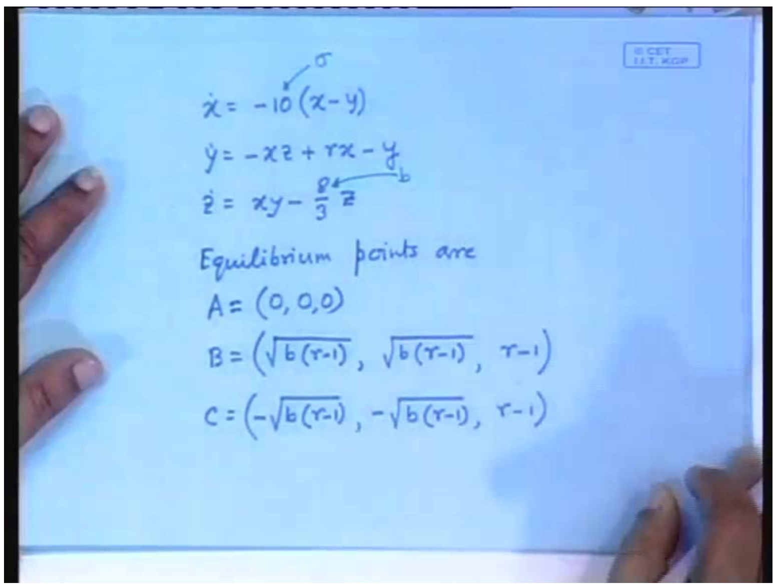
Chaos, Fractals and Dynamical Systems
Prof. S. Banerjee
Department of Electrical Engineering
Indian Institute of Technology, Kharagpur
Lecture No. # 04
The Lorenz Equation - I

In the last class I gave you this problem.

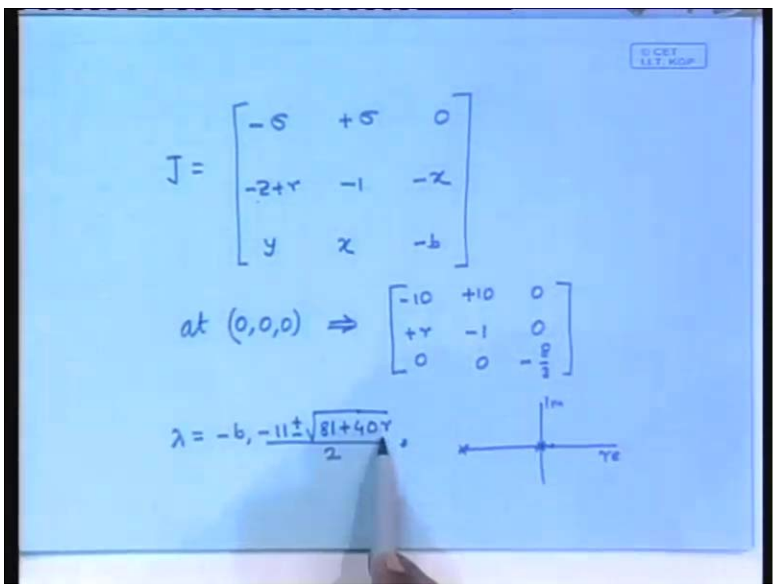
(Refer Slide Time: 00:00:57 min)



Now we will just substitute the values so that you can easily calculate. So this is the state of equations that we started with and I asked you to do two things. First, as you have seen for any dynamical system, if the system equations are given, it essentially defines the vector field and we should first try to get a grip on the character of the vector field and for that we have understood that the method is that we first locate the equilibrium points. Then we locally linearize around the equilibrium points and try to understand the behavior around the equilibrium points because these are pivots in a state space. These are the places where you have good understanding. So first understand those parts and then in terms of that, if you can figure out how the behavior is going to be, then it's fine. If it is not possible to figure out then we will look at the rest of the state space. So go by that process because that will give you some understanding. First where are the equilibrium of this set of equations?

Obviously (0, 0, 0) is one but that is not all because these guys are there. So it should give rise to other equilibrium points. Can you find out? So let us call them A B and C three equilibrium points. The equilibrium points are A (0,0,0,). You get the left hand side zero. B is (root b(r-), root b(r – 1), r – 1). The third equilibrium point, I can see that it should be symmetrical. C is (-root b(r -1), -b(r-), r-1). The moment we have it that immediately tells you a few things. First notice that if r is less than one, what happens? This fellow gets imaginary and the position of equilibrium point must be real because it is in the real state space. eigenvalues can be imaginary or complex whatever but the position should be real and therefore these equilibrium points B and C do not exist until 'r' the parameter reaches a value of one. So that can be inferred immediately. But A is always there. As you change the parameter say starting from a value that is less than one, suppose you are increasing the parameter, then you find that this was there and this was there. We will look at the stability later. This fellow was there existing and at that particular point, these two fellows come into existence earlier it given not there. Only that much can be inferred from these equations. In order to infer more, we need to look at the stability of the equilibrium points and the stability of the equilibrium points are obtained from the Jacobian matrices. So get the Jacobian matrix.

(Refer Slide Time: 00:06:20 min)



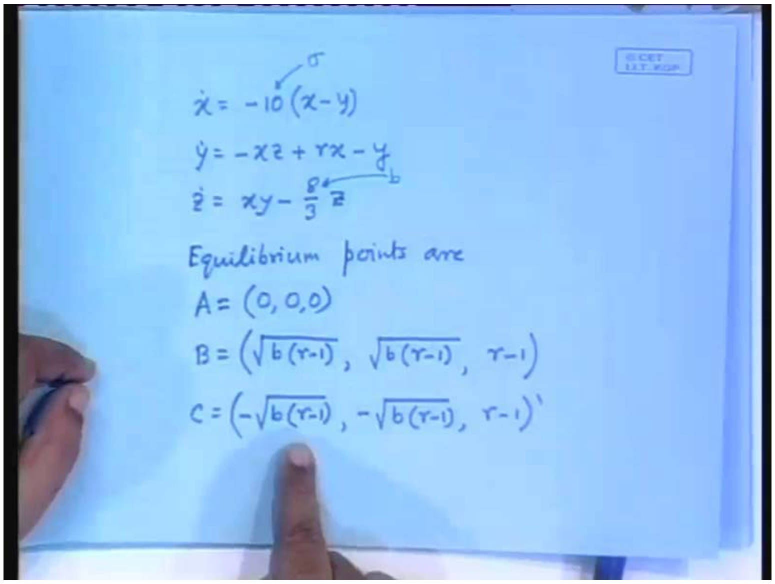
We will obtain it in terms of a general thing and then we will put the values. So this fellow was our sigma (Refer Slide Time: 06:43) and this fellow was our b. let us do it generally so that you can use other values of sigma and b also. So the first term will be - sigma + sigma 0. Second equation with respect to x, it is - z + r. with respect to y,-1 and with respect to z, -x. similarly here it is y, x and –b. now we know these values. we have taken some values here. We have kept r as the variable parameter but for each equilibrium points we can substitute these values x y and z.

Now if you substitute the value of (0,0,0), then what you get? You have the matrix -10 +10 and 0. Here it is +r -1 and 0 and the 3rd row is 0 0 and -2/3. So can you find out the eigenvalues of this in terms of r? There will be three eigenvalues. Lambda is –b, -11+/- root (81+40 r) /2. Now let's see what does this tell us. At r is equal to one, you get 0 and -11. This one is square about then 0 and -11. So there are two eigenvalues. One is widely negative. So I can comfortably assume that that fellow will remain negative but the other fellow is zero. Zero means exactly at the border line between stability and instability. For r further negative then what happens? There would be a for some values of r for which this will be real. Some values of r for which it will not be real. So find out for which values it will be stable. For r is equal to one we have found that it is minus eleven and zero. So if you draw the the real and imaginary parts, one is here and another is here. As you change the parameter I can see that either r values larger or smaller, it should go this way and therefore the equilibrium point will become unstable.

So here is something that is stable eigenvalue. This part is also a stable eigenvalue and the other part which is now having the value zero is the suspect case which can make the system unstable. For which value of the parameter does it become unstable? It is one. There are three equilibrium points A, B and C. this fellow was stable at r =1. This fellow becomes unstable and these two fellows come into existence. So you are changing the parameter. One equilibrium point was stable. So you are happy about its behavior. Start from any initial condition it will go into that and at that time all the Eigenvalues were all real and therefore it would nicely go into it. At this particular value of r it becomes unstable. So what are the Eigenvalues? Then it would remain real. We simply substituted at r = 1 because we had some hunch that something happens at r = 1. We substitute r = 1 and realize that that is a critical case. So move it this way or that way it is going to be unstable. But when it becomes unstable, is it still a real pair of eigenvalues or is it a complex conjugate pair of eigenvalues? Here it is still real pair of Eigenvalue. They are still real pair of eigenvalues.

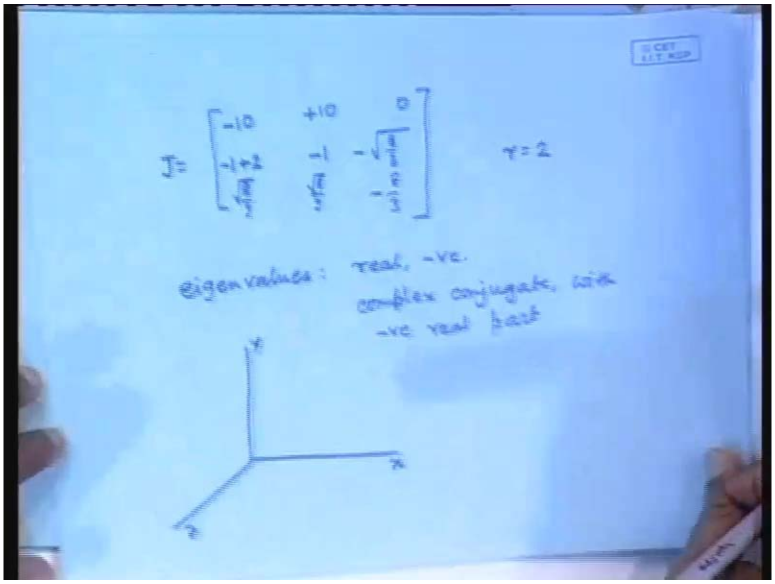
So you have got the fellow unstable at that point if you start from that that initial condition or somewhere close to it will go away. This is a stable eigenvalue. The other one that was close to -11, that's also stable eigenvalue. There would be Eigen directions associated with them and any initial condition along those eigendirection will not go away. They will come close to the equilibrium point even though the equilibrium point is unstable. But a slight deviation along the unstable eigendirection will grow exponentially and it will go away along that direction. So it is not just unstable anyway. It is unstable in a specific way in a particular direction. Now let us try to understand the question: if it is unstable along that direction where does it go? In order to answer that question you have to find out the stability of these two fellows who have come onto existence.

(Refer Slide Time: 00:16:44 min)



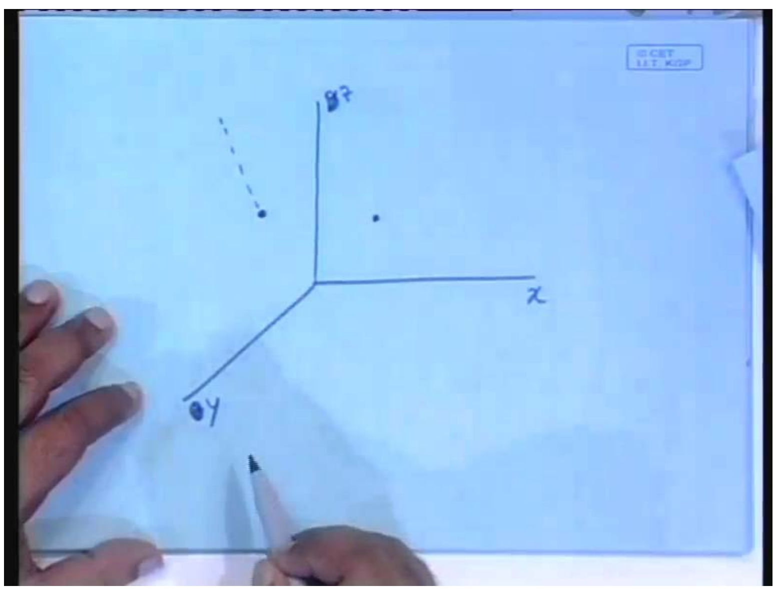
If they are stable they will ultimately land of there and stay there. So find out their stability for that. All you need to do is to simply substitute these into here (Refer Slide Time: 17:00). At this stage unless you put values you will find it a little difficult to handle. Calculate the stability when say, the value of r has been pushed to an extend that is greater than one say, two. So what will you do?

(Refer Slide Time: 00:17:37 min)



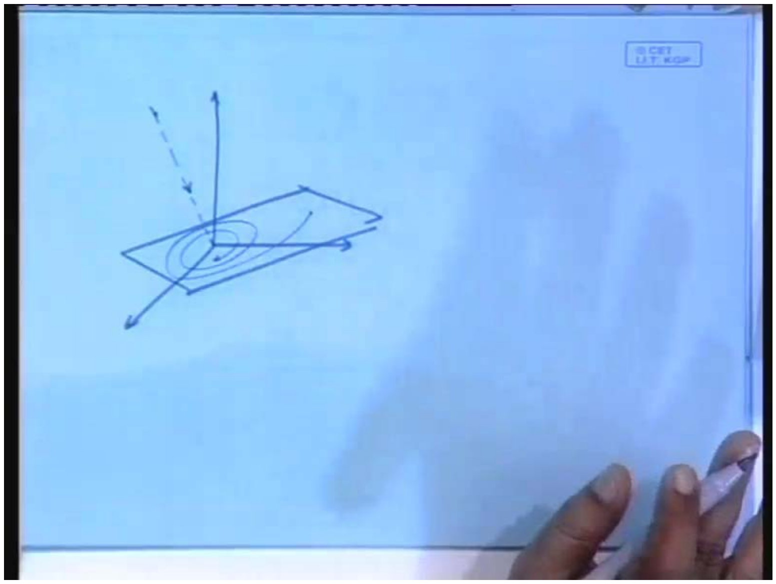
Now my Jacobian matrix is -10 +10 0. -1+2, -1, root (8/3), root (8/3), root (8/3), -8/3. So you can always find the eigenvalues and tell me whether this particular equilibrium point is stable or unstable. Can you figure out the type of eigenvalues that you are going to get. One thing is to get the values of the eigenvalue. You can always plug it into Matlab and get it. So I am asking you can you figure out whether it there going to be real or complex or stable or unstable? That's suffices our purpose? For our purpose that will suffice the character of the equilibrium point. I don't really need to know exactly the eigenvalues. I think you will get one real negative eigenvalue and two complex conjugate eigenvalues with negative real. So at this parameter values where I have assumed r = 2, that means I have pushed it beyond the r =1 level then these two equilibria are stable but with a spiral character. Now try to understand what happens. I am just writing the character of the eigenvalues. First Eigenvalue is real negative. The other two are complex conjugate. Can you infer then in the 3D states space the kind of orbit that you are likely to see, where are they? These two equilibrium points, where is the position say b, where is it if you substitute the values? It's root 8/3 positive.

(Refer Slide Time: 00:24:33 min)



So along the x coordinate, you have got something, along the y coordinate you have got something and along the z coordinate you have got something. If these are the two equilibrium points, now if you have the real eigenvalue, there should be an eigenvector along which this fellow will be stable. So you can see that? Suppose this is the eigendirection along which it is stable. Eigenvector associated with the real eigenvalue. They should also be associated an Eigen plane associated with the complex conjugate Eigenvalues. How to calculate that? These are concepts that possibly you need.

(Refer Slide Time: 00:26:30 min)

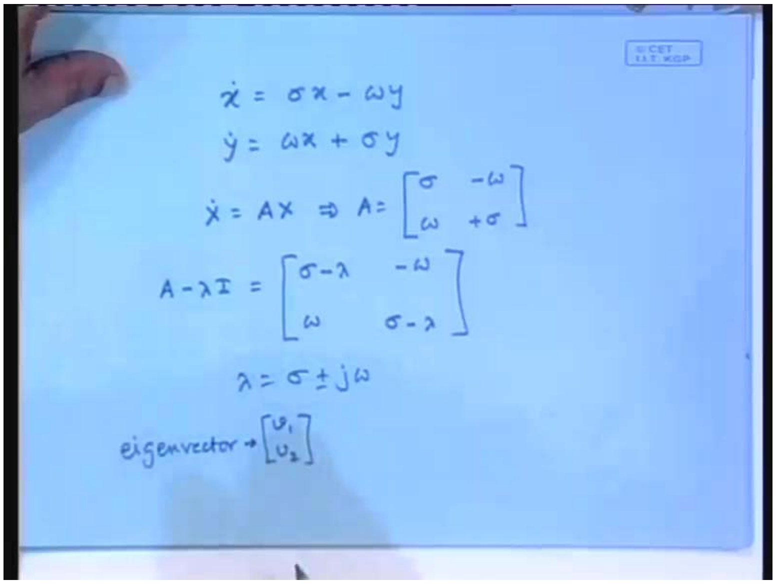


Here there was a complication that the the equilibria are not at origin. So let us assume that you have got an equilibrium at the origin. Suppose this is one stable direction associated with the real Eigenvalue which means that if you have an initial condition here, it will exponentially decay. Suppose there are a pair of complex conjugate Eigenvalues. There should be some kind of a plane associated with it so that any perturbation along that direction will die down like this. any deviation from this plane will die down because of the action of this stable Eigenvalue and any deviation along this plane will die down because of the action of the complex conjugate Eigenvalues with negative real part so that if you start from any initial condition away from here it will behave like this and it will ultimately go towards that. This is an Eigen plane in the sense that if you start any initial condition on this plane, it will forever remain on this plane. In that sense it is an Eigen plane. How to identify that Eigen plane?

Try to recall the way you solved the problem with the pair of complex conjugate Eigenvalues. What did you do you? First if you write down the characteristic equation, in the case of a two dimensional system, you got a quadratic equation, in the case of the 3D system, a cubic equation. Whatever it is but its solution gave you the complex values. Now these are Eigenvalues. How did you calculate the eigenvectors? You plugged in these eigenvalues into the equation (A-lambda) I X = 0. Thereby you obtain the eigenvectors. Now these eigenvectors in this case will also turn out to be complex conjugate. So, Eigenvalues complex conjugate if you obtain in the same way, blindly, I have this eigenvalue and therefore I try to obtain the eigenvectors. You will get complex conjugate eigenvalues. Then what was the logic?

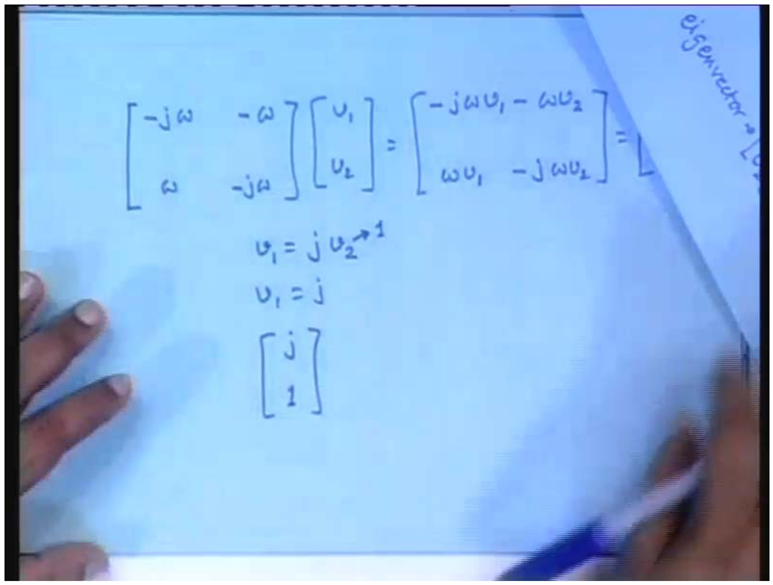
The logic was that you would say that this complex conjugate Eigenvectors were consisting of a real part and the imaginary part like P + JQ and then you would say that P + JQ is a linear combination of the P and Q and therefore P and Q are both independent solutions and thereby you would say that any solution can then be constructed out of these two real solutions. That was the essential logic. Let us solve a problem simple problem.

(Refer Slide Time: 00:30:43 min)



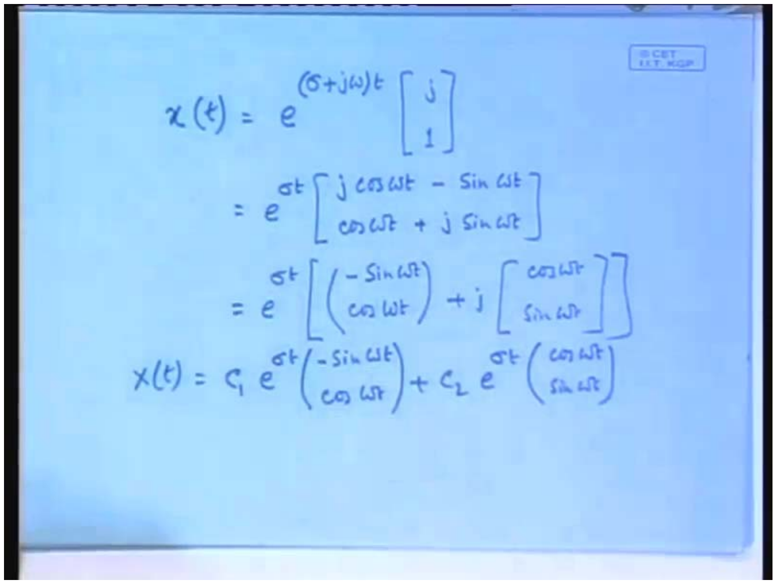
x dot = sigma x - omega y and y dot = omega x + sigma y. from this you will have x dot = Ax where A is a matrix sigma -omega + omega + sigma. Then in order to calculate the eigenvalues, you would write (A – lambda I) = sigma – lambda, -omega, omega, sigma – lambda. The determinant of this would have to be 0 so that equation then becomes lambda = sigma +/- j omega. The equations have been written such that you get it essentially. This is very familiar. So I assume that you are feeding in the familiar domain. You will have to find out the eigenvectors associated with this eigenvalues.

(Refer Slide Time: 00:33:20 min)



Say an Eigenvector is v1 v 2. Then you would write - j omega - omega omega -j omega times v 1 v 2 =0. So that will be written as - j omega v 1 - omega v 2. This is omega v 1 - j omega v 2 = 0. That’s what we write. Now you see they both are the same equations as happens for all Eigenvector equations. But then from here it is possible to identify the eigenvectors. So what is the eigenvectors? From this line you can write v 1 = j v2. That is the eigenvector equation. So it involves j. in that sense, it is a complex conjugate now. How would you identify this? We have said that the eigenvector is this vector. These two are related by this. Eigenvector is any vector along that direction. So it will be very convenient for us to simply say that either v 1 or v 2 is 1 and it will be convenient to say v 2 =1. So v 2 = 1 & v 1= j. so what do we have? So here we started with one eigenvector one eigenvalue. Its conjugate take the negative one. It will turn up to be complex conjugate. So we can safely work with this Eigenvector associated with Eigenvalue. Then what do you say? How do you proceed? When you do that in the 3D, you have to do exactly the same way. So only difference is that this will be a three dimensional equation, you will get a three dimensional thing but nevertheless ultimately it will lead to conceptually the same thing.

(Refer Slide Time: 00:36:12 min)

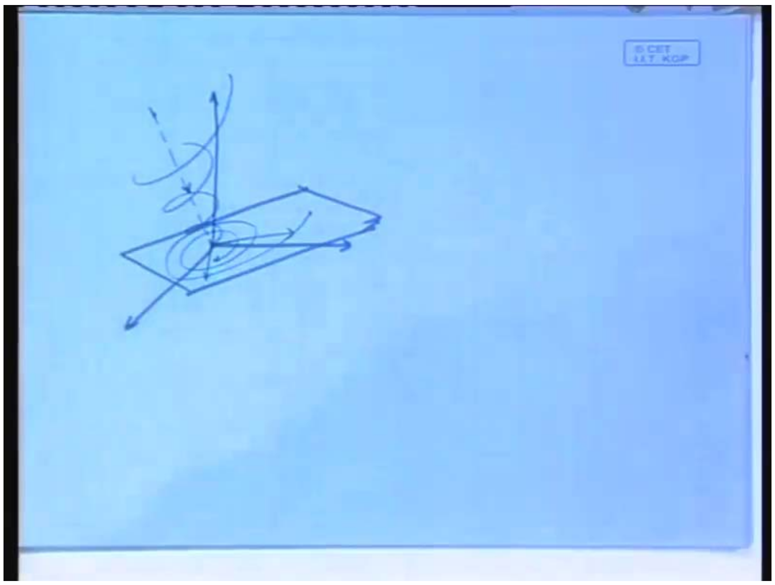


So we have here x(t) as the solution of differential equation is the e to the power Eigenvalue t times Eigenvector. That is one solution. Eigenvalue is sigma + j omega t times the Eigenvector which is j1. That is the solution. So this is one solution. Now we can break it into two parts: e to the power sigma t and e to the power j omega t and we know the e to the power of j omega can be written in terms of sines and cosines. So just do that. Then you will get e to the power sigma t [j cos omega t - sin omega t cos omega t + j sin omega t]. What have we done? We have separated out into two parts: e to the power sigma t times e to the power j omega t e to the power j omega t. we have written as cos omega t + j sin omega t and then multiplied with this vector you got this.

Now we can separate this out into the real part and imaginary part. We will get e to the power sigma t times (- sin omega t cos omega t + j cos omega t and sin omega t). So here we have separated out. Then we say that let this part be called p and let this part be called q (Refer Slide Time: 38:50). This complex number is nothing but a linear combination on the p part and q part and so we can say that ultimately x(t) therefore would be possible to write it as c_1 e to the power sigma t times. This minus sin omega t cos omega t + c_2 e to the power sigma t cos omega t sin omega t. That was the logic. notice that we had complex conjugate Eigenvectors but we went by the same logic that in the ultimate solution, one of the possible solution is e to the power Eigenvalue t times the Eigenvector but then that allowed us to separate out the real part and imaginary part and ultimately we get a real solution. That is solution of this. Now this is a 2D system. In a three dimensional case, all that will happen is the matrix will be a three dimensional matrix - 3 x 3 matrix and you will get a cubic equation. Ultimately from there you will solve. A cubic equation may lead to one real Eigenvalue and two complex conjugate Eigenvalues or all real Eigenvalues.

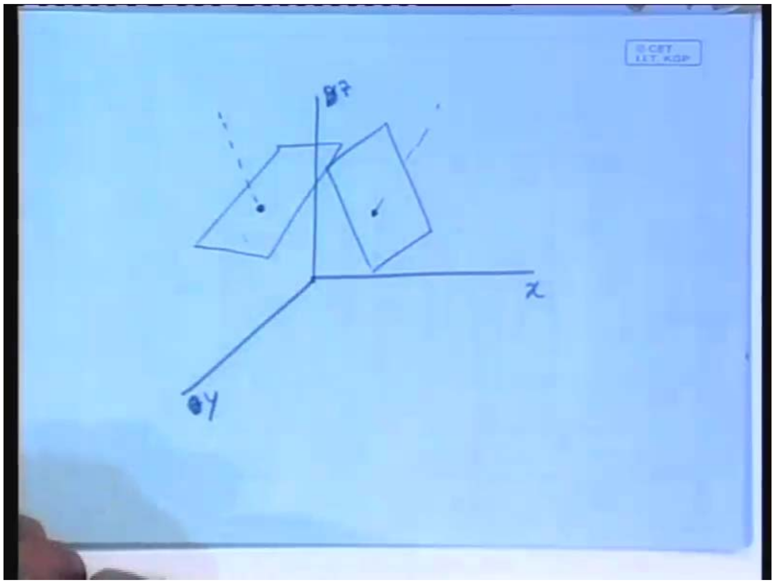
They cannot be in any other solution. Now if you have one real Eigenvalue and two complex conjugate Eigenvalues, if all are real Eigenvalues, it is very trivial to obtain the eigenvectors. They are all real eigenvectors but it is not trivial. That is why this concept comes. What happens if you have a complex conjugated pair of Eigenvalues and one real? This situation that I have just depicted here. Then how do you obtain this plane (Refer Slide Time: 41:20)? Notice the logic that I was to following here. What was the logic that the real part and the imaginary part individually give two solutions I have notice real part and imaginary part of the eigenvector individually give solutions? Real part is a vector in the real space imaginary part that that real component of the imaginary part is also real vector. So you will be able to identify two vectors in this plane.

(Refer Slide Time: 00:41:57 min)



There can be only one plane passing through these two vectors. The method of approach is that if you have a complex conjugate pair of eigenvalues, obtain the eigenvectors and obtain the real and imaginary part separately. The real part will give you one real vector the imaginary part will give you another real vector. The plane passing through that will be the Eigen plane. Have you have you understood the concept? Often this particular concept is not given in the differential equation classes but for us, in order to understand what happens in the states space, it is definitely necessary first to understand this Eigen directions and Eigen planes. Next class I will show you the eigenvalues. From the evolution of a system you will be able to see here is an Eigenplane. I can see that. But before seeing doing that by simulation is blind. But what is really necessary is to grasp the understanding that if you have a three dimensional system and if you have the one real eigenvalue and two complex conjugate eigenvalues, there must be one real eigendirection and one real Eigen plane.

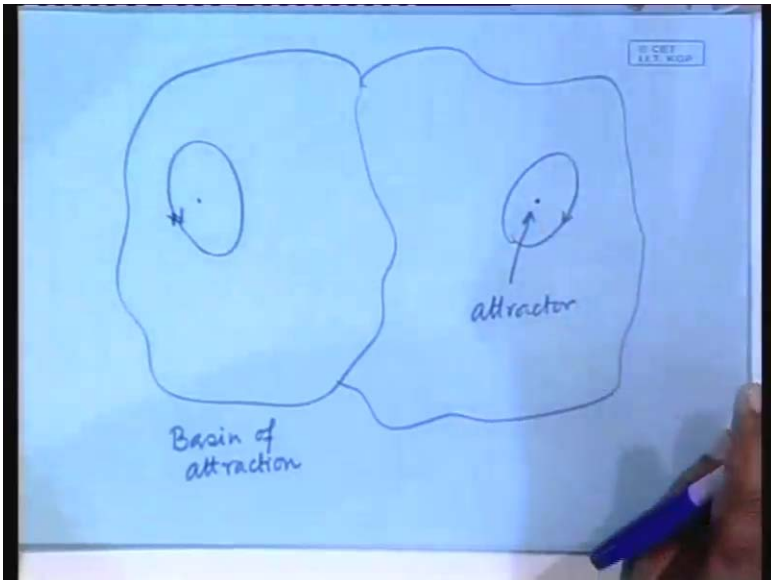
(Refer Slide Time: 00:43:51 min)



So in the problem that we were originally attacking, that is the problem with this system, here also there would be a real Eigen plane and I don't know where it is but let me schematically draw like this. This means that any deviation from here will die down but any deviation from here will also die down (Refer Slide Time: 44:13). It will be an inward spiral. Similarly here now because of this symmetry, we can say that here the directions would be symmetrical to that direction because all the values are symmetrical. So we will say that here is also another plane. So this fellow was an equilibrium point that was unstable from here. It goes away along that unstable eigendirection and when it lands somewhere here it always goes on to either this or that. Both are stable. Will it go here or here (Refer Slide Time 45:04)? You really don't know. It might go here it might go there (Refer Slide Time: 45:13). There is the fun of non-linear system that there can be two equilibria both stable and you do not know which one it will go to. In fact it can go to both the equilibrium points depending on where the initial condition is.

If the initial condition is somewhere here (Refer Slide Time: 45:44 to 45:53), common sense will tell you that will most likely go to this one. if the initial condition somewhere here, it is most likely it will goes somewhere here but this also tells you that the whole state space should be divide into compartments. One compartment and those initial conditions that will ultimately go here, another compartment those initial conditions that will ultimately go here. So even though we have not calculated it today and I ask you to actually calculate the Eigen directions, first analytically work that out that and then do the simulation. Don't do the simulation first. Then you won't develop understanding. but that has at least convinced that there must be Eigen directions like this and there must be Eigen planes like that (Refer Slide Time: 46:42) and depending on where I start from, this state space is divided into compartments and that sort of tells which equilibrium point ultimately I will converge at. Now these have certain names.

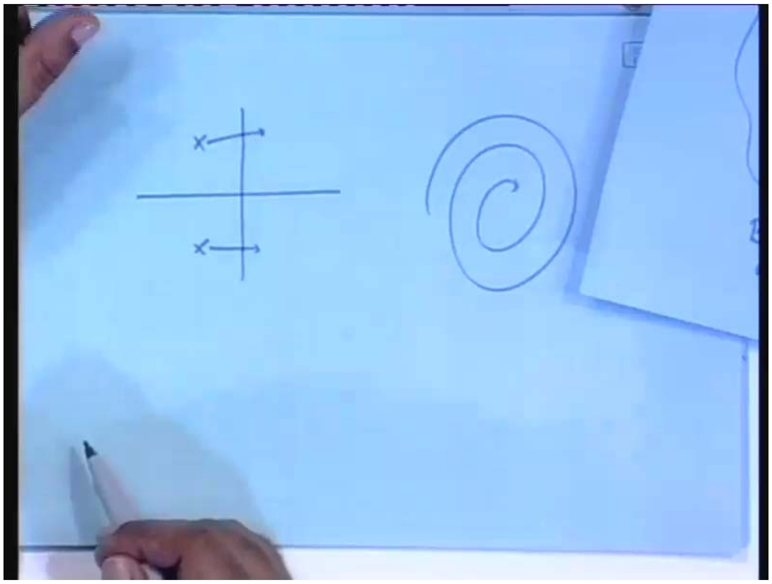
(Refer Slide Time: 00:47:10 min)



A situation where there are two equilibria and this state space is sort of divided into compartments so that from this compartment it goes here from this compartment it goes there (Refer Slide Time: 47:22). These regions are called the “basins of attraction”. So from this basin all initial conditions will ultimately hold on to this equilibrium point. This basin ultimately all hold on to that equilibrium point. Now what is the attraction thing going on here? See, in gravity you have heard that everybody is sort of attracting everything. So sun is attracting the earth. The earth is attracting the moon and everything is attracting the others. So, here in this state space. you can, in similarity visualize these points are sort of gravitating bodies and anything in this vicinity will be attracted to it. In that sense, these equilibrium points that are stable equilibrium points are also called attractors. You might say that the sink type of equilibrium point is an attractor. Well, in linear systems, that is the only type of attractor you have heard of. These two are that type of attractors. But in the non-linear system, you have heard of another type of attractor called the limit cycle. The limit cycle itself is an attractor.

We have discussed in the last class that a limit cycle is itself an attractor. In the sense that if you start outside, there is a spiral that goes inwards. If you start inside the limit cycle, there is a spiral that goes outward. Ultimately it holds on to that. In that sense limit cycle is also an attractor. So linear systems theory will tell you that there is only one type of attractor - “the point attractor”. Non-linear systems theory will tell you that there is also another type of attractor- “the periodical attractor” that goes on and all oscillators as I told you are attractors. Now from the stable equilibrium point, how does the limit cycle kind of behavior develop?

(Refer Slide Time: 00:50:42 min)



If the eigenvalue here where complex with negative real part so that the behavior would be incoming spiral kind and with the change of the parameter, if they move like this so that at a certain parameter value, they become outward spiral because of the non-linear behavior of the system, this behavior that I was talking about - the outgoing spiral behavior that pertains to only a very close neighborhood of the that equilibrium point thereby the limit cycle develops. So what has actually happening at the equilibrium point if I calculate the Eigenvalues? It is this. Eigenvalues that was complex with negative real part were moving this way so that the real part becomes positive. For our three dimensional system that I have given, does something like that happen? Try to find out. This equilibrium points, you can find the eigenvalues. So I will ask you to to workout. Do these eigenvalues, for some parameter value, do something like this? If that happens, then I would expect, after sometime these behaviors to develop. So that can be worked out from looking at the local linear neighborhood. beyond that these two equilibria are separate and each of them have their own basins of attraction and beyond that if you change the parameter, you will not be able to infer what happens from linear system. So the linear systems theory will tell you what happens near the equilibria point and you will be able to infer the occurrence of limit cycles. I don't want you to do this simulation. That we will do later. Try to do things as much as possible by analysis. That's all for today.