Chaos Fractals and Dynamical System
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Lecture No. # 37
Dynamics of Discontinuous Maps

Considering the situation where the determinant in both the sides of the piece wise smooth map were less than unity. What does it physically imply? It implies that if you take some area then in successive iterations of the map, the area will shrink. Essentially that is the meaning of the determinant being less than unity. The determinant being negative means, the area flips to the other side. It's like a dish, it flips. But in general we have been considering the situation where you have modulus of the determinant less than unity. Just recall that in case the determinant is negative then it does not allow a complex conjugate pair of Eigen values. If the determinant is negative, the Eigen values are always real either it can be a regular saddle or a flip saddle or regular attractor, flip attractor. Is regular attractor possible? No, probably only flip attractor is possible. More or less we have dealt with all these possibilities.

Now let us come to the situation where which admits the possibility of the determinant becoming greater than 1. Now obviously if the determinant is greater than 1 in both the sides then it is sort of expanding in both the sides. Take any unit area, it will expand in all the subsequent iterates and as a result the attractor will not exist. It will go to infinity but supposing the system is having one side in which the determinant is positive. Under what condition can it happen? For example in most power electronic circuits there is a switching. The switch is turned on or off and that is what gives the piecewise smooth character of the system. We are trying to figure the physical meaning of the determinant becoming greater than unity.

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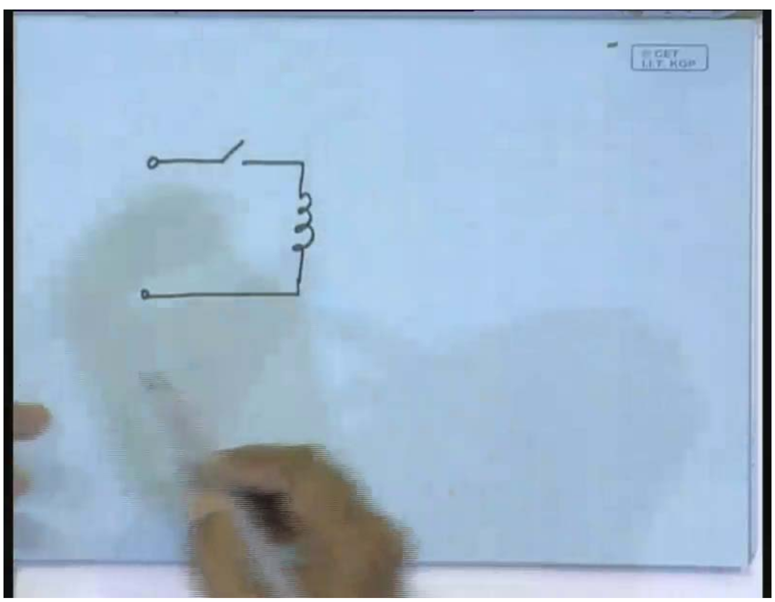

Is it possible in a physical system? You just imagine the situation where the switch remains on perpetually. What happens then? Obviously if the switch remains on and there is only an inductor connected across it. For example a circuit like this. The power supply is here, there is a switch and then you have got an inductor here. Say then what will happen? The inductor current will go on increasing. In most of the configurations of the power relating circuit, if the switch is perpetually on you have got this kind of a configurations. As a result of which the state actually runs to infinity. But it does not because there is a switching. For some part of the time, the energy is stored in the inductor. For some part of the time, energy is released from the inductor and you have an equilibrium. But if one of the compartments of the discrete time state space considers that situation where it remains in the on state then obviously that region will have a determinant greater than unity. That's actually happens. If you really analyze these cases, you will find that's happening.

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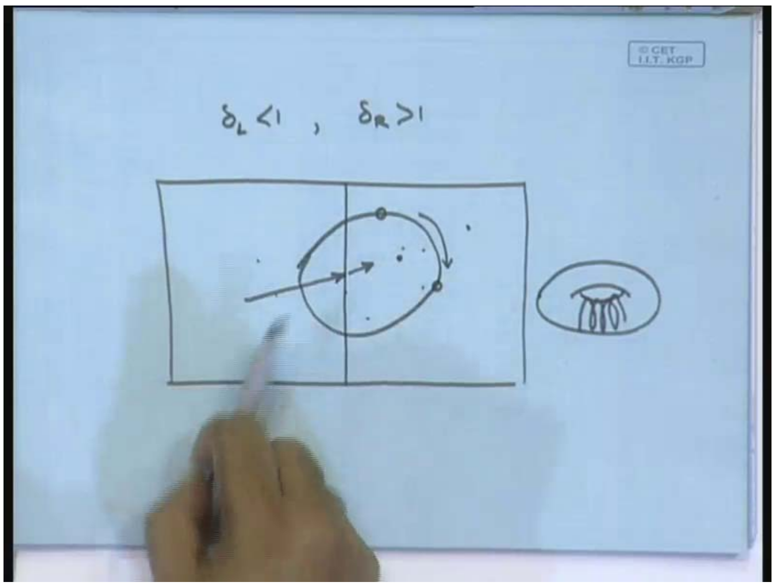

Physically we will have that kind of situation but then if you have the determinant there in one side positive, obviously there will not be a periodic orbit with all the points in that side. Why? Because it goes to infinity but nevertheless let's consider. We are considering situations where the determinant of the left hand side is less than one and the determinant of the right hand side is greater than one. Let us consider the state space like this and you have got the partition with border line here. Now as usual when mu is less than 0, its negative then the fixed point in the left hand side and as mu is moved, it moves and hits the border at some point, at mu is equal to 0. Then it goes to the other side. So long as it is here, supposing you have chosen the parameters so that the fixed point is stable. So what happens? This fellow is stable, this fellow is stable, this fellow is stable but it hits the border and then when it goes here, obviously it admits no stable solution because the determinant is greater than one. The behavior must be unstable.

Suppose you have chosen a range of parameters where in this side, it is complex conjugate. What does it mean? Normally a complex conjugate Eigen value with the fixed point sitting here would mean what. Complex conjugate Eigen values with determinant greater than 1, it will mean a spiraling outward and under normal circumstances that will run to infinity in a spiraling way. It will go like this but then as it goes in a spiraling way, it has to encounter the boundary and it has to go to the left hand side. The moment it does so, it encounters a situation where the fixed point of the left hand side has moved to the right and now it is a virtual fixed point where a virtual attractor. This fellow is attracting. Any initial condition to the left hand side will be attracted to the right because of that attractive virtual fixed point. The outward motion therefore cannot continue at infinite term. It cannot really go to infinity, it has to be arrested. It is sort of constrained to be a bounded orbit. Here you have got an outward spiraling behavior but we come to the conclusion that the orbit overall must be bounded. How can that be?

That can happen only if it will develop some kind of a closed loop such that any initial condition that is inside, will go to the outside and convergent to it. Any initial condition that is outside will come into it and converge on to it. How can that be possible? Say supposing you have got an initial condition here. What will happen? That fellow will be attracted to the virtual fixed point somewhere here and as a result it will move closer to this orbit and as it goes to the orbit, it will be locked there. Why should it move inwards? It will not really, it will tend to go outward but then after some time it has to move to the left hand side and then it has to move inwards. That is why it produces an inward motion also, in the ultimate result producing a closed loop.

A closed loop in the discrete time means quasi periodicity or a torus in the continuous time. Ultimately this has given rise to a torus but the orbit is actually spinning around and you are observing it on the Poincare section. As we have already learnt while we were dealing with that kind of situation that it can either be a mode locked periodic orbit or it can be a quasi-periodic orbit. The quasi periodicity will mean that the two frequencies are incommensurate. Even though you don’t see the two frequencies here, you have to understand that it can either be commensurate or incommensurate. How would you measure that? It is measured in terms of how many rotations it takes along this, before it comes back to the same position, in case of a periodic orbit. Every time it comes through the big circle, you have got one point. If it is here in one iteration and he is moving to this point in the next iteration, it means that it has gone over the big circle once and has moved by this angle. That is how it is to be understood but ultimately we come to the conclusion that there can be a closed loop, there can be a torus. In that case what will the bifurcation diagram look like as you vary the parameter m?

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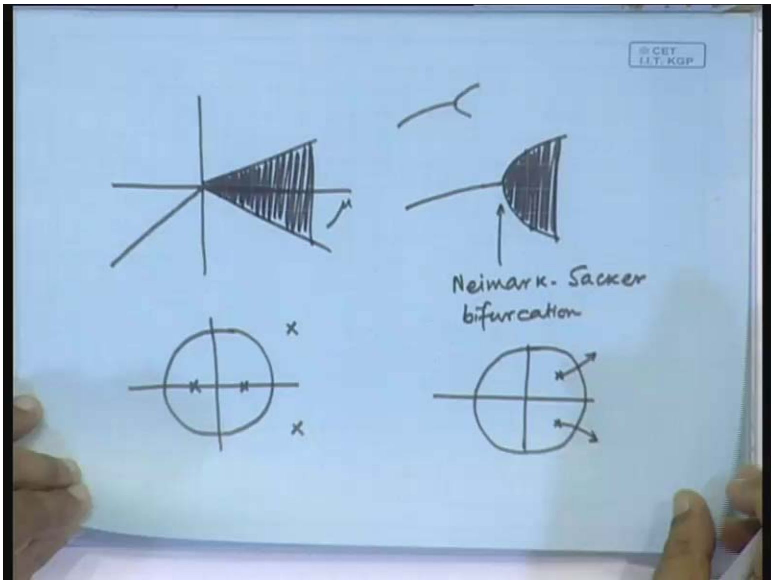

As you vary the parameter mu, you have a periodic orbit hitting the border and then turning into either a mode lock periodic orbit that occurs on this closed loop or a quasi-periodic orbit. Let's first draw the quasi periodic thing. It will be something like this. This will be a quasi-periodic orbit. Apparently looking at the bifurcation diagram, it will look like a chaotic orbit but it is not chaotic orbit in the sense that here if you measure the Lyapunov exponent, it is identically zero. In that sense it is not a chaotic orbit, it is a quasi-periodic orbit, it is an orbit on a torus.

We come across a situation where you can have a transition from a periodic orbit directly to a quasi-periodic orbit and we already know that is possible. What is its smooth equivalent in the smooth map? Can you have this kind of a situation? Yes, you can have but there the behavior looks something like this. What is this bifurcation called? No, it is not a period doubling. It is a Neimark sacker bifurcation. Because a period doubling would be like this but here it has gone directly in to a quasi-periodic orbit. It is a Neimark sacker bifurcation.

Now here if you plot the behavior of the Eigen values, it will be something like this. In case of a smooth map it will be something like this that initially you had Eigen values inside the unit circle, when it was here. It was spirally inwards and finally converging on to this orbit, Eigen value is here and then the Eigen values exited the unit circle like this. In this case what has happened? This can be called a non-smooth Neimark sacker bifurcation where the Eigen values are behaving all most the same way but initially they were inside the unit circle and then jumped outside the unit circle. Initially they could also be like this. Initially when it is here, they could also be on the real line one positive, one negative all these things are admitted. Then after the crossing the border, they discretely jump to outside unit circle so that is the difference.

In one case they move smoothly resulting in a smooth bifurcation like this, from a periodic orbit to a quasi-periodic orbit. Here it is a non-smooth jump leading from a periodic orbit to a quasi-periodic orbit and as you have seen that if you have a jump like this that could be a quasi-periodic orbit when the two frequencies embedded in it happened to be incommensurate.

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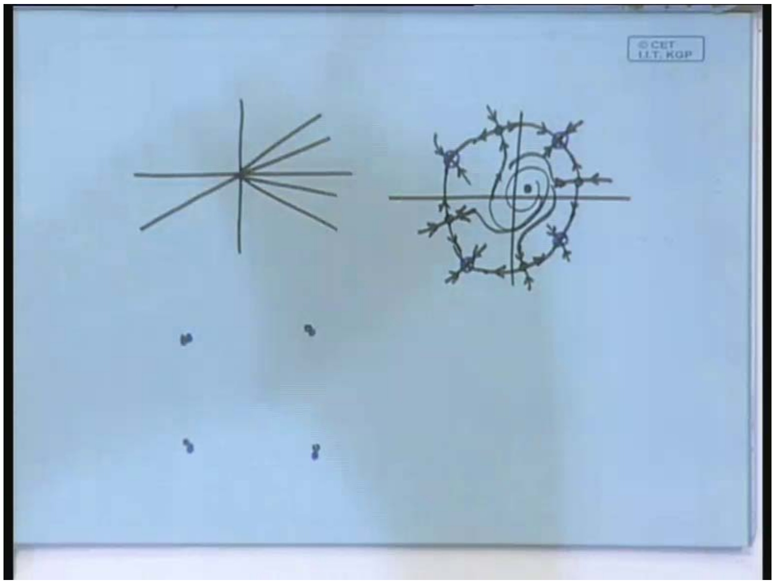

It could also be a commensurate frequency in which case you will have a transition from a periodic orbit to another high periodic orbit. What is a distinction? You have already come across this where there was a transition from a periodic orbit to a high periodic orbit. We already have come across this. Only a couple of classes back, we have talked about that. What is the distinction between that period four orbit and this period four orbit? The distinction is that this period four orbit occurs on a torus and in discrete time that occurs on a closed loop. We have to understand what forms that closed loop. Here we said that it forms a closed loop, so we have to figure out what exactly forms this closed loop?

If you have a period four orbit like this then say the period four orbits are like this. Then apart from this period four stable fixed points, there must be period four saddle fixed points. This fellow is a saddle fixed point and inside, there must be an unstable fixed point sitting which had a spiraling outward behavior. So his behavior is spiraling outward. This is an unstable fixed point, so you will have a stable manifold and an unstable manifold. Here also you will have stable manifold and unstable manifold. Here also you will have the unstable and stable manifolds and here also you will have the same thing unstable stable.

Now it is not difficult to figure out the behavior of this spiral because this can go like this. This can go like that, this can go like this and this can go like that (Refer Slide Time: 16:43). So outward behaviors connect like this. These go in to the stable fixed points. Here all the errors are pointing towards that, here also all the errors will be pointing towards that, here also all the errors will be pointing towards that, so these are the stable fixed points. Let me draw in a different color. These are the stable points that you will actually observe like this. But what forms the torus, what forms the closed loop? It is actually the unstable manifolds of the saddle fixed points and the specific stable fixed points. It is a union of the unstable manifolds of the saddle fixed points and the stable fixed points. That completes the closed loop.

The moment I say that, you will be able to figure out that this happens only when there is this kind of a structure means there is an unstable fixed point and the stable fixed point, stable, unstable, stable, unstable and all that they have to exist. They actually are born through some kind of a saddle node kind of bifurcation. This saddle node bifurcation could be smooth type, can be non-smooth type. The point is that they are always born through saddle node bifurcations. Like what? Imagine the situation… let me show. Supposing here is a point, here is a point, here is a point, here is point. I can see a point but actually imagine that right then, a pair of fixed points have been born they are very close to each other right now. But as you change the parameter, they move away from each other. As a result you see this structure being born. That means this stable fixed point and this unstable fixed point at some point, some parameter value were born together in saddle node type of bifurcation. Either it was a smooth saddle node bifurcation or a non-smooth kind whatever it is. But they were born at some point of time and then as you change the parameter further, they move away from each other giving rise to the structure.

Again as you change the parameter further, this fellow will move away from each other but they cannot do so indefinitely. Why? Because suppose these two are moving away from each other, after some time this will encounter this one and they will collide. That is another saddle node bifurcation. A saddle is now colliding with a node and then these particular structure will then disappear. The point is that each periodic orbit therefore will exist for a definite range of the parameters. Which parameters? Here the parameters are in this case, the trace of the left side, the trace of the right side. As you move them, as you vary them then this structure will change and at some point of time, this period four orbit may be born. At some point of time, some point of parameter it may be lost and when it is lost, you get the quasi periodic orbit.

Again another orbit may be born, again another orbit may be born. As a result you see a succession of periodic windows, mode locked windows sandwiched between quasi periodic behaviors. That is what is observed in smooth systems also and that is observed in the non-smooth systems also. We have an account of a situation where a mode locked periodic orbit or a quasi-periodic orbit can be born out of a periodic orbit. A periodic orbit suddenly changes to a quasi-periodic orbit or a mode locked periodic orbit. That is also possible through a non-smooth border collision bifurcation. We are more or less through with this part of the work. So far we were considering situations where the map was continuous.

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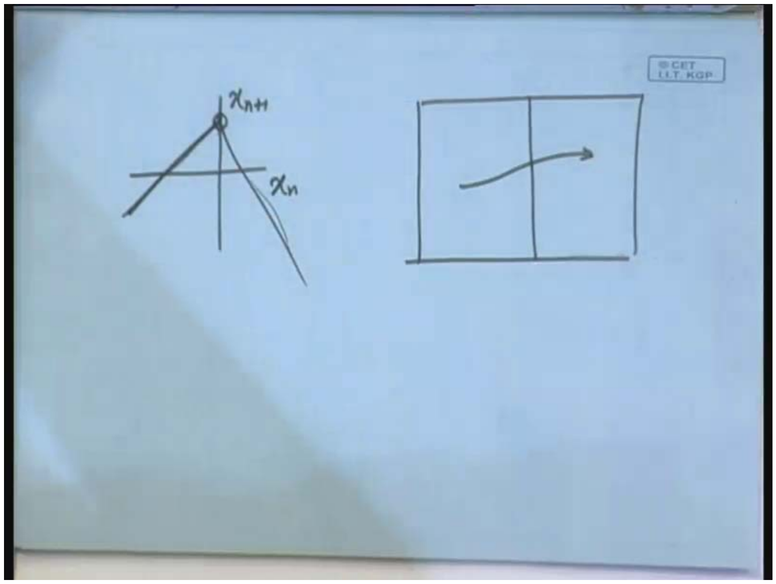

While we were dealing with the one dimensional map, we considered situation something like this. At this point it was continuous. Here is x_n and here is x_{n+1}, then at this point it was continuous. When we considered the motion of this then at this point, still we considered continuity. Under what condition can that assumption be broken? That assumption was the state space is continuous. What does it mean? It means that here there is an equation of this line, here there is another equation of this line. If you substitute the position zero then whatever you get as the y coordinate, here also if you substitute x equal to 0, you get the same thing as the y coordinate.

In terms of this, what does it mean? You have an equation for the left hand side, you have an equation for the right hand side. If you substitute the position of the border line on the left hand side equation, whatever you get, if you substitute the position of the border line on the right hand side equation you will get the same thing, that is the meaning of the continuity. Under what condition can that be lost? Let us first physically probe that problem.

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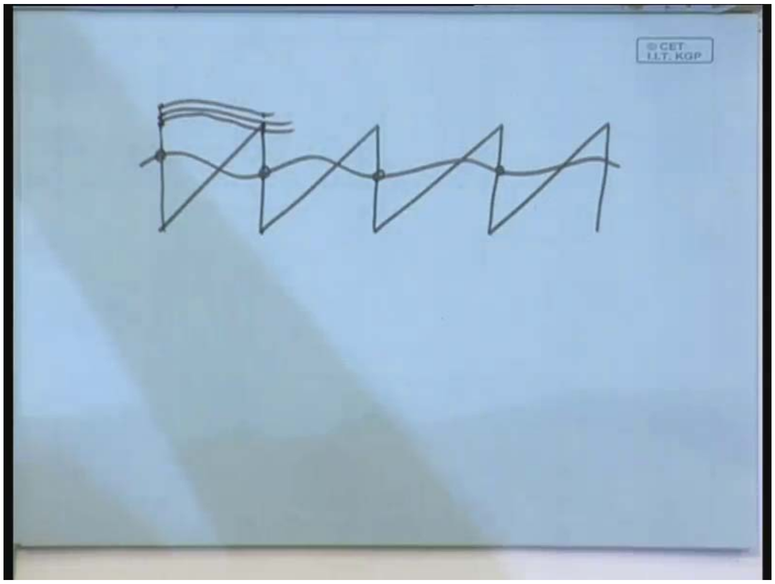

One of the situation that we have already talked of elsewhere in this course is that of a switching circuit where you have a triangular wave and there was a some kind of a wave form with which the triangular wave is compared to generate the switching signal. When does a border collision occur in this case? When you are actually observing the state here, here, here and all that? A border collision will occur only when these two extremities are grazed. Imagine that it is grazing like this and then as you change the parameter, it goes like this and as you change the parameter it goes like that. How would you intuitively understand the problem of continuity and discontinuity in this case?

Notice that if you start from this point, at this time it goes to this and it comes here. Naturally if you want to express this point as a function of this point, you have to go through an equation for this period and then an equation for this period and ultimately you come here. If you are considering this orbit then you do not have to do that. You only have to consider the equation for this period and you have to come here. This side and that side are given by two different functional forms giving rise to a piecewise smooth map. But now we are considering the question of continuity and discontinuity.

You would notice that if you are considering this particular case where a orbit just grazes and if you perturb that initial condition by infinitesimal amount, towards the upwards direction and by an infinitesimal amount downward direction then ultimately when you reach there, they are still moved by a infinitesimally small amount. Is that clear? Because if you go that way, it will follow a different initial condition but along the same equation and that initial condition being removed from the first one by only an infinitesimal small amount. The final condition will also be removed from the other final condition by infinitesimal small amount. So a small perturbation here implies a small perturbation there. Here also the same thing. A small perturbation this way will mean a small perturbation this way. You can bring down that perturbation to 0, here also it will bring down to 0 that implies continuity.

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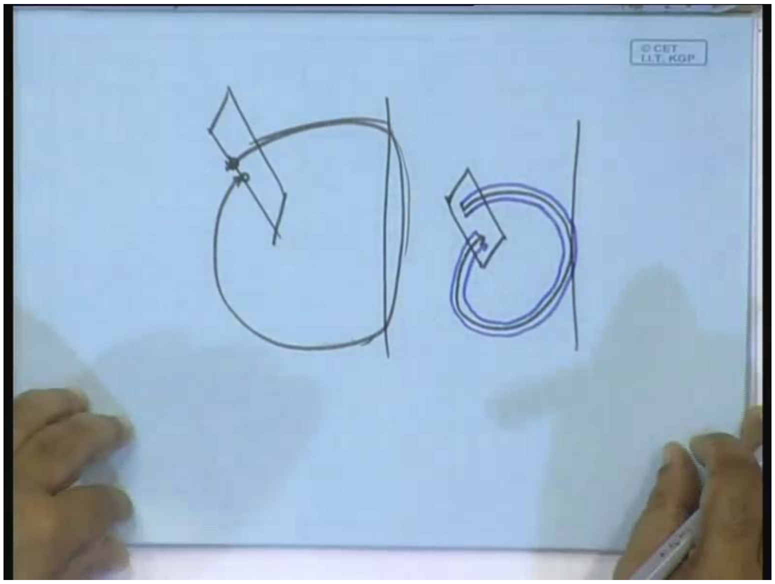

Imagine a situation where you have an orbit with a border line and the orbit goes and hits the border and then comes back here, somewhere. Say this is my observation plane, you observe it here and then it comes here. Will this behavior be continuous? If you move it slightly then it goes like this, comes back like this and then obviously it will move only by a small amount. Now imagine the situation where it just grazes, an orbit just grazes. See it starts from here, it's just grazes and comes back to this point on that Poincare plane. If you now move it slightly this way and this way, what will happen? This orbit will not graze and will miss it and as a result it will come here. This orbit will graze, will go slightly to the other side and will come back. As a result the slight perturbation here still remains a slight perturbation here. This map will also be continuous because you have moved it slightly and the final condition also moved slightly.

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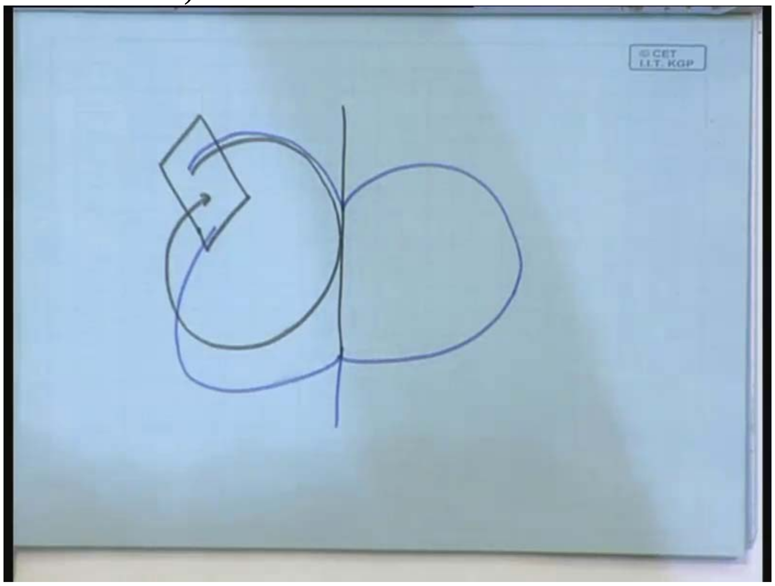

But imagine a situation where you have the switching plane here, the Poincare plane say here. You start from a point and it just grazes and comes back somewhere here. But when you allow the orbit to go to the other side, it doesn't go like this. It goes to entirely different point which means that if the vector field in the other side is such that the moment it goes to the other side it goes completely in a different directions and which when you bring the perturbation down to 0, it does not really fall back or go close to this one which means that a slight change here will mean a large change in the final condition that implies a discontinuity in the map. In many physical situations that occurs and that is why this becomes meaningful to study that. How will you study such a situation? Firstly when we studied the continuous maps what did we do?

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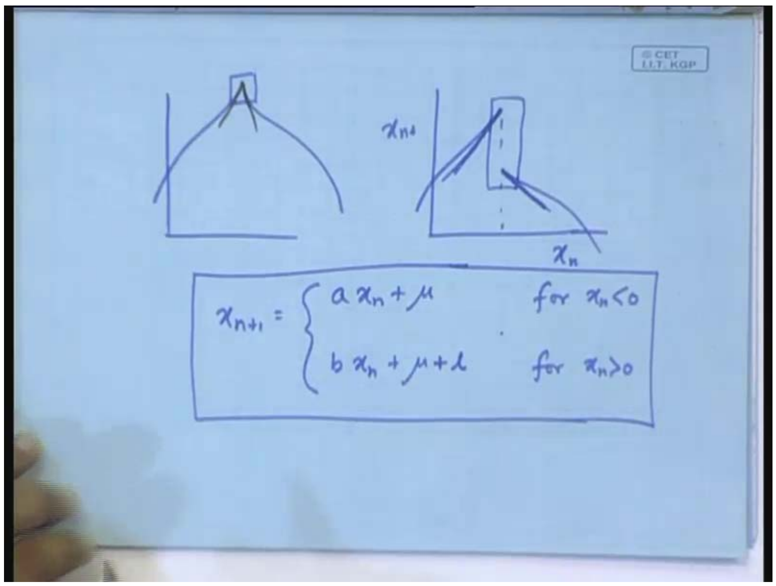

If you are studying a one dimensional continuous map then we said that let the map be something like this, whatever it is and then I am not really interested in what happens in this part or this part. I am basically trying to figure out what happens when a fixed point crosses this and we said that when that happens this particular event can be probed if you simply consider the… and that we call the normal form of this border collision bifurcation. In this case what is the situation? In this case, situation is that you have got one side and the other side like this, there is a gap. This is the mathematical understanding of discontinuity. If you have x_n here, if you have x_{n+1} here and here is the border line means take a point x_n slightly to the left of the border line, it goes here. If you take a point slightly to the right of that border line, it goes here; two different values. If you bring the perturbation close to 0, still this distance does not goes to 0. That is the meaning of discontinuity. Some of the authors have called it the map with the gap. There is a gap in the map.

How would you probe the behavior? Obviously if you are specifically probing that behavior which happens close to this, then it will suffice if you simply take the local linear approximations here. This being approximated by a linear map, this being approximated by a linear map with a gap.

The way we obtained the normal form in case of the one dimensional map, we can apply the same logic to obtain this map but with a gap with a discontinuity. Simply we can write that map as what was our original map, x_{n+1} is equal to we said that it will be ax_n plus mu and bx_n plus mu. This was for x_n less than 0 and for x_n greater than 0. This was our definition of the continuous map in the normal form. Just change the continuity, put x_n is equal to 0 both become mu, so it is continuous. In order to bring in the character of discontinuity here, all we need to do is to say add a value here, add a number here say I will put it either here or here. Let us put it here. Now let us probe this simple map, the behavior of this simple map. This is a normal form of a piecewise smooth discontinuous map. You would immediately notice a few very startling feature of this first.

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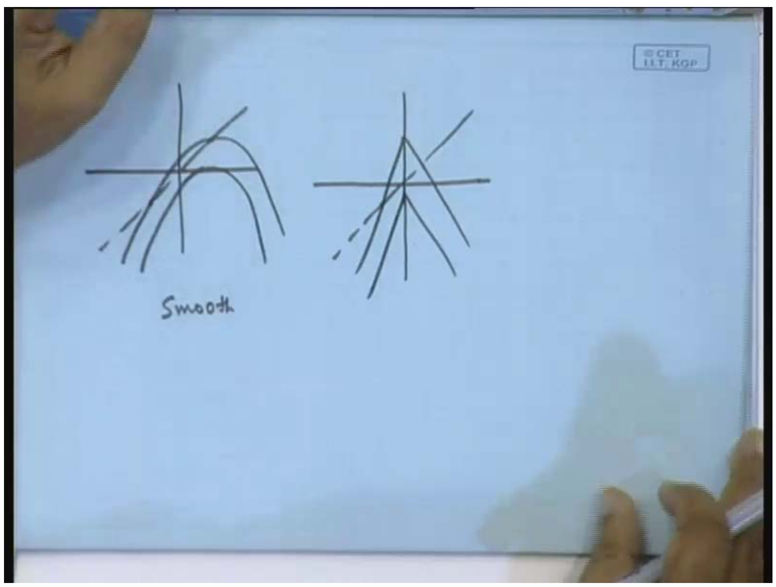

In case of the smooth map for example you learnt that you can have a behavior like this. That means graph of the map was earlier like this for some value of the parameter. Now it has become like this. As a result there has been a birth of a pair of fixed points. Out of that one is always unstable, the other is always stable for a smooth map but always the birth is in pairs. In a continuous but non-smooth map, what is the equivalent situation? There you could have something like this but let us contrast it. You could have a situation like this. This is for one value of the parameter and then as you change the parameter, it goes like this. As a result the pair of fixed points are born but now both are unstable. Yes, we have seen that but remember in this case also two were born at the same time, you can't help it. We have seen that in case of two D maps, this can be valid there can be the birth of a single turn unstable fixed point. But a one D yes, you have to have two fixed points born at the same time.

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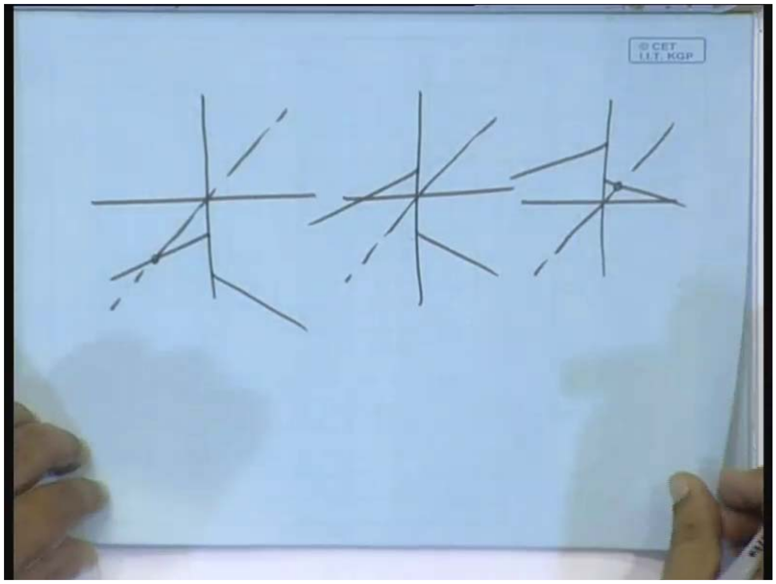

Now imagine the situation of a discontinuous one dimensional map. First let us consider this. Here is a 45 degree line. Here you have a fixed point nothing else, a stable fixed point. As you change the parameter there is no fixed point. A 45 degree line goes through the gap, no fixed point. Still there can be stable periodic behavior in this case, we will come to that. In this kind of a situation there is a possibility that there is no fixed point, yet there is a stable periodic behavior. As you change the parameter further the situation is… here is a fixed point and if this slope is less than unity it will be stable.

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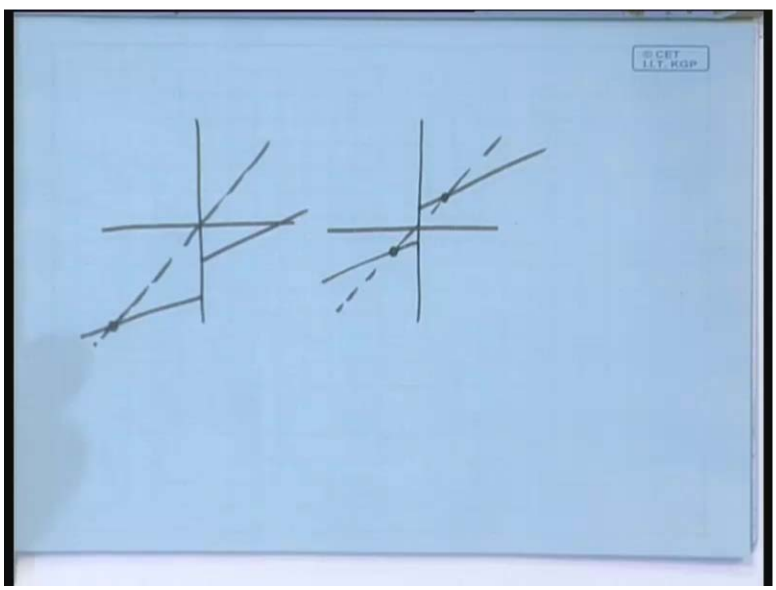

Let's look at that another condition. Imagine that here is a fixed point and here. Where is the fixed point? It is here, there is no other fixed point. The behavior is stable. As you increase the parameter mu, it goes up and then you have… and you have two fixed points now. There was one fixed point and one fixed point has been added, a single fixed point has been added. Here the birth of the fixed point is not accompanied by a birth of an unstable fixed point. A stable fixed point alone could be borne in a discontinuous map. So that is one peculiarity, you have always keep in mind, that is possible. Can't there be a pair of fixed points like this in a smooth map?

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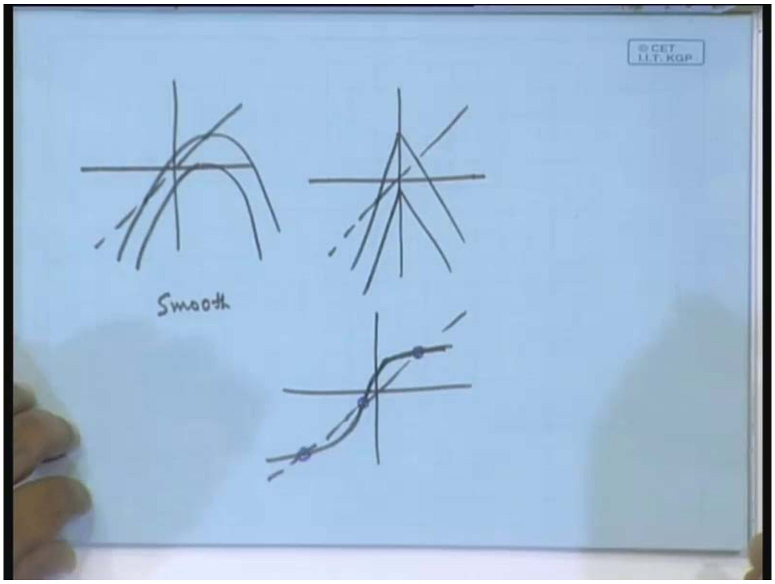

Let us consider this situation. A smooth map, a pair of fixed points. Is it possible to have pair of fixed points, stable fixed points? Yes, very much. Imagine this situation where you can have a smooth map like this. Notice that here you have got a fixed point here, you have got a fixed point here and a unstable fixed point here. You can have a situation like this but there must be an unstable fixed point sitting in between. Why? Because if these two fixed points are stable fixed points, they must have their own basis of attraction and the basin of attraction can only be created by an unstable fixed point. This fellow is the unstable fixed point then which is actually the basin boundary. Because any point to this side will go to this fixed point. Any point to this side, will go to this fixed point so that is the Basin boundary. Basin boundary is then produced by the unstable fixed point sitting here. But what about here?

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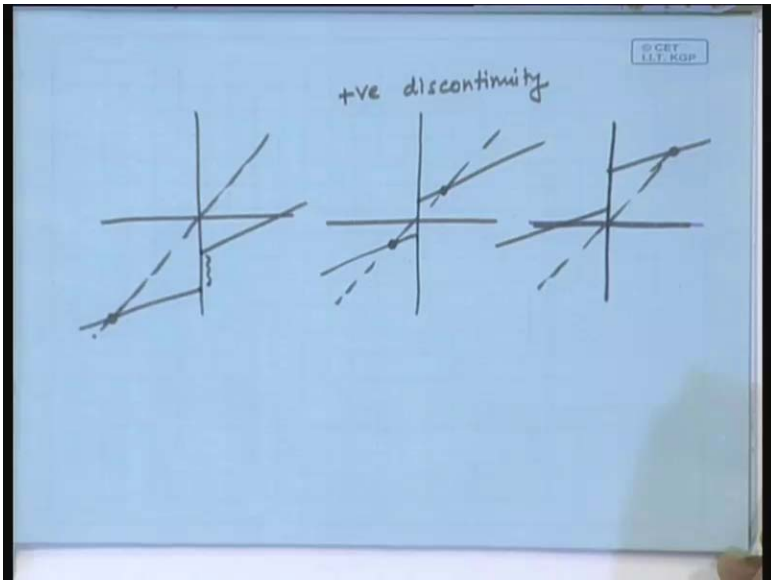

In between there is a gap and there is no unstable fixed point sitting here. Who forms a basin boundary? The discontinuity itself forms the basin boundary, the border itself forms a basin boundary. In that case any initial condition to this side will go here, any initial condition of that side will go to there. So this boundary itself becomes a basin boundary. Discontinuity acting as a basin boundary is another typical peculiar feature of the discontinuous map. Third important feature is that the behavior of the same map for a positive discontinuity and a negative discontinuity will be different. Let's just imagine in successive stages of this particular situation.
Let's draw it in black.

If you move it further, this will be the behavior. When it was below, the mu was having a small value, for intermitted value and for larger value this was a stable fixed point then become two stable fixed points then become unstable fixed point. Just imagine here the discontinuity was positive. Here you get a positive push in order to land here and then you start, so it is a positive number.

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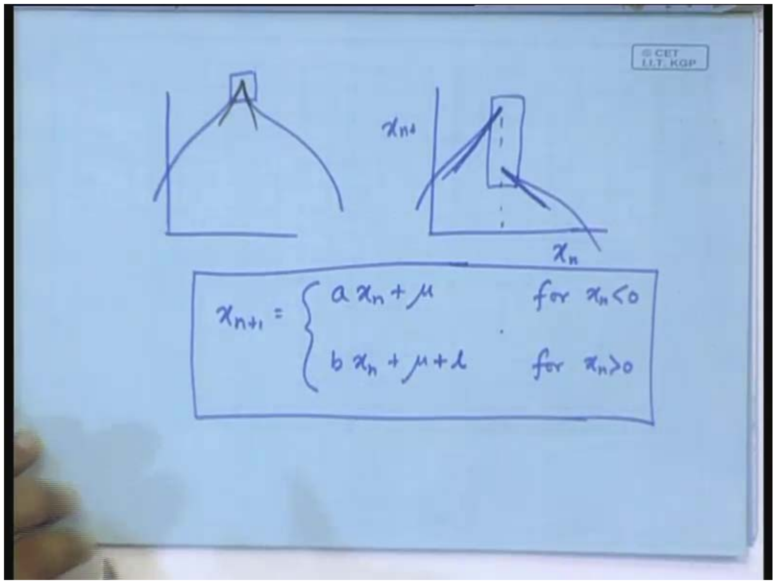

Imagine here put x equal to 0, you have got mu and in the right hand side mu plus l. This value is mu and this value is mu plus l, a positive discontinuity. This is the situation for the positive discontinuity. Can you imagine the situation for a negative discontinuity, same situation, same behavior.

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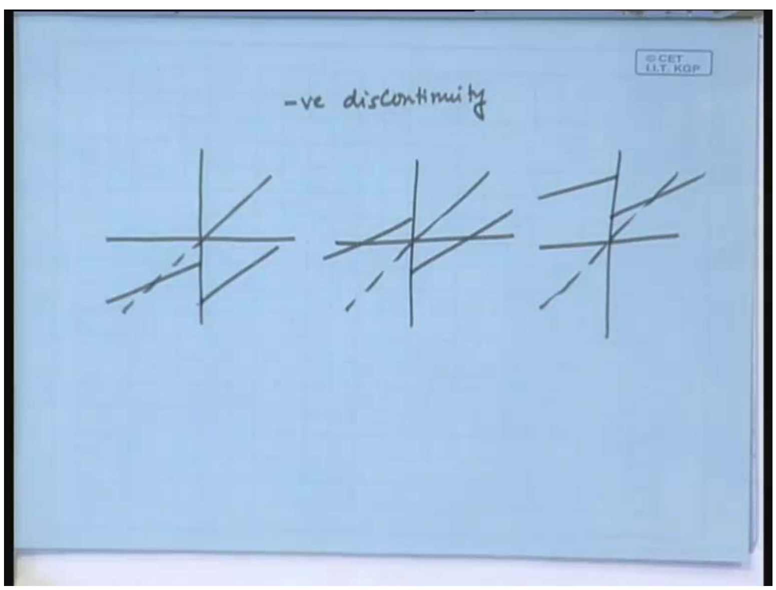

There negative discontinuity, here it is like this. Here it is like that. This negative, so here is one fixed point. Second stage, vary mu further you have… third stage.

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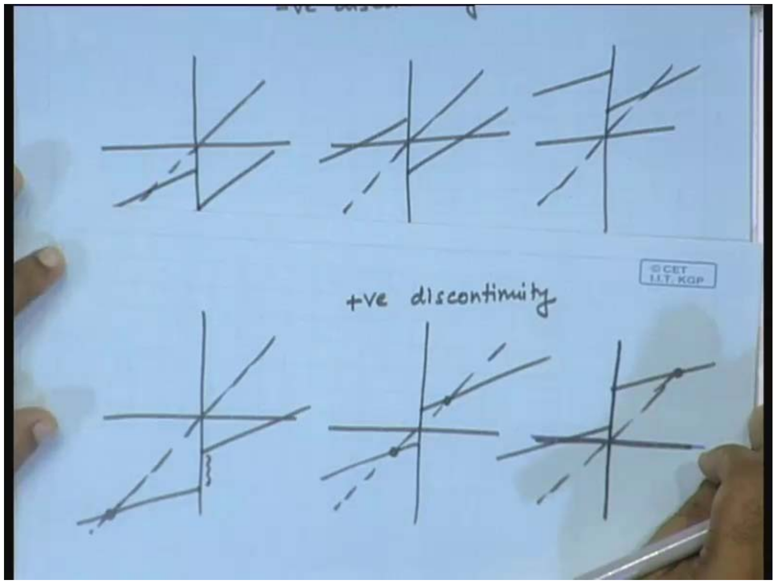

Now keep this two side by side, you will be able to see what is the difference. Can you see now? Yes, you can see both. You see the behaviors are different really, depending on the sign of the discontinuity. If the discontinuity is positive, it is a period one to two. You don't know what. I can see that the behavior will be stable because both sides are converging yet, there is no fixed point. We will take a look at what happen there. But here there is again a period one fixed point. In this case it is a period one to two stable period one fixed points, not a period two and here it is clearly a period one fixed point. The behaviors are different. In this case we will need to consider the situations for positive discontinuity and negative discontinuity separately. Now let us come to exploring each situation but for this in order to go little faster, I will use the computer.

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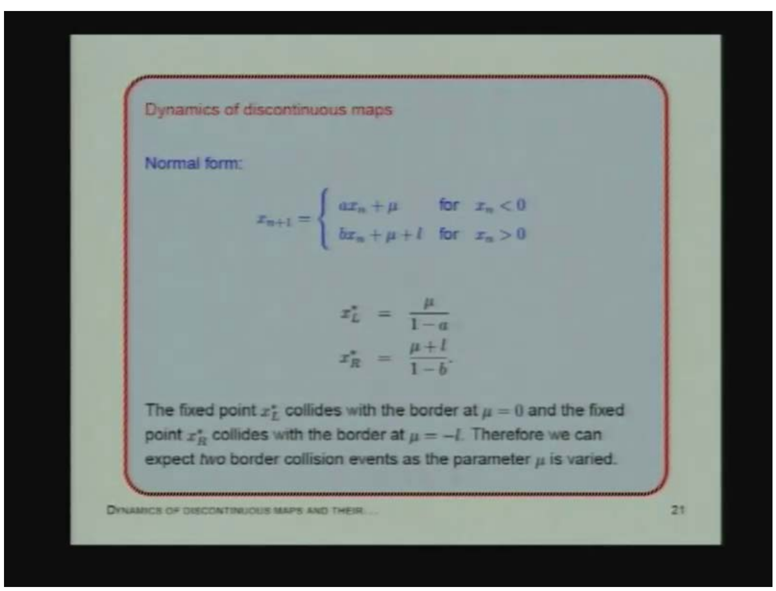

Can I take a look at the computer? Yes. This is the equation for the map that I am using. Can you read? If you use these equations then the left hand side fixed point will be given by this and the right hand side fixed point will be given by that. You can very easily obtain it simply by equating the x_n here and then with this equation you will get this, with this equation you will get that. Very easy, one line algebra so I am not spending more time on it. We are now taking a look at the situations for different values of a and b, the way we did earlier but for different signs of L. But remember one thing that this particular situation is where look at the map and let's come back.

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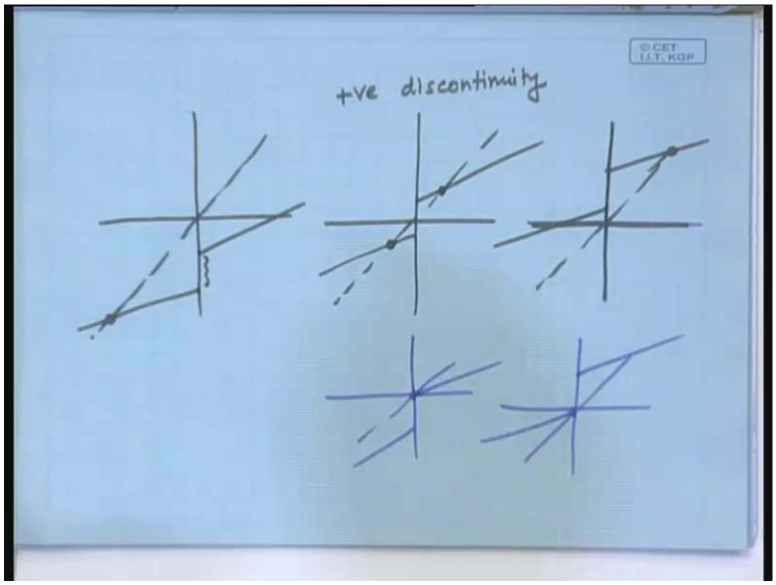

When do we have border collision? Let's consider this, let's take this and here. Obviously the situation where there is a border collision position and there is another border collision position.
Here is a fixed point which is now on the border, here is another fixed point which is now on the border which means that as the mu changes, the whole map moves up. There are two parameter values for which there will be border collision. Now when does this happen and when does this happen? Here the border collision condition is obtained by the left hand map where we have already seen the border collision happens at mu is equal to 0. Here minus l, mu is equal to minus l. This one collides with the border when mu is equal to minus l. There will be two border collision conditions in this case, we have to consider them separately.

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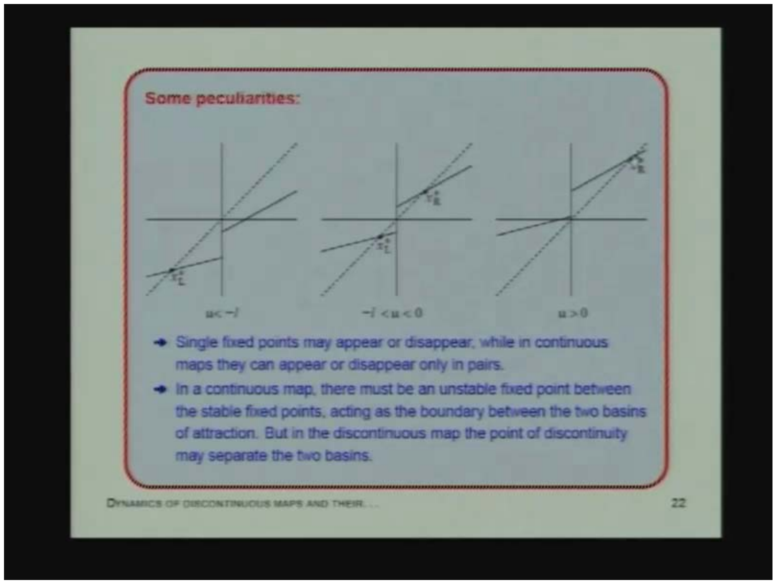

We have seen this particular situation where there are two fixed points, there is one fixed point. So single termed fixed points are been born and all that. Let us continue. Let's consider each of the situation now. You have seen that there are situations where I said that we don't know what will happen. Because the fixed point does not exist yet, I find that in this part the slope is less than zero. In this part the slope is less than 1, in this part the slope is less than 1, meaning that in both sides the behavior is contractive. Take any chunk they will subsequently contract. Take any chunk of initial condition like this, it will continue because the slopes are less than unity. If they contract then obviously that will converge into a periodic orbit. What will be the periodic orbit? That is obvious condition, obvious question and in order to probe that first we consider which periodic orbit? Obviously period two. Then period three and so on and so forth.

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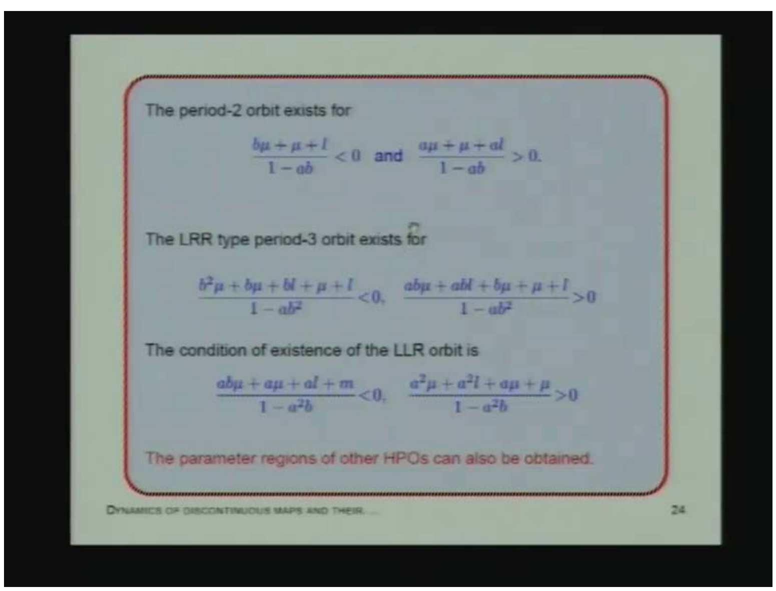

Now here if you look at the computer screen, if you really solve the equation for finding the period two orbit which I have demonstrated twice in different types of maps and I don't want to demonstrate once again. That is why brought in the projection so that I can once for all give the values. See if you obtain it, you will get these values and this is the position of the fixed point and this fellow exist. Position of the fixed point means there is a l point in the left hand side, there is a r point in the right hand side and the fixed point will exist so long as, the left point is less than 0 and the right point is greater than 0. Simple logic.

Similarly if you are trying to obtain the condition of existence of the period three orbit, you will consider first a LLR orbit and a LRR orbit. For the LLR orbit, you will find a condition for which the LL, second L is less than 0 and the R point is greater than 0 that is the condition. Similarly for LLR, this is the LRR orbit. This is the second R point, this is the L point and the next R point. Let's explain that once.

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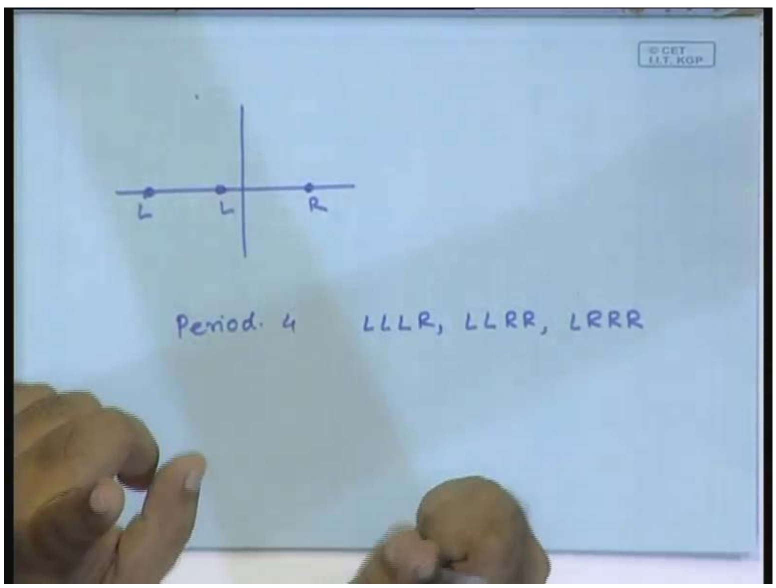

In the LLR orbit, out of that one is away, second is closer and third is here. See if you are considering, which one can collide the border? Obviously there is no point considering this points colliding the border. We will consider this point and this point and that is what we are doing. But this tells that there is a parameter range given by this and this, over which the period two orbit will occur. There is another parameter range for which the LRR periodic orbit will occur. There is another parameter range given by this that means this is one range, one extremity of a range and this is another extremity of the range over which LLR orbit will occur. There will be two ranges where we are expecting to find a period three orbit.

How many ranges would we expect for a period four orbit? Depends on, in how many ways you can write a period four orbit. A period four orbit can be LLLR, LLRR, LRRR. Other things are not possible, LLLL is not possible. A period four fixed point with all the points in the left hand side is not possible because it is a linear map there. For period four there are possibilities LLLR, LLRR, LRRR three possibilities. In period five how many possibilities are there? In general you will find there is n - 1. So you can expect a period two orbit to occur in one range. A period three orbit in two ranges, period four orbit in three ranges, period five orbit in four ranges and so on and so forth.

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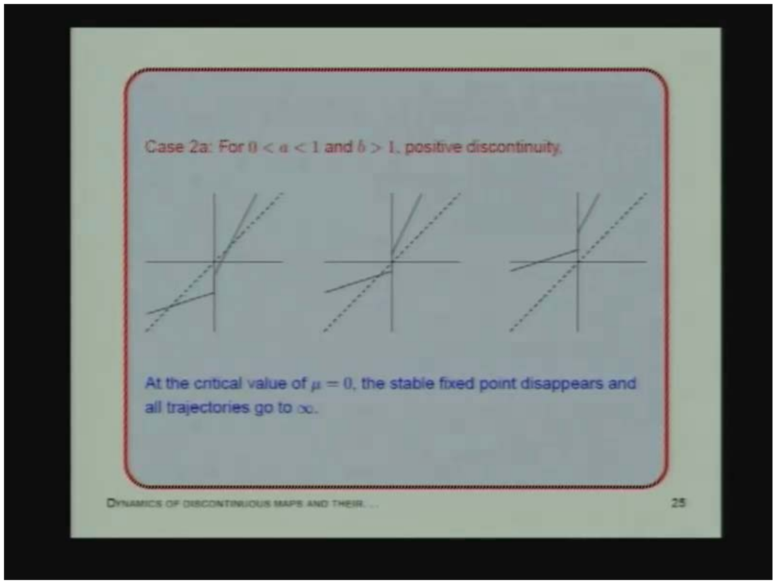

Now we will illustrate what it means in terms of the bifurcation diagram but let us consider each of these cases now. We will go in to more details in the next class but let us start at least. Here is a condition when the a is between 0 and 1, meaning that the slope here is less than 1 and b is greater than 1 which is like this. So as you increase the parameter here, there was a fixed point here, there was a fixed point here which was unstable. Now it becomes this one a fixed point and then it became like this. So at this critical value mu is equal to 0, this stable fixed point disappears and trajectories go to infinity, simple case really. A fixed point suddenly going to infinity.

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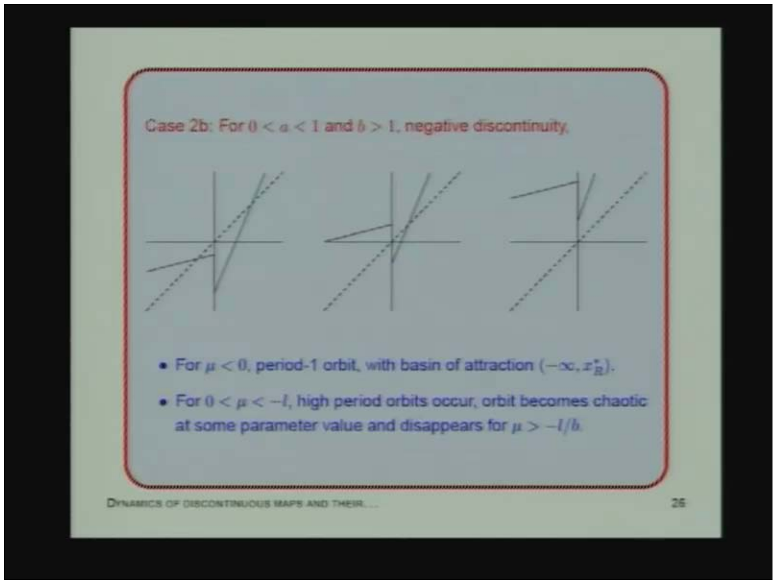

But if the discontinuity is negative, earlier we were considering positive discontinuity. Now we consider negative discontinuity. Initially for mu less than 0 it is like this. When it is between 0 and L it is like this and then it is like that. Good, no problem. Here there is a stable fixed point, no problem but here in this situation there is an unstable fixed point. Naturally we have to ask the question will a periodic orbit occur. In addition to that there is a possibility that the product beyond a certain multiple, here is a slope less than unity, here is a slope greater than unity. So if you consider ab, it could be less than unity. If you consider ab square, it could still be less than unity but if you consider higher number of b that means larger number of iterates in the R side obviously beyond some value it will exceed one. In that situation what do you expect? Do you expect chaotic orbit? So unless this condition, there can be chaotic orbit so on and so forth.
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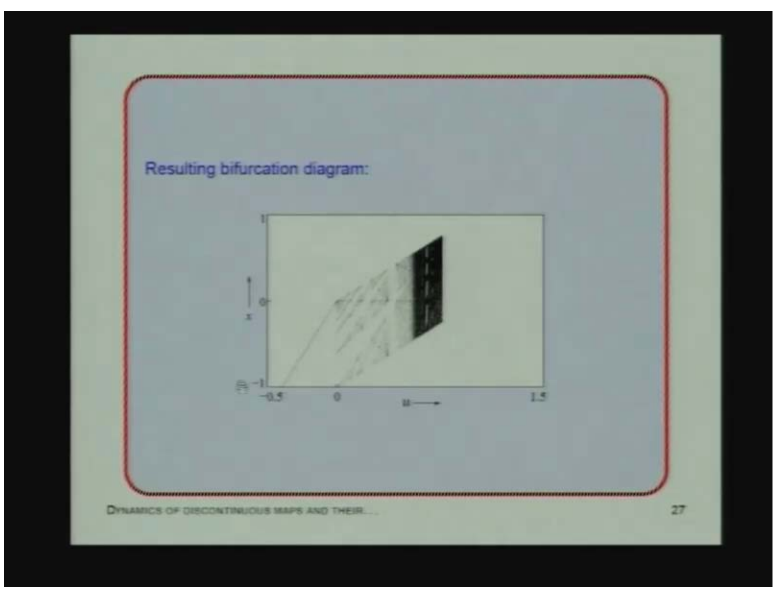

So when the discontinuity was positive, we say period one vanishes but in this case there will be a succession of high periodic orbit and chaos like this. At this point it collided with the border and there was a succession of high periodic orbits finally vanishing. Take a very close look at it, if you can see carefully. I would suggest that you obtain this bifurcation diagrams yourself and then check. From the theory we are anticipating, there will be one range for period two, two ranges for period three, three ranges for period four so on and so forth. See there is a one range for period two and so on and so forth. That will in general be true. Here there is a period adding, there is a range for period three, there is a range for period four and so on and so forth.

Notice one thing that if you have any iterate in the right hand side, since the behavior is like this it will actually move to the left hand side. Normally the orbits will be LLR type so the higher number of R iterates will normally not occur. You see the behavior given by the range extremities of the period two behavior, extremities of the period three behavior, extremities of the period four behavior all that can be obtained from these equations.

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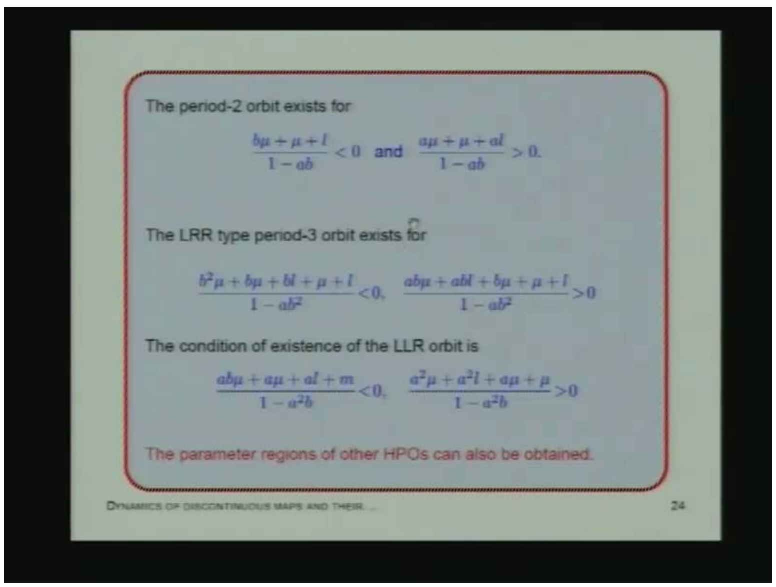

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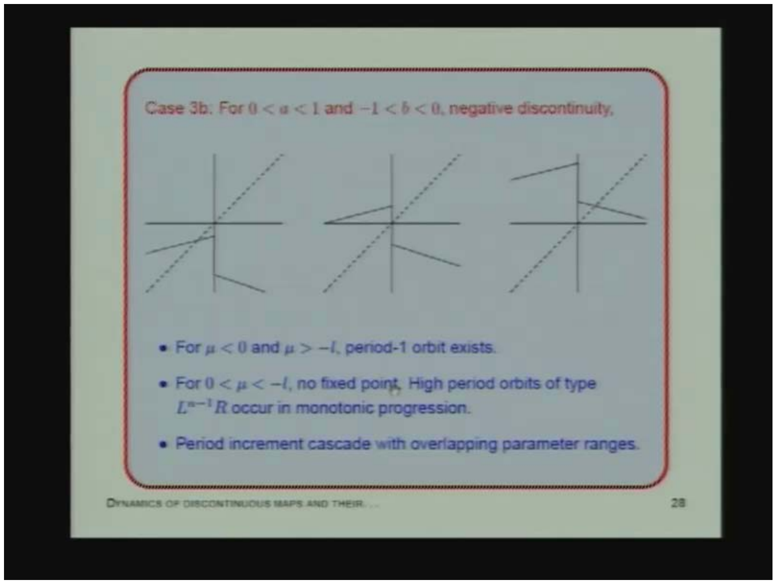

If you consider situation like this, for example where you have the a side left hand side with a slope less than unity and the other side also between minus 1 and 0. That means it has a negative slope but less than unity. Then after sometime it goes to a situation where there is no fixed point, yet both the sides are contracting. What kind of behavior will you expect?

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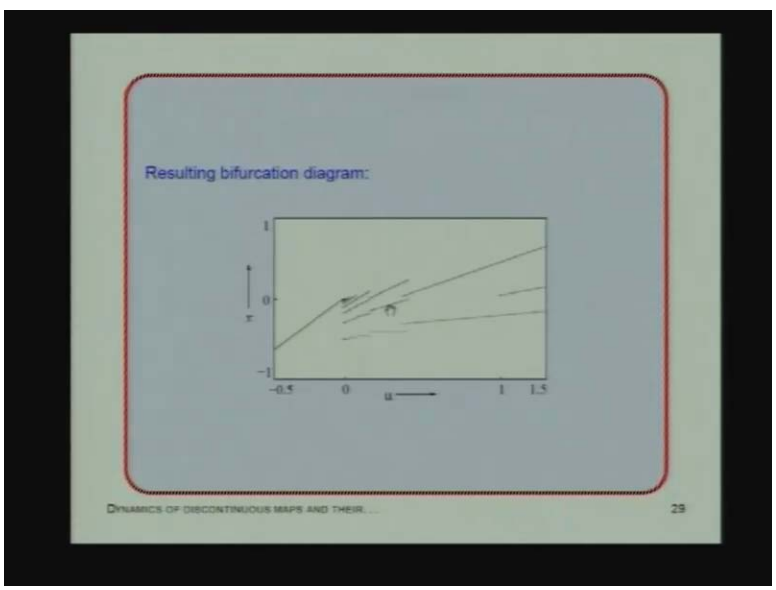

Like this? (Refer Slide Time: 54:44) Why? Because since this side is negative, its implication is that if there is any initial condition in the right hand side, in the next iterate it will go to the left hand side because of the flipping behavior. It has a flipping behavior.

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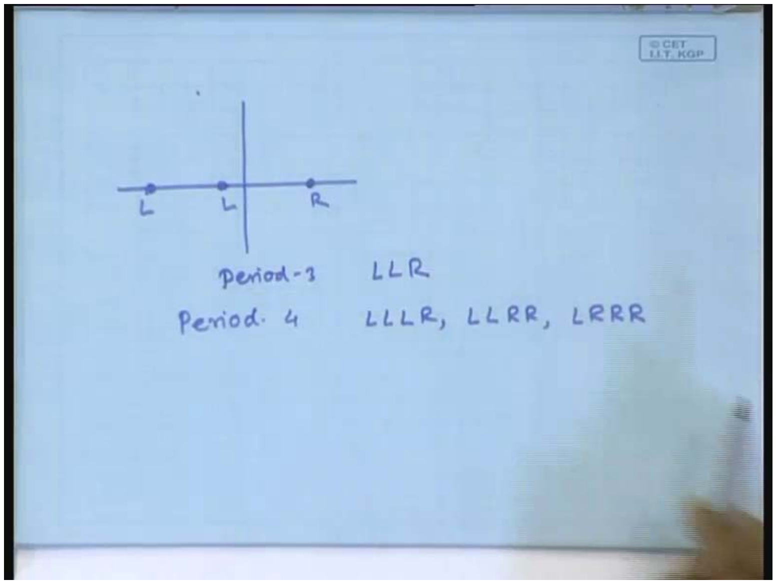

Since any iteration in the right hand side goes to the left hand side in the next iterate, it means that you can only have this orbit but not this orbit and not this orbit. Period three you can only have LLR which means that there will be one range for period three, one range for period four, one range for period five and all that. That you can calculate from those equations.

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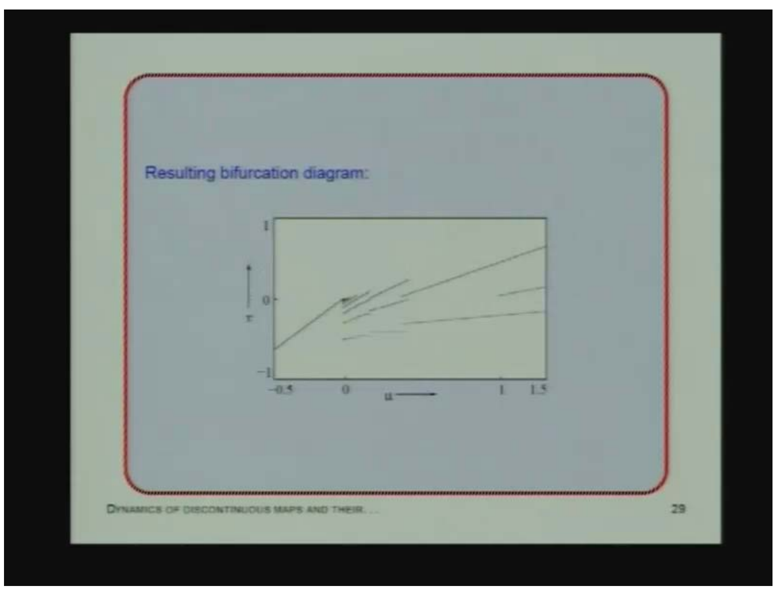

The resulting behavior is like this. There is one range for period two, one range for period three, one range for period four and so on and so forth but that will ultimately add up to an infinite periodicity here. It is a period adding cascade which will add up to infinite periodicity but here infinite periodicity will not mean, not imply a chaotic orbit. Is that clear? It is a bit tricky because we had earlier said that a chaos is nothing but an aperiodic orbit. This is also aperiodic orbit, infinite periodicity means aperiodic orbit, yet it is not a chaotic orbit. Why? Because since every part of the state space is contractive, the Lyapunov exponent must be less than 0. If it is less than 0 this fellow even though it is a high periodic orbit or infinite periodic orbit, it’s not a chaotic orbit. So with the details, we will continue in the next class.

Thank you.