Chaos Fractals and Dynamical Systems
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Lecture No. # 35
Bifurcation in Piecewise Linear 2D Maps (Contd.)

I had proceeded to subdivide the two dimensional parameter space into compartments by the logic that we will divide in such a way that in each compartment, a specific type of fixed point hits the border and turns into another specific type.

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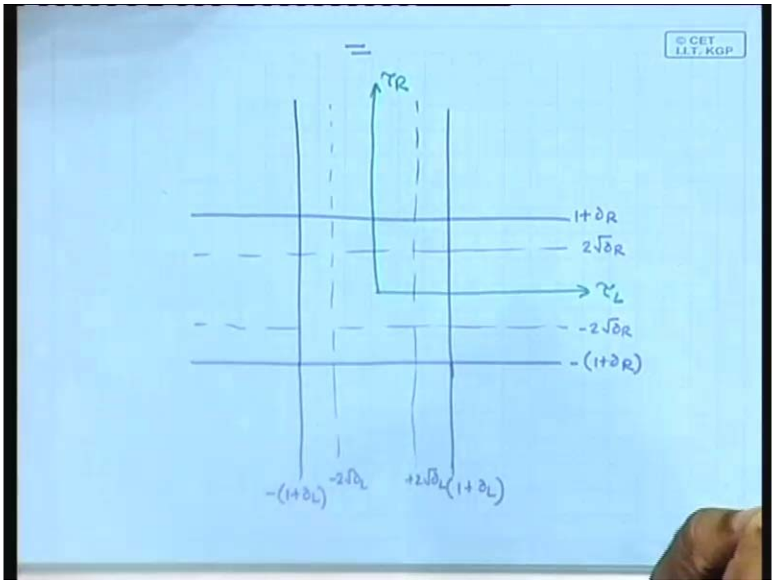

What we had done was to divide the parameter space as depending on… This was your tau_L axis and this was your tau_R axis and these values were one plus delta_R. This was the gross division in between there would also be these subdivisions depending on which, you decide whether it is a spiral or a regular fixed point. This is at 2 root delta_R, this is minus 2 root delta_R, this is minus 2 root delta_L and plus... We have decided that in this compartment a stable fixed point, hits the border and remains the stable fixed point. In this part of the parameter space an unstable fixed point, hits the border and remains the unstable fixed point.

These are the situations where there is no really change in the stability status. We had also decided that in this part where the tau_R is below 1 plus delta_R and tau_L is above 1 plus delta_L, in that part you will have a birth of a pair of fixed points. We call it the border collision type saddle node bifurcation or border collision fold bifurcation and we decided that in this part here, we have a stable fixed point coming and hitting the border and becoming an unstable fixed point. What type of unstable fixed point? It is below this that means your tau_R is less than minus 1 plus delta_R that is it will be a flip saddle. Last time we were discussing this part, before we broke off we were discussing this part. Since a stable fixed point comes and hits the border, it becomes unstable fixed point.

The natural question is will any high periodic orbit becomes stable and we started the discussion with period two. We had seen that this stability is given by the period two fixed point that means in the state space there is one fixed point to the left another to the right.

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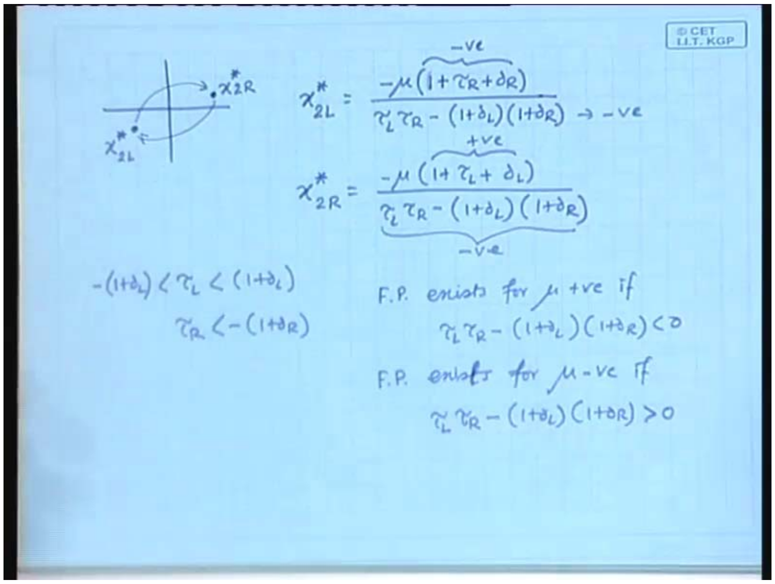

This point maps to this point and this point maps to this point. We had called it $x_{2\,L}$ star and this point was called $x_{2\,R}$ star and we had obtained the values which can be easily obtained. I have written the values as $x_{2\,L}$ star is equal to minus mu 1 plus tau_R plus delta_R divided by tau_L tau_R minus 1 plus delta_L. It will be good exercise for you to do it yourself. These are rather easily found because all you need to do is to start from a point here, map it to the right. Again from here map it to the left and that way you obtain the stability of this point. Your x_{2R} star, it would be obtainable just by simply changing the L's to R's and vice versa, 1 plus tau_L plus delta_L divided by tau_L. This is already symmetrical so nothing has to be changed and that basis we had proceeded or at least try to infer about this existence of this period two point.

We are talking about this range. In this range the condition is tau_L which is between minus 1 plus delta_L and tau_R is less than minus (1+ delta_R), that was a condition. Now if this condition is true then what is this particular number? It is negative. This says that this term is negative and by this it is positive so this term is positive. Obviously take into account that this term is negative, this term is positive. First consider this term to be negative that means 1 plus delta_L 1 plus delta_R is bigger than tau_L tau_R, this term is negative. If this term is negative, the denominator is negative, so it cancels off. The sign of this depends on mu and this and this fellow is negative. When mu is positive, you need this to be here, in the negative side.

No, it's not related to stability, we are talking about the existence. We had obtained the existence condition, not stability condition because as yet we are not talking about the stability at all. Stability will depend on something else, I will come to that later. This is negative, this is positive this fellow is negative. Then if this fellow is negative for what value of mu will this fixed point exist? Positive, then this term will be negative, this term will be positive. So the FP exists for mu positive. If tau_L tau_R minus… only when this is true then this is true. Else what happens? If this is positive then it exists for mu positive. No. For mu negative, yes. The fixed point exists for mu negative if the reverse is true.

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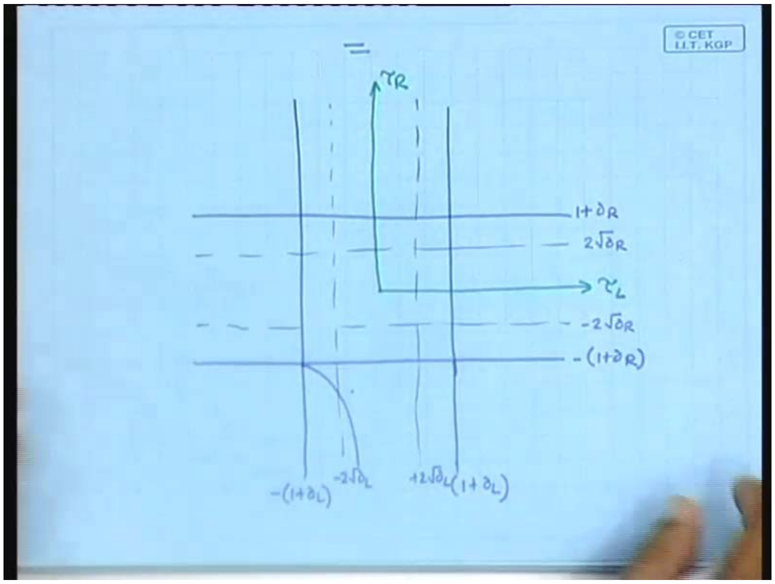

This particular equation would be quite similar to what you have seen earlier in the one dimensional case. It will be something like this. This equation, it means that if you take a point somewhere here then the period two orbit will exist after the border collision. That means from theory you will anticipate something like this to happen in the bifurcation diagram.

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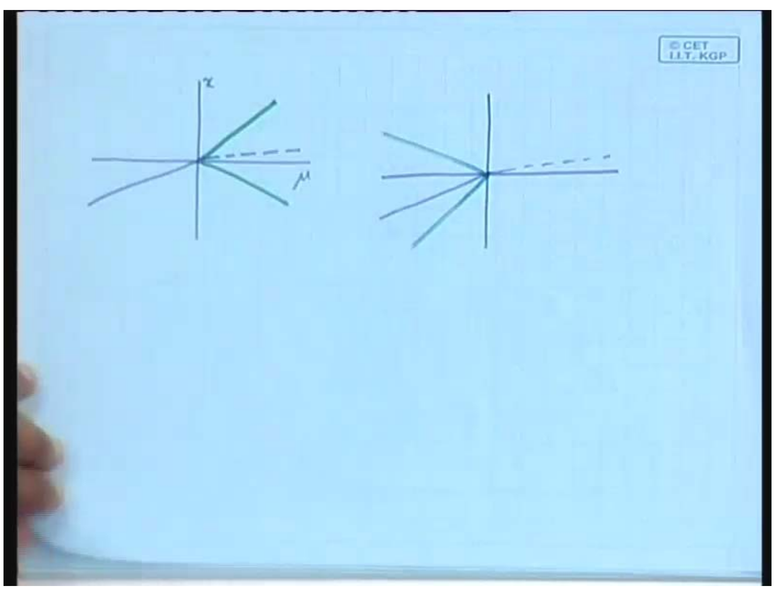

Then a period two orbit will; that means for mu greater than 0 this period two orbit will exist. I cannot as yet commit whether this fellow will be stable because we have not yet considered the stability condition. But at least we are sure that this fellow will exists. But what about this part? Here the stable period one fixed point came and hit the border but the period two orbit was existing in the same side and so period two orbit was existing in the same side. Therefore the behavior will be… therefore that periodic orbit will be stable or unstable? I don't know, I have not talked about that yet. For the present time I will mark that like this but we will later clarify that.

Here the period two orbit is a mu, for mu is equal to 0 when this is true. Now let us consider the stability condition. The stability condition is where you have one point mapping to this point and then it is coming to this point. What is the stability of this orbit? The stability of this orbit will be given simply by here is a matrix that maps to here and then comes back here, therefore this matrix times this matrix and its eigenvalues.

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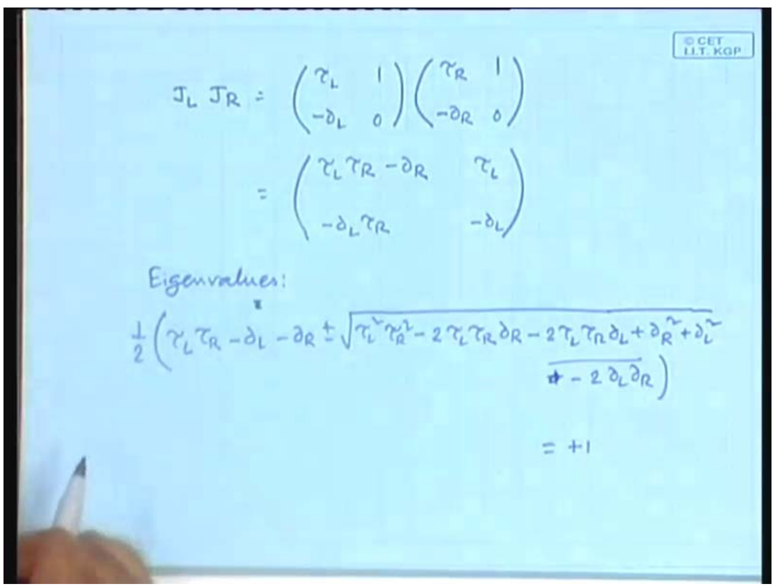

We will have to talk about the first Jacobian matrix is J_L say and J_R that one which is tau$_L$ 1 minus delta$_L$ 0 times tau$_R$ 1 minus delta$_R$ 0. This I gave you the last time, have you done that? It will be tau$_L$ tau$_R$ minus delta$_R$ tau$_L$ minus delta$_L$ tau$_R$ and its eigenvalues would be a big one. The big one I have given you the last time. I will not give you the same big one today. Instead I asked you to simplify that. If you simplify that, that simplification is rather simple. Because what you had was, eigenvalues would be half tau$_L$ tau$_R$ minus delta$_L$ minus delta$_R$ plus minus root over tau$_L$ square tau$_R$ square. This you can easily get simply by putting this equation into a symbolic computation program like a mathematic or maple and ask you to give the eigenvalues. Rather simple stuff, minus 2 tau$_L$ tau$_R$ delta$_R$. I am doing this because you have to be used to such manipulations, minus symmetrical thing 2 tau$_L$ tau$_R$ delta$_L$ plus delta$_R$ square symmetrical plus delta$_L$ square plus there will be minus 2 delta$_L$ delta$_R$, this whole thing so this continues.

Now if you simplify this, the condition for stability is this fellow. This fellow is equal to plus 1 or minus 1. Say I consider it equal to plus 1 then how I will manipulate it? I will bring these two here first then I will bring this fellow here first then I will square it and then cut off many things and then that simplification will ultimately lead to something very simple. That is what I asked you to do.

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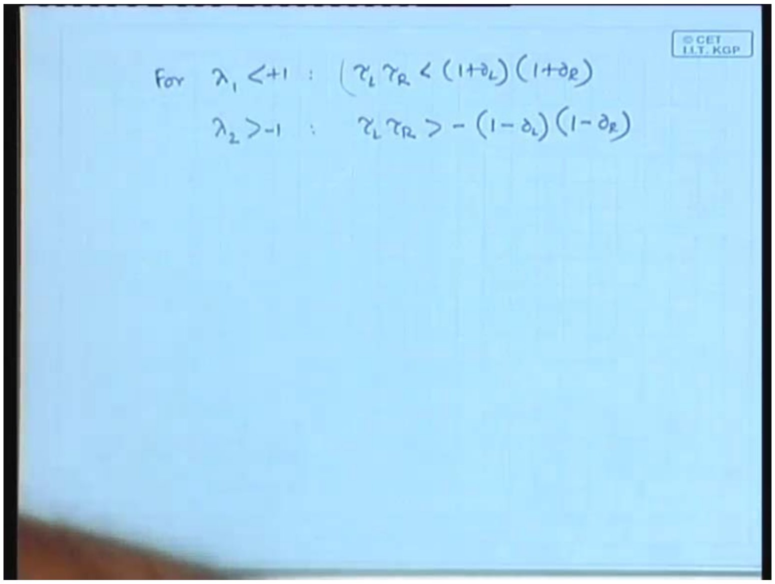

Now if you do that you will find that the conditions simply boiled down to… The condition will ultimately boiled down to, there are two cases plus 1 and minus 1. This has to be less than plus 1 then the condition would be τ_L τ_R less than and the other condition is smaller one should be greater than minus 1 that gives you τ_L τ_R greater than minus 1 minus δ_L. Now notice that we already have come across this fellow because when we talked about the existence your denominator was… When we said this term was less than 0 means this term is greater than this.

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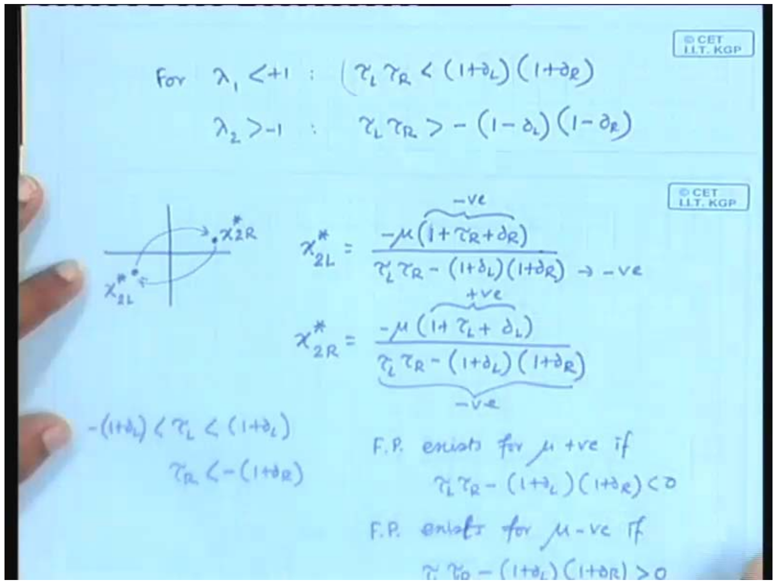

This is the condition, so stable. Stability condition is essentially the denominator of this equation.

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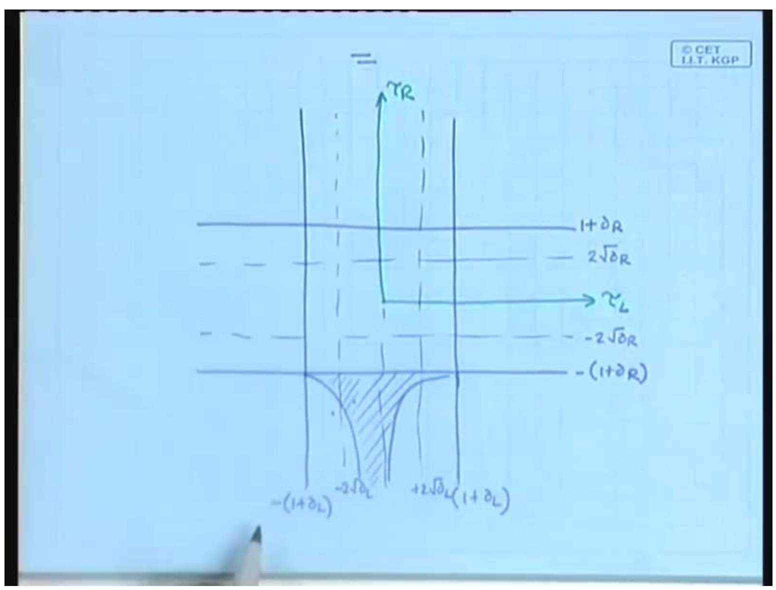

What does it mean? It means that when we talk about this point, it is not only existing but also stable but here when we talk about this point, it is unstable. The period two fixed point that was existing for the same side mu negative but that would be unstable.

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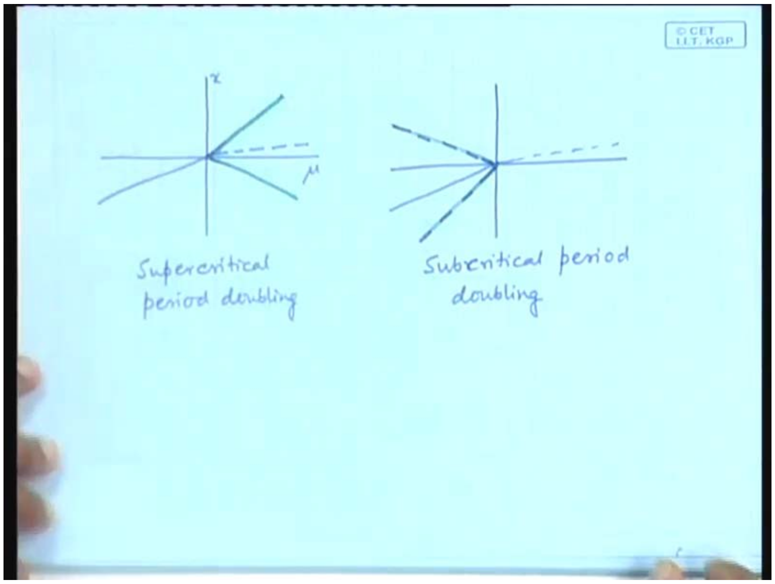

Now we have decided that this diagram that we drew, this orbit must be an unstable orbit because its eigenvalue would be greater than plus 1.

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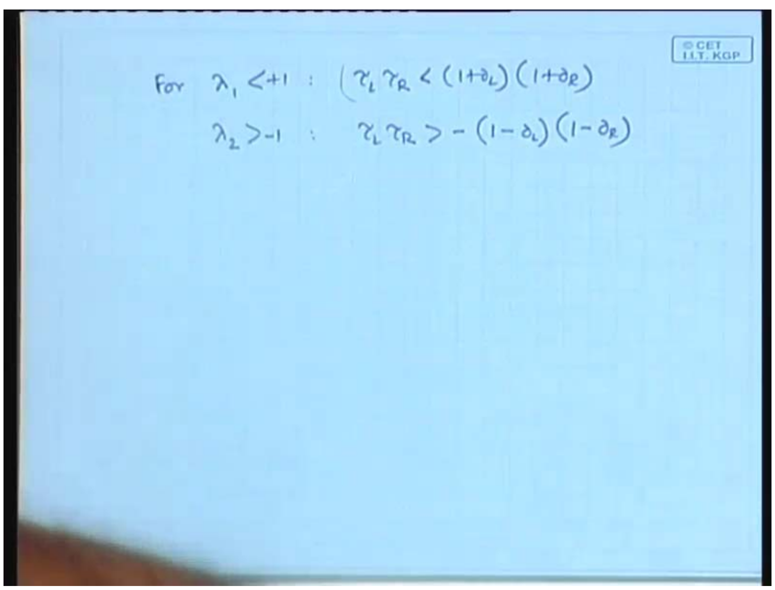

Obviously there is another condition which is this. Violating this condition, this period two fixed point will become unstable. It will continue to exits but it will become unstable. When this is satisfied, when this becomes equal and that equation again we can anticipate from what we have learnt in the one D case would be something like this, this kind of a line. In this range we can expect the period two orbit to actually occur. Here it will not occur because firstly it is a subcritical period doubling.

This is a subcritical period doubling and this is a supercritical. Again like we discussed in the case of the one D here, these two orbits immerged at acute angle not the 180 degree angle, here also it is so. This fellow has become unstable. Let's try to understand what actually happens in a physical system, in this case. If any physical system encounters this kind of a bifurcation, what will happen actually? A stable periodic orbit was there, it hit the border and then beyond that the period one orbit is unstable. Period two orbit does not exist and actually then this system will collapse, so that is the collapsing condition. Again this subcritical period doubling and whatever we are inferring about that refers to only the closed linear neighborhood of that border crossing fixed points.

In that sense it will go to infinity but that does not means globally it will go to infinity. In the local sense it will go to infinity. In this case the system will actually collapse. Yes, you have a question. His question is that how do you make sure that a higher periodic orbit will not be existing? Yes, we need to understand where will the higher periodic orbits exist. Let's look at that. When we talk about the higher periodic orbits which is the one that we consider first? We should consider first period three. Let us consider period three and when we consider period three then what kind of orbit will you consider?

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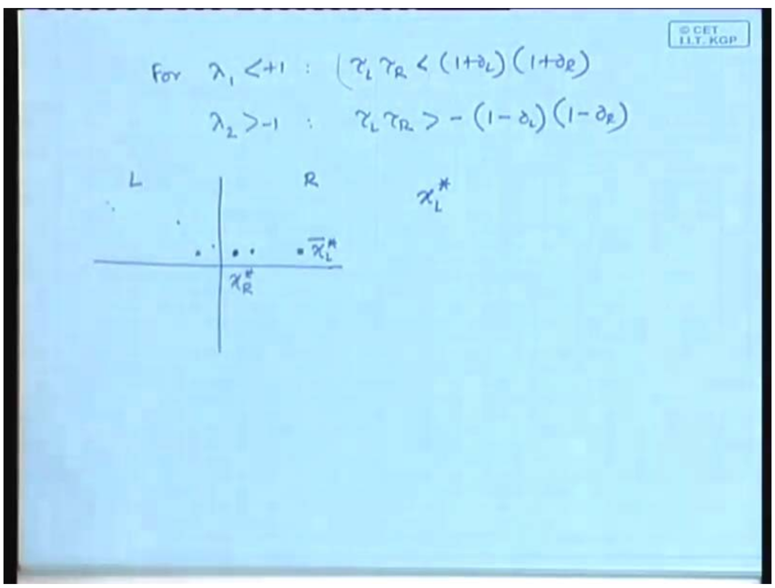

Let's formulate the argument this way. In this part the tau_L is between minus 1 plus delta plus 1 plus delta and therefore it was stable. In this part it was stable, in this part it was unstable. So long as mu less than 0, we don't really need to bother about it because I know that the period one fixed point is stable. So we have considered mu greater than 0. When mu is greater than 0 then what is the status of the fixed point that was actually obtained from the left hand side equations.
We have said no, there was a x_L star which for mu greater than 0 it will be somewhere in the right hand side it is the virtual fixed point. A virtual fixed point but that fellow is now an attractor. Any initial condition to the left, we will see there is an attractor sitting there, somewhere so it will be attracted to that. Successive iterates will approach actually that nonexistence fixed point.

You can expect this kind of orbit to be there and finally it will try to approach that but the moment it comes here it encounters what? In the right hand side it is a flip saddle and it will flip. If it flips it goes to this side but in this part, there is an existing fixed point. Suppose the fixed point is existing here and this fellow is mapped here. What will happen? It will flip to the other side and as a result it will come back here. You can see that when we consider a period three orbit it was possible to consider a LLR orbit, L is this side, R is this side or LRR orbit. But out of these two, it is more logical to consider a LLR orbit because I can see that you can have more number of iterates as this fellow approaches this virtual fixed point.

This is x_L star bar, bar means virtual and suppose here we have the x_R star but the moment it comes to the right hand side, it flips to the left and therefore we can anticipate that we should logically consider an orbit that is LLR orbit. Only one point in the right hand side. How we will obtain the existence of that? His question is how do you know that the period three orbit does not exist? Yes, sure.

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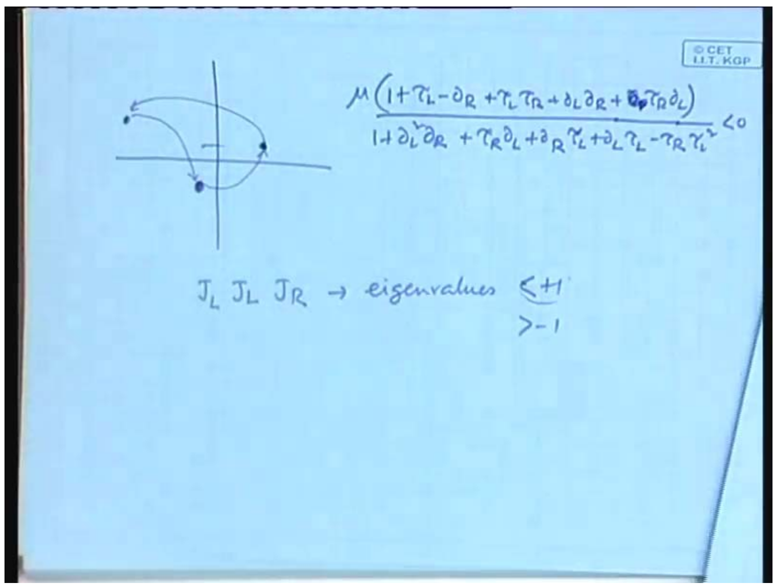

We will consider an orbit something like this, a point here another point here and one point to the right and it will map in this way. In considering that we can easily obtain all the three points. What we will do? We will start from this point, apply the left hand map once, apply the left hand map another time and come back here. Apply the right hand map and come back here. Then we will say x_{n+3} is equal to x_n, y_{n+3} is equal to y_n. Solve the equation, you get the fixed point here.

Similarly you start from here, apply the left then right then left. Do the same procedure, you would obtain this point. Start from here, all this can be done easily by hand. Up to the period three, it is not all that difficult. Once you obtain that then what do you do? You would notice that we are trying to consider when this fellow will exist. Existence means this point should be in the left, this point should be in the left and this point should be in the right. Is it possible for this point to hit the border before this point?

Obviously this point was mapping to the right and this point was still mapping to the left and so if anything hits the border, this fellow should…We should actually consider this fellow's x coordinate. The x coordinate of this fellow and that is what we will actually determine whether this fellow is in the left hand side or right hand side or this periodic orbit exists or not. What about this? Can I hit the border? If it does, say it moves to this side I would have to say that there is a period three fixed point with all the points in the left which is not possible because its side is a linear side. In a linear system you cannot have a high periodic orbits, so you cannot have period three also.

The critical situation is here, this fellow hitting the border and that condition you can easily obtain it as, you have got a 1 plus tau_L minus $delta_R$ plus tau_L tau_R plus $delta_L$ $delta_R$ plus tau_R $delta_L$ whole times mu divided by 1 plus $delta_L$ square $delta_R$ plus tau_R $delta_L$ plus $delta_R$ tau_L plus $delta_L$ tau_L minus tau_R tau_L square. This should be less than 0. Now if you actually substitute the parameter values like say, take $delta_L$ $delta_R$ both to be 0.5 or something, the determinants. These are dissipative systems so consider both sides to be having determinants to be less than 1. If you substitute that then you will find that this particular thing does not occur to the left of this. It always occurs to the right of the period two, so it will occur somewhere here. This is one that is obtained from the existence.

There is another condition of stability and what is the stability condition? Stability condition is obtained by Jacobian of the left times Jacobian of the right. Its eigenvalues should be less than plus 1 and greater than minus 1, that is the condition. Now this condition you will find that this is same at the denominator but this condition gives an additional thing so you get two contours, two extremities and that defines in which range the period three orbit will actually occur.

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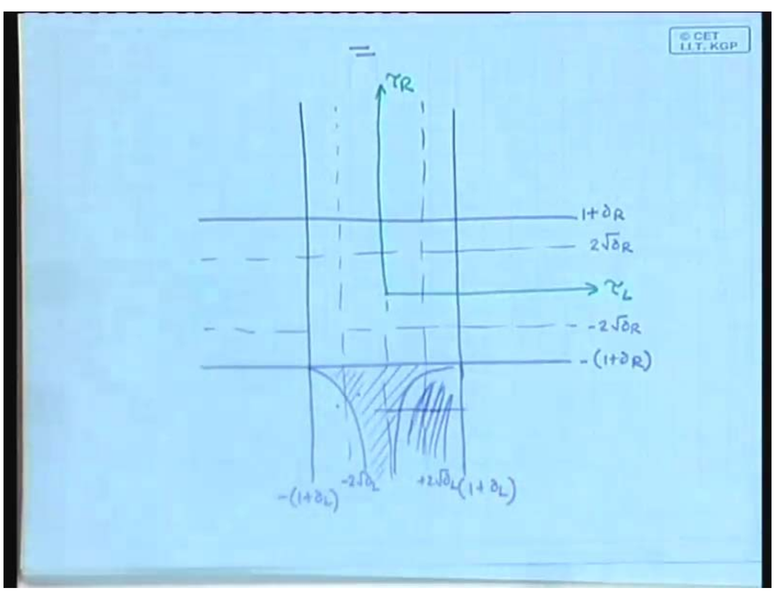

Similarly the process that I have just outlined, you can apply that same logic to the period four, to the period five and all that and you can find that there would be distinct ranges where this fellows will exist.

Again as we saw in the case of the one D, if you increase the parameter like so, you will find a period two orbit in between there is nothing. Again a period three orbit in between there is nothing, here is a period four orbit in between there is nothing. Is it possible? No, because there is no scope of this fellow going into infinity because of the existence of the attracting fixed point in the right, any point to the left cannot go to infinity this way. It will see an attractor and move always to the left. Any point to the right will flip to the left and therefore it cannot go to infinity. If it cannot go to infinity then obviously something will occur and the something is, if none of these periodic orbits are existing in this part, this will be chaotic. So period two chaos, period three chaos, period four chaos and all that and that way it will leave this point.

You can anticipate a similar type of period adding cascade. As you increase the parameter this way, the type that we saw the last day on the computer but that was for the period one dimensional system but for the two dimensional system also a similar situation will occur. The only complication is that there can be overlapped between these orbits, as a result they can be coexisting attractors. Something that doesn't happen in one D case. In a two dimensional case these ranges of existence and stability where this fingers actually occur, they may overlap and when they do there would be a range in which both period two and period three occur and there may be a range where period three and period four occur and all that. In those parts you will find coexistence of attractors. It is also possible to have coexistence of a periodic orbit and chaotic orbit. There is a lot complication here. Let's not go in to that complication.

Essentially we understand that under such situation we are likely to see a period adding kind of cascade. Now let us understand this part. This part we know, what happens in this part? A period one stable fixed point and a period one unstable fixed points are born. It is like a saddle node bifurcation. Simple, but in this part what happened? Let us consider this part. What is the condition?

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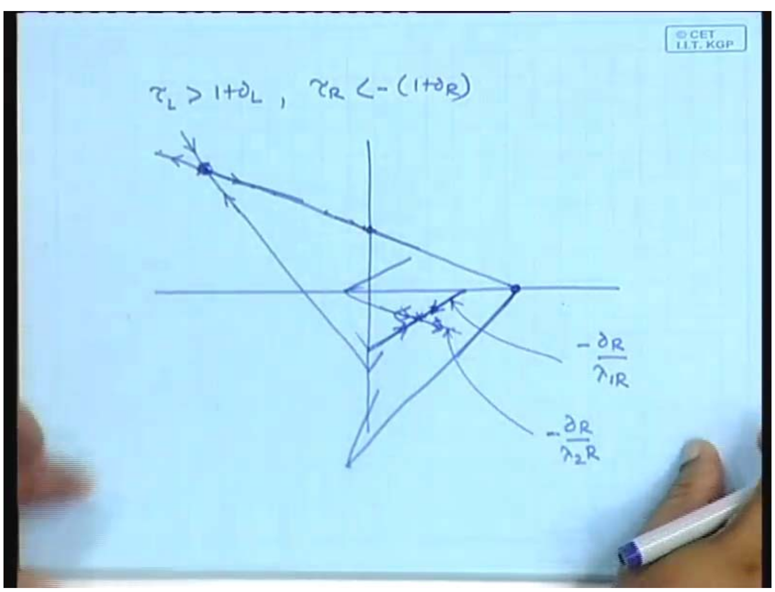

The condition is in this part. Tau_L is above 1 plus $delta_L$ and tau_R is below minus… (Refer Slide Time: 32:15). I don't have the slides here but in the last class we derived the condition for that to occur. That means two fixed points are born. That condition is satisfied here but now since both the sides are unstable sides, both the fixed points are unstable. Two unstable fixed points are born. What will happen? To see what will actually happen let us draw the character of the state space. Two fixed points are born, one say is here and another say is here, both unstable. This fellow is a regular saddle, this fellow is a flip saddle that's all because one is greater than this one. This condition is regular saddle, this fellow is flip saddle. If you have a regular saddle, you can draw its stable and unstable manifolds. The stable of an unstable manifolds will originate at the eigenvectors. At this there will be an eigenvector like this and there will be another eigenvector like that. Say this is the stable eigenvector and this is the unstable eigenvector.

Now a very important point pertaining to non-smooth systems something that I did not talk about so far. How will this unstable manifold proceed? The unstable manifold will proceed along this and unstable manifold has the property that any initial condition on the unstable manifold will remain on the unstable manifold that is the definition. This point will map, this will map it is a regular so it will map like this. But it had come here and it has continued but then this point when it maps to another point here, it maps by the left hand side map. Here also it maps by the left hand side map but here it maps by the right hand side map.

Now where does this point map to? We have already seen while we were deriving the map for this normal form that any point on the y axis maps to the x axis. This point must be mapping to this straight line here, so this point must be mapping to this point. Any point to the left should be mapping by the left hand side map. Any point to the right will be mapping by the right hand side map and therefore obviously the slope after this point cannot be the same. It will have to fold. Is the logic clear? Because this point is mapping to here and this point is mapping to somewhere here, this cannot continue in the same line because you are now using a different map, different linear map and so this must have a different slope. It means that the unstable manifold will fold at its intersection with the x axis and then this fold point will again map to some point so suppose this one continues and at this point, this point maps to this point.

See this is the fold point, this point also must be a fold point. It is also a fold. All the forward iterates of the fold point will be fold points. The unstable manifold folds at the intersection of the 45 degree line with the x axis and any forward iterate of a fold point will be also a fold point. That gives a very complicated folded structure to the unstable manifold. What happens to the stable manifold? Just the opposite because what is the definition of the stable manifold. It is that the stable manifold approaches that fixed point and therefore by the inverse map, it goes away from the fixed point. If you imagine the inverse map, its character will be the same as this.

The point is that if any point on the y axis normally maps to the x axis, by the inverse map any point on the x axis will map to the y axis. By the same logic this fixed point, this stable manifold will fold at its intersection with the y axis. Because you are actually applying the same logic but applying by replacing the forward iterate by the backward iterate and while you take the backward iterates then x axis maps to y axis. The logic that we said is that the moment it crosses the y axis, the slopes are different and see the y point maps to the x point and therefore it will have a fold at the x axis.

The whole thing happens on the y axis. This fellow will fold at the y axis like this. There is a structure given here. Now you would notice that after this point is understood let's talk about the similar stable and unstable manifolds of this flip fixed point. What will happen? This fellow will also have a stable and unstable manifold say the unstable manifold is like this and this stable manifold is like that. Then what will happen to this unstable manifold? This will be stable manifold and this will be the unstable manifold. The stable manifold will have a slope like this and it will fold here (Refer Slide Time: 39:00). Unstable manifold will fold somewhere here on the x axis but then this unstable manifold and the stable manifold, they have some slopes and that slope can be easily understood. Why? Because this slope is nothing but the slope of the eigenvector.

If you give this matrix to a program like mathematic or maple and I ask you to obtain the eigenvector, it will give the eigenvector slope. You will find that the slope of this stable eigenvector, its slope is minus $delta_R$ divided by $lambda_{1R}$, 1 is for the first eigen value and R is for the right hand side. This slope is this much and the other one, the unstable eigenvectors its slope is… determinant by the eigenvalue itself that is the slope of the eigenvector. We are considering condition where this is true. The determinants are less than one, we are considering the situation and these are the slopes. It is not difficult to see that then this fellow will fold. No, this diagram will cause difficulty lets blow it up.

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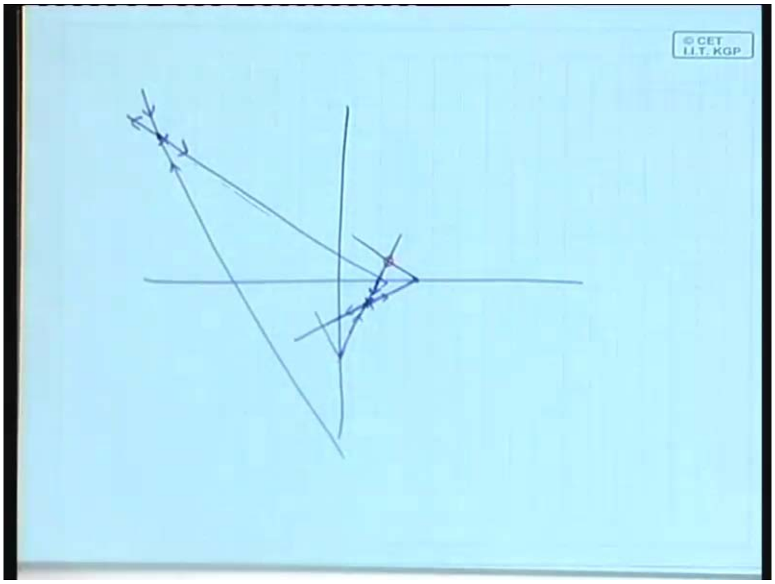

We are considering the fixed point here whose unstable manifold is here and the stable manifold is somewhere here. The unstable manifold will fold here and like this but the stable manifolds will not fold here and will continue. Which means that at this point what do we have? We have the intersection between the stable and unstable manifolds and whenever there is an intersection between the stable and unstable manifolds, what happens? There has to be an infinite number of intersections. We have already seen that. This means that somehow these two manifolds will be convoluted, will be folded and go any possible way so that there are infinite number of intersections.

We have seen that under such conditions there is a horseshoe kind of structure, the existence of a chaotic orbit. This logic tells you this is a stable manifold which will fold here. Again there will be intersection and stuff like that. There will be large number of intersections, infinite number in fact and the logic behind the intersection is that here is an unstable manifold which you will fold here. Here is the stable manifold which will not fold here, which will fold here. The stable manifold has a larger slope by that value and the unstable manifold has a smaller slope therefore after the fold, there must be an intersection. Therefore there must be a chaotic orbit because of that this intersection. This is all very well understood.

Now there was another fixed point sitting somewhere here. What will it do? Its unstable manifold was coming like this and its stable manifold was coming like that and the unstable manifold will fold at the intersection here. This is the unstable manifold and this is the stable manifold. This unstable manifold then has an intersection with the stable manifold of this fellow. The unstable manifold here will have a stable manifold of this fellow. What is the consequence?

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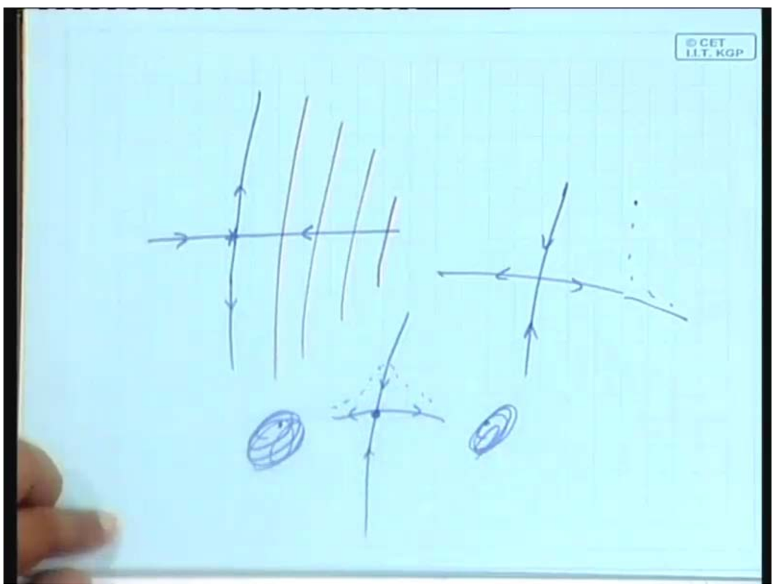

While we were talking about the smooth map then also we have treated this problem. Suppose there is a saddle fix point, it will have a stable manifold and it will have an unstable manifold and suppose any line intersects the stable manifold. Then what will happen? You can take the forward iterate of this line means all the points are iterated by this map. It will map somewhere because it is a stable manifold, there is a contracting direction like this. In this successive iterates it will approach the unstable manifold here and finally asymptotically it will converge on to the unstable manifold.

Now here notice what is happening, we have the same situation here. Here is the stable manifold and this line has intersected with the stable manifold. Naturally the immediate conclusion is that this unstable manifold will asymptotically converge on to the unstable manifold here. They will ultimately become one and the same unstable manifold of the system, though they initially started out as the unstable manifolds of two different fixed points.

Now in any system all the iterates are attracted to the unstable manifolds. We have seen that just to refresh your memory. If you have a point here that is initial condition, where will it go next?
It will be attracted to this side because of the stable manifold and it will go there. So it will ultimately go like this. Ultimately you see it is coming closer and closer to the unstable manifold and it is going away and away from the stable manifold. The unstable manifold ultimately acts as the attractor. Now here we have a situation where there are two unstable manifolds and both point to the same thing, a chaotic orbit here. It will ultimately land as a chaotic orbit and because in the state space, the initial conditions are attracted to the unstable manifolds which ultimately, all lead to the same structure here. There must be a chaotic orbit here.

This chaotic orbit was unique, you cannot have another orbit. It must be unique. It is to be unique chaotic orbit. Will it be stable? Stability condition is different. You already know that if there is a fixed point here, there is another fixed point here and suppose this is something stable. So some orbit here, some orbit there and in between there is a regular saddle fixed point sitting with the stable manifold here and the unstable manifold there. Then we have already learnt that the stable manifold acts as the basin boundary. Each will have a basin of attraction, this will have its basin of attraction and the separator will be the stable manifold. Why? Take any point on this two sides. They will be asymptotically going away from each other. This stable manifold act as a repeller, it will actually divide the basis of attraction, we have learnt that. Here also the same thing will happen but that requires a regular saddle, here we have a regular saddle. So its stable manifold will divide the basin of attraction. One is here, the other is at infinity.

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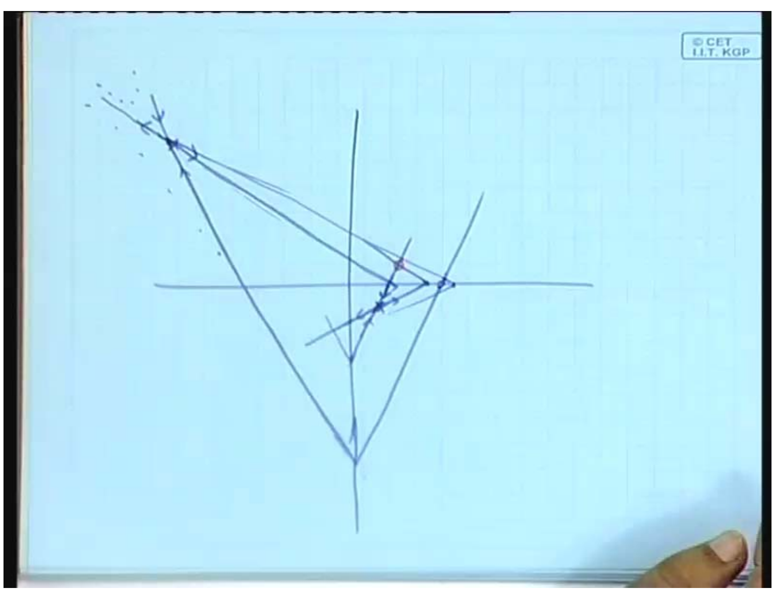

Any initial condition here actually goes to infinity. By the action of this unstable manifold, it goes to infinity. Any initial condition here also goes to infinity. Any initial condition here also goes to infinity. There is an attractor infinity, there is an attractor here and they coexist and they are divided by this stable manifold. This is the dividing boundary and most interestingly this fellow folds like this. We have seen that the stable manifold will fold. See what happens then.

We have got a situation where this orbit comes and hits it and we know that this leads to the attractor. We also know that this fellow folds and comes here, in case this is bent like this then it will intersect the stable manifold, before it folds. It will intersect the stable manifold before it folds. Means the unstable manifold and the attractor existing on the unstable manifold will then go out of the basin of attraction. It is a boundary crisis and you can easily obtain the condition for the boundary crisis. You know the slope here, you know the slope after it bends, naturally you can find this cutting point. You know the slope here, you can find the cutting point. This orbit will be stable so long as this point is to the left of this point. Here also by logic you can obtain in closed form the condition of existence of this chaotic orbit. We had obtained that remember, in case of the one D system. You could obtain that it was very simple. In this case it is lot more complicated but nevertheless it can be obtained and it yields an equation that is also quite similar, it's like this.

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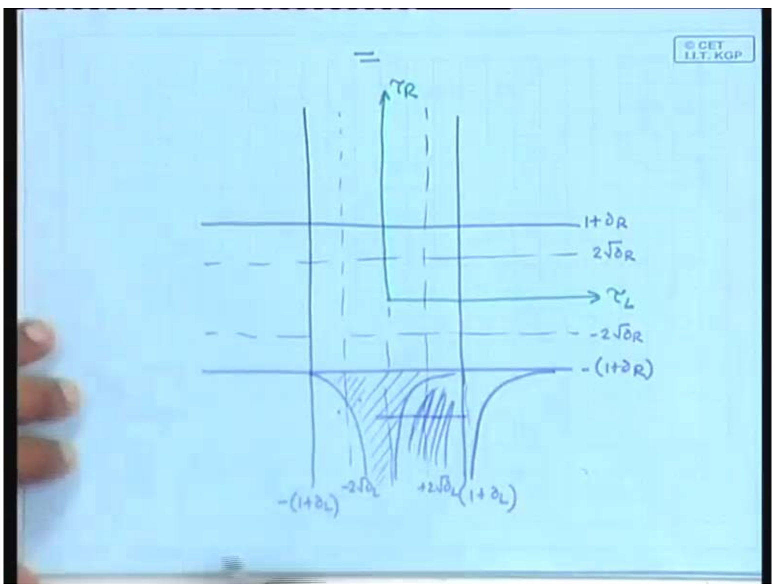

In the one D case we had obtained this but its condition was very simple but in the two D case that condition is relatively more complicated.

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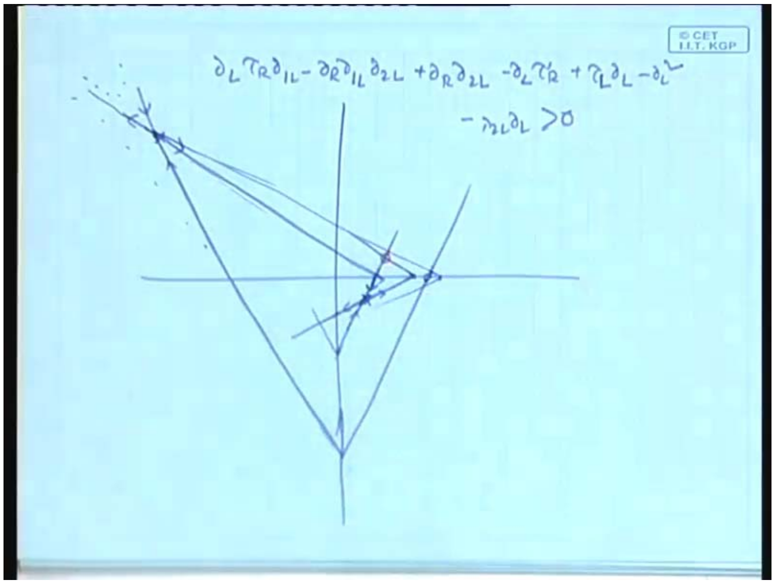

It is delta$_L$ tau$_R$ lambda$_{1L}$ minus delta$_R$ lambda$_{1L}$ lambda$_{2L}$ plus delta$_R$ lambda$_{2L}$ minus delta$_L$ tau$_R$ plus tau$_L$ delta$_L$ minus delta$_L$ square minus lambda$_{2L}$ delta$_L$ that has to be greater than 0, then it is stable. Now this condition may be looking formidable but I have explained how it is obtained.

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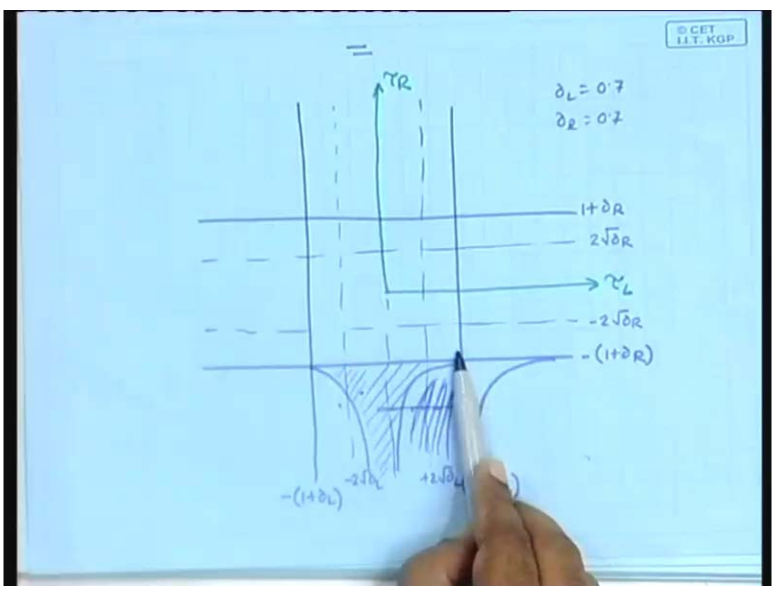

I will give you the exercise that you have predicted what will be the behavior in all these parts. Take a suitable value of the determinants, say the determinant to the left is say 0.7, determinant to the right is also 0.7 both dissipative and then you take the values of trace in the left and in the right in such a way that once this condition is satisfied, vary mu and see what happens.

Your prediction is that it will cause a period doubling. In this part period tripling, in this part period one to period four and so on and so forth. In this part period one to chaos directly. In this part nothing to chaos because at this point a couple of periodic orbits are born, period one orbits are born and both are unstable. So born means it is not existing before mu is equal to 0. In this part you will expect that the periodic orbit is born. Now if you want to write the program to obtain the bifurcation diagram, for mu negative it will run to infinity and your program will collapse.

You will have to start from mu positive and see what happens if you reduce the mu. You will find a chaotic orbit slowly shrinks and become 0 and then it vanishes. These things you should yourself try out by writing a program. This program is rather simple, you could also use one of the existing programs. no problem but it is always advisable to do it yourself to convenience that yes that is what happens. We have more or less covered these parts, as I told you that this part looks simple but there is a lot more complication to that which I will come to later. I will discuss about that in the next class. So that’s all for today.