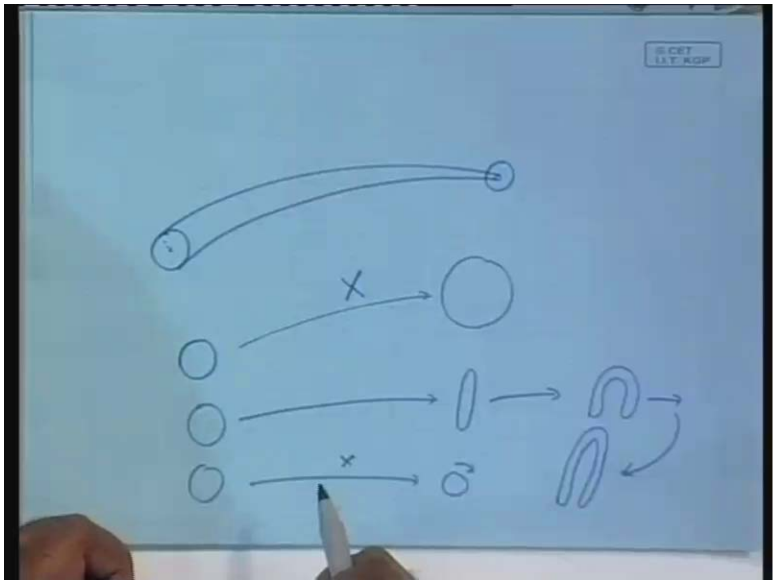

Conversation between student and professor: his question is that: if you are measuring somewhere here and had obtained an error ball and you predict using the set of differential equation and say it becomes this much and now you observe it, you still have the error ball this much. See we are making the assumption that from here to here, the time is not so large that the technology is improved so that you can measure the error with a larger accuracy, a smaller error ball. I am not assuming that. I am assuming that the technology is more or less the same so that if you observe now you will have some error ball if you observe it now it will have same kind of same size of error ball but the character of the system is such that if you write down the differential equations and start from all the initial conditions here they will land up here. So even though you do not know where it is in the initial condition, here you can predict with confidence that it should be in this small ball at this point in that sense we are not really underestimating the error. It is the character of the system that prediction becomes more accurate than observation now. That is a character of all stable systems. While studying contour theory you didn't realize this but this true for all stable systems stable dynamic systems. So we have a situation where have come across a system in which there is sensitive dependence on initial condition. We have just seen that in front of your eyes. Sensitive dependence on initial condition means that two initial conditions within this error ball will go apart as the time flies. Obviously it cannot be like this. If it is unstable, then what will happen? It will go to infinity. It doesn't go to infinity. It remains bounded. So now we can specify the character of a chaotic system. Chaotic system means bounded aperiodic behavior. It's not periodic. The same state is not coming back. It is not periodic. Same state is not coming back yet there is sensitive dependence on initial condition.

So three essential properties of chaos. They are bounded, aperiodic and with sensitivity dependence on initial condition. Just bounded will not serve the purpose because sink type equilibrium point is also bounded. Aperiodic will not serve the purpose. I will come to that later because there are orbits which are bounded, aperiodic but there is no sensitivity dependence initial point. So all these conditions satisfied is chaotic. Let us see how this this condition can happen. This means somehow you have started from this initial condition. Can it take the shape like this in the chaotic system? Obviously it will not because if it takes the shape, as it goes on goes on increasing and finally it will go to infinity. That is not possible. That is not the character of this. So this is not true. Is it possible to have it like this? This means it takes the shape on an ellipse. If they take the shape of an ellipse, then the major axis might keep on increasing and go to infinity but still we will to do something about it. Can it be like this? It becomes smaller and only turns as would happen for complex conjugated eigenvalues? No because in that case it will shrink to zero. So this is also out of question. So something is happening like this. But then we still do not have the answer as to how can that remain bounded. It will inevitably go to infinity after sometime. What happens is a trick. If it takes a longer shape, then after sometime this can also bend. Then this will keep on increasing which means again it will take the shape and this whole thing can bend. That means when it bends, it looks something like this (Refer Slide Time: 41:11).

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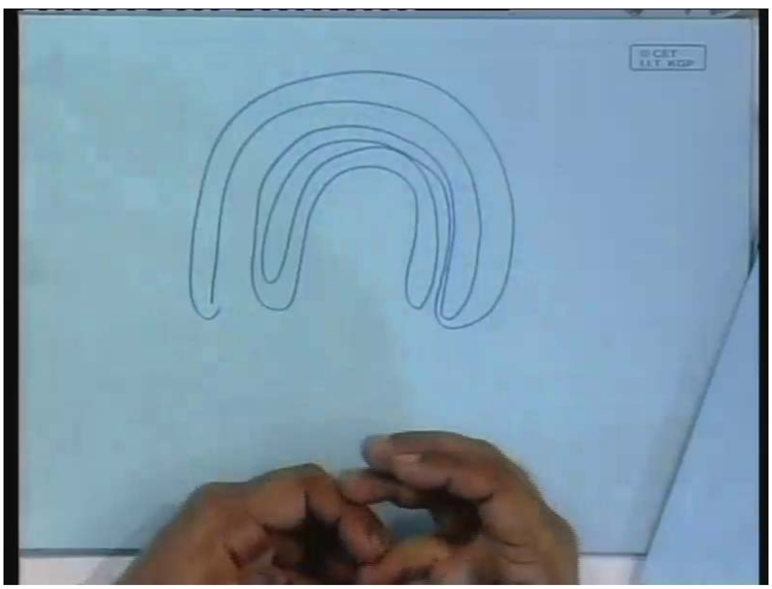

If you take this, stretch it and bend it, you will get this kind of shape. Have you ever seen your mother making parota? Role the stuff and fold it again. The whole thing is rolled and again folded. She does that because in the folds oil would go in and that makes it very tasty. But this is what is happening in a chaotic system. It is stretched and folded. So the state space of the chaotic system is a serial process in which the state space is stretched and folded. So that process goes on.

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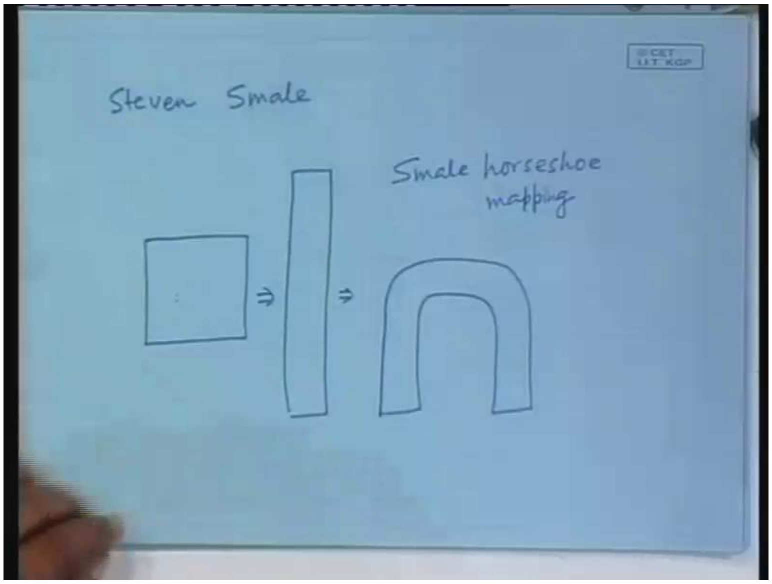

In fact, this is clearly illustrated by a process that was proposed as a mathematical iteration procedure by Steven Smale. It is same Smale who got the field medal for his work on other areas. He is a great mathematician. He said, “let us start from a square”. I will define a process of iteration with this square and the way that topologies work, I will do all sorts of processes with it without tearing it. So I will hold it like this and will stretch it. So in the first process he will stretch it which means that this will become longish but this side it will shrink. In the next process, he will take it and bend it like a horse shoe. Now this is a new shape which can be again taken as the starting point for this process. So this shape will again we stretched and folded. Try to understand what will happen in the next iteration.

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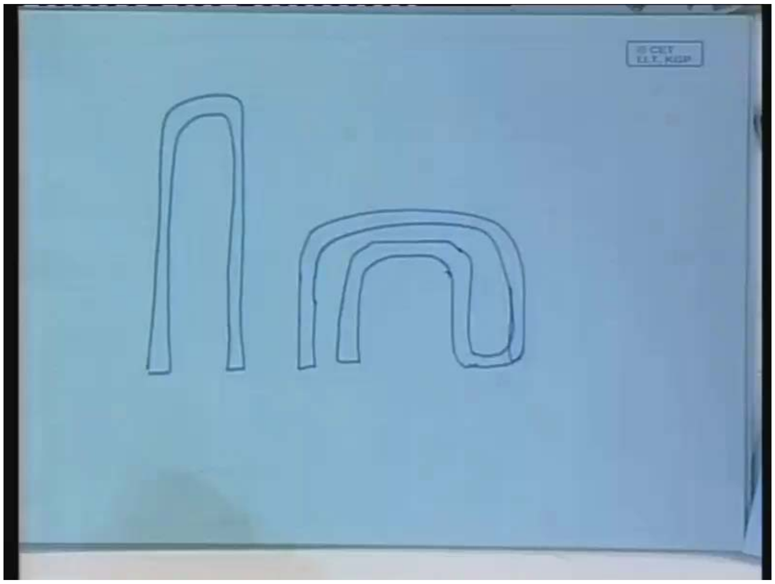

This fellow will be stretched like this. But now it will be even thinner. This fellow will be taken and folded back. Can you now realize what is happening? So we will start with this. This fellow has gone here. So it will go like this here (Refer Slide Time: 44:57). Then this will go like this, this one is bent and if it is bent like this, it will take the shape. So stretching and folding. Again put this here, do this process. Do this at infinite time. What do you have? These will be infinitely thin but infinitely layered structured but we will have the property that to initial conditions in this direction will keep on separating out. The distance will keep on increasing. There will be a sensitive dependence on initial condition without ever running into the situation that has run to infinity. Still it has remains bounded. So this is called this Smale Horseshoe Mapping. The process by which we can understand what goes on in the state space of chaotic systems. Smale Horseshoe Mapping means that this match to this through this process. Again this is continued as an iterative procedure. Ultimately you will end up with this infinitely layered structure that you see in the states space of chaotic system.

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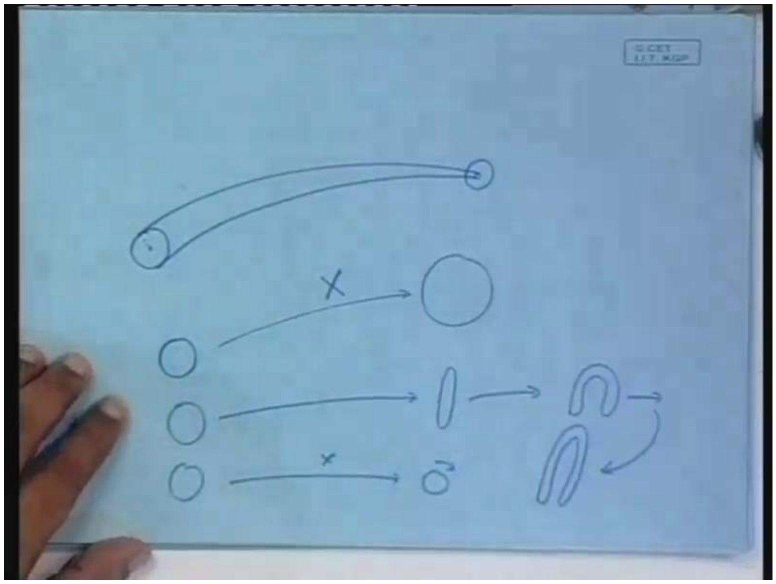

There is another immediate conclusion that in order for this kind of behavior to happen, what is a character of the equilibrium point? If there is an equilibrium point in the system, what is the character of this equilibrium point? It has to stretch. Stretch means there should be one direction in which it is shrinking. Another direction it is increasing. Shrinking means a negative Eigenvalue. Increasing means a positive Eigenvalue - e to the power lambda. It should be positive and therefore there should be some fellow sitting here. It has a character of a saddle. So a saddle equilibrium point must be responsible for this kind of a behavior. In such systems there would normally be at least one saddle equilibrium point. Now we have more or less understood what chaos is. This word is actually somewhat of a misnomer. This word was coined by James Yorke.

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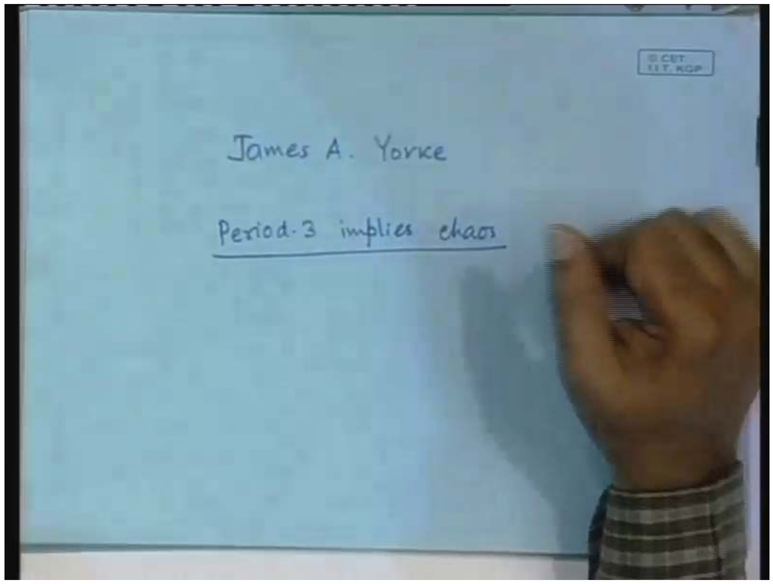

He wrote a paper titled “Period-3 Implies Chaos”. The paper is very provocatively titled. We have seen that period 1 system can be there period two systems can be there and he prove mathematically there if a system allows period three behavior means that it also allows chaotic behavior. so that was his claim but than this is the first time that the word chaos was used and since the word chaos is used in common parlance, more often in IIT than elsewhere, that is why it is it it generally takes a takes a meaning in over heads. But here it has a completely scientific connotation. the scientific connotation is that it is that kind of a behavior of a dynamical system where the behavior is bounded, sensitively dependent on initial condition and at the same time, it is not periodic. look at it for another point of view that you can have a periodic orbit; period 1 orbit you can have a period 2 orbit at that that we have already talked you can have a period three orbit you can have a period many orbit and you can also have a period infinity orbit. so this is nothing but that period infinity orbit – bounded. That is the character of chaos. so what is a implication of the sensitive dependence on initial condition? The implication is that prediction fails because if you want to predict you what is the tool? The tool is that you first get hold of the initial condition, plug it into the system on the differential equation and predict. the solution of the system with differential equation is your prediction. if there is a sensitive dependence on initial condition, even if the initial condition is having very miniscule error, it will lead to completely different final results and therefore prediction will be completely useless. that is why after Lorenz’s work, it is now more or less understood that weather prediction is very much useless. there was no prediction but still why do these guys use super computers to try to predict? If it is true mathematically that prediction is very much useless in such systems, why do these guys spend millions of rupees on buying super computers and trying to do something? it is because the two actually did follow each other sometime which means that prediction is good within a certain time frame. Weather prediction therefore is not infinitely good. You cannot really keep on predicting the weather. But you can predict within a day or two and that is exactly what they try to achieve or if the error is very small.

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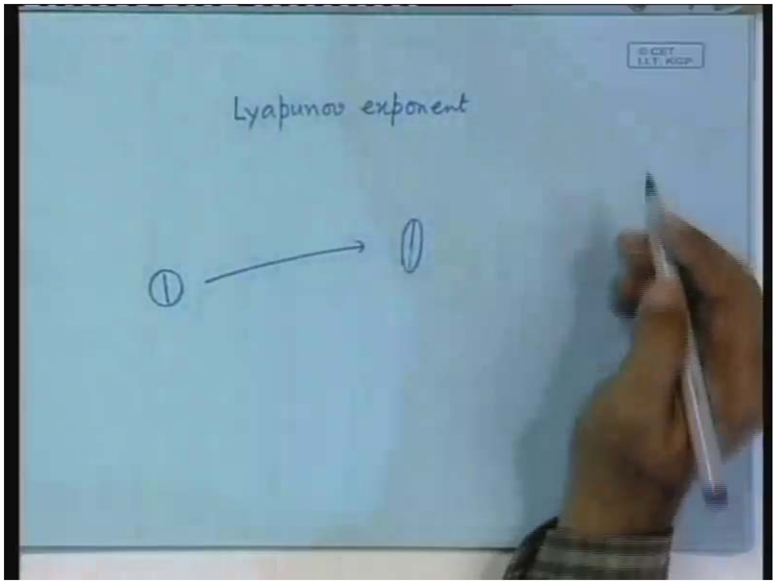

Say if this (Refer Slide time: 52:49) error ball is evolving into an ellipse but the rate at which at this direction expands is not very large. That's what you mean .now this is related by what? you can see that this expansion of this line into that line is the result of e to the power lambda 2 with lambda positive. So if you can measure this lambda, we can sort of specify the rate of expansion and your point is correct that the extent to which the prediction will be good will depend on that lambda. But this is a non-linear system. So you cannot measure lambda everywhere. We learnt how to measure the Eigenvalues at the equilibrium points and these are not happening at equilibrium points. It is just evolving. What do you do about it? You could, for all practical purposes keep on following that orbit. Suppose you just calculate the orbit and for every point you keep on obtaining a Jacobian and calculate the eigenvalues and take the average. that's a reasonable estimate of the rate of expansion and if that is not very large to the positive side a small positive number may be, then you would say that, "Okay, even if I know that system is chaotic, I can still predict to say this may reduce", with confidence. Therefore following your lead I can tell that the measurement of this rate of expansion is a vital importance when it comes to deciding whether I can predict or not or to what time frame I can predict. This is a very important number. In the next class we will take up the issue of how to measure this number. Notice that it has the character of the Eigenvalue but it is not really the Eigenvalue in the sense that this system is a non-linear system. Eigenvalues can be computed at every point but here we talking about the average positive Eigenvalue so that is not really the Eigenvalue. It is sort of a eigenvalue average over the orbit not just everywhere in the state space. As the orbit goes, I am calculating the Eigenvalue everywhere and then averaging the positive one out and you are getting a number. these numbers which have the character of the the Eigenvalues but related to the non-linear system are called the Lyapunov exponents. Obviously in a three dimensional system, there will be 3 Lyapunov exponents. Out of that if one is positive, I would say that there is sensitive dependence on initial condition. so one of the conditions of chaos is that at least one Lyapunov exponent must be positive. That's all for today. We will continue in the next class.