Chaos Fractals and Dynamical Systems
Prof. S. Banerjee
Department of Electrical Engineering
Indian Institute of Technology, Kharagpur
Lecture No. # 34
Bifurcations in Piecewise Linear 2D Maps

In the last day we concluded by showing that any given non-smooth system, if you have a border line like this and if you have a border crossing fixed point then you obtain the eigenvalues here just before the border crossing and you obtained the eigenvalues there just after the border crossing. Then you can obtain the normal form from there which is given as x_{n+1} y_{n+1} is equal to… This is for x_n less than 0 and for x_n greater than 0 this will just be substituted by the right hand side quantities. It should also be possible to obtain the same result simply by a matrix manipulation of the Jacobian matrix obtained here and Jacobian matrix obtained there.

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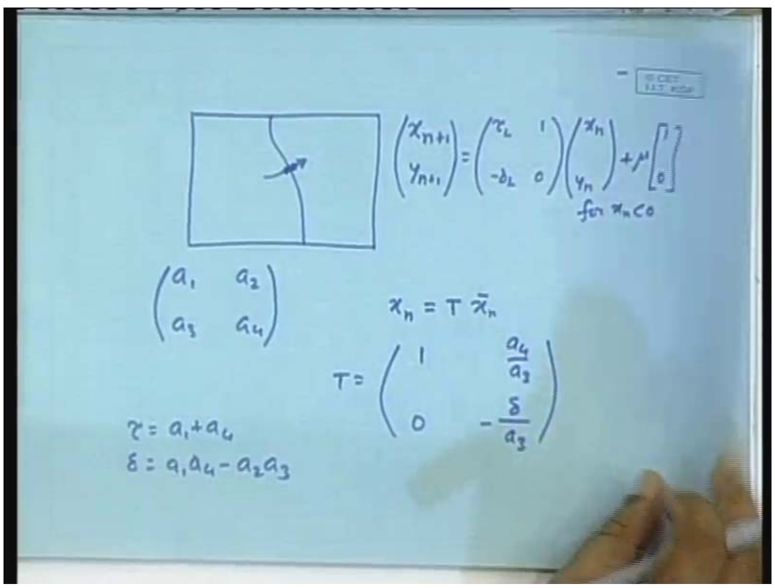

Suppose you have obtained the Jacobian matrices like this. In this side say you obtained the Jacobian matrices as say a_1 a_2 a_3 a_4. Then you want to obtain something like this. What you have to do? If you transform the x vector x_n y_n as, initially it was say x bar and then you want to transform into x. So x_n can be some matrix T times x_n bar where x_n bar were the original ones. Now if the T is defined as 1 0 a_4 by a_3 and here it is the determinant divided by a_3, this will be minus. Then you can easily see that using this transformation, this matrix can be transformed to this matrix.

In this case the trace is $a_1 + a_4$ and the determinant is a_1 a_4 - a_2 a_3. This is another way of doing the same thing that we did. We can do the same thing using a different logic. Finally we have obtained the normal form and then using that, we are trying to understand the behavior of non-smooth matrix not the complete behavior but we are exactly trying to obtain and understand the behavior of the non-smooth bifurcations, the border collision bifurcations. The logic is that in the neighborhood of the border collision bifurcation, it suffices to consider the local linear neighborhood of the border crossing fixed point.

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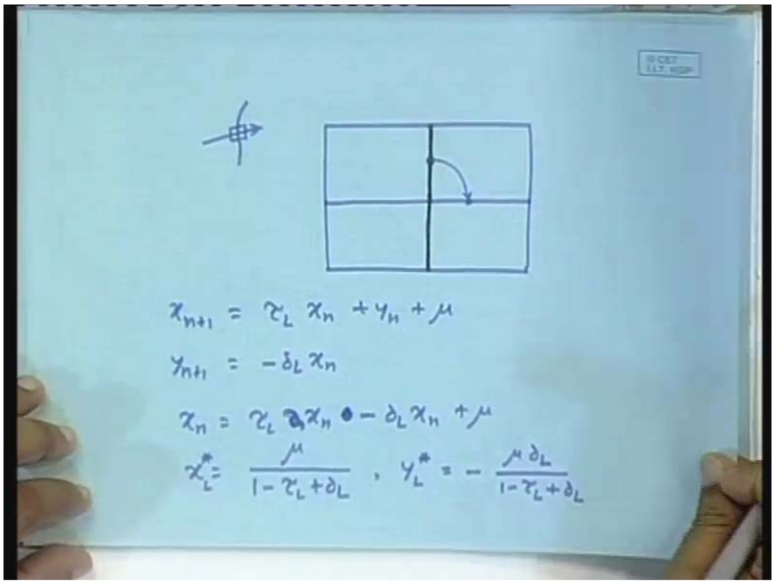

Since the border crossing fixed point crosses like this, you have to consider one side in the left another sign in the right and this small box is then transformed and it is now written in the normal form. So where are we? We have obtained a changed state space description representing the local linear neighborhood where the border line is exactly the y axis and this is the x axis. So this is not only the x axis and the y axis, this fellow is also the border line. In this part it has got one description, in this part another description as we have seen, it is given in terms of the trace and the determinant of the left hand side and this is given in terms of the trace and the determinant of the right hand side. We have also seen that one character of this map is that any point on the y axis maps to the x axis. All the points on the y axis would map to the points on the x axis and where would 0 0 map? To itself. So 0 0 will map to itself and your y axis will map to the x axis.

Now when you say 0 0 will map to itself, it is not exactly correct because of this fellow. Only when mu is 0, 0 0 will map to 0 0. That means a fixed point is exactly located at the origin else not. We will have to find the location of the fixed point in this case. So using the expression for the map, we can now obtain the location of the fixed point. Now you can easily do that because your equation was supposing we are considering the left hand side then your x, this is the equation. Let us write it individually. x_{n+1} is equal to tau_L x_n plus y_n plus mu and y_{n+1} is equal to minus delta_L x_n. That's all. How to find out the fixed point? You will say that x_{n+1} is equal to x_n, y_{n+1} is equal to y_n. If y_{n+1} is equal to y_n, you can substitute it here. You can say x_n is equal to tau_L x_n, I will substitute it here minus delta_L x_n plus mu. So from here you can obtain x_n is equal to mu by 1 minus tau_L plus delta_L.

How to obtain the y_n? Simply by substituting it here, you get y_n, so x star this is of the left hand side x_L star, x_L star y_L star is the fixed point as calculated from the left hand side equation. This is equal to minus mu delta_L divided by 1 minus tau_L plus delta_L. It's not difficult to see that, similarly you can obtain the quantities of the right hand side that means you can calculate the fixed point that exist in the right hand side and that would be given in absolutely similar way.

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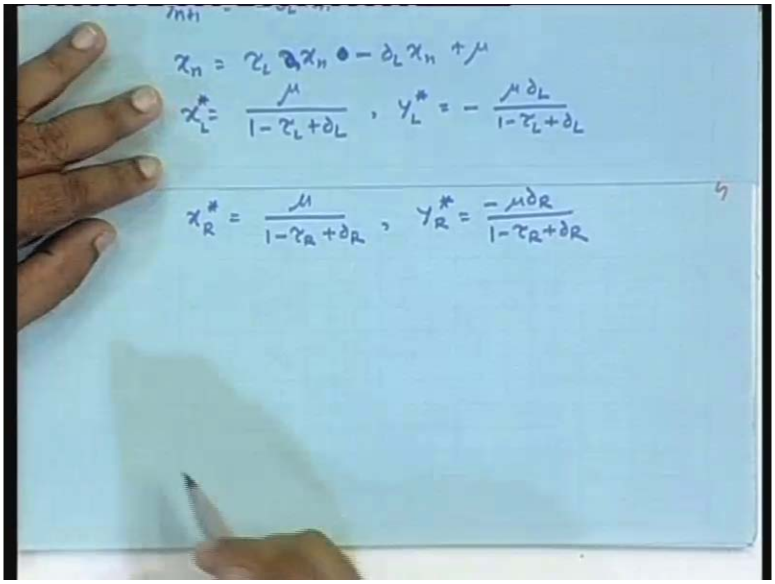

x_R star is equal to mu by 1 minus tau_R plus delta_R and y_R star is equal to minus mu delta_R 1 minus tau_R plus delta_R. These two are the equations for the left hand side and right hand side fixed points. Now notice that this term and this term they contain mu, they contain the trace and the determinant. The trace and the determinant may have a positive value and the negative value that is giving a positive value and negative value to the denominator. At the center mu can be positive and negative. As a result the x_L star could be positive or negative. Now we have calculated this x_L star using the left hand side equation, using the equations of this side. Obviously this fellow will exist so long as the x_L star happens to be a negative number. That means this fixed point is in the left hand side else you could run into a situation where you have calculated using the left hand side equation. The fellow does not exist in the left hand side.

If this quantity is a positive number, the fixed point is actually here but you have calculated using the equations of the left hand side. What do we infer then? We infer that this fellow doesn't exist, this fixed point does not exists. But notice if your initial condition is here then this initial condition and it's further iterates we will see that as a fixed point. Even if this fellow doesn't exist, it will see that fixed point say it is here and supposing this fixed point is the attracting fixed point, it will be attracted to that and its dynamics will be guided by that. Even though the fixed point does not exist, it has an influence on the dynamics and therefore we will call it a virtual fixed point. If it exists, means x_L star this magnitude is negative, x_L star y_L star that fixed point exists and this we call it a real fixed point else it is a virtual fixed point.

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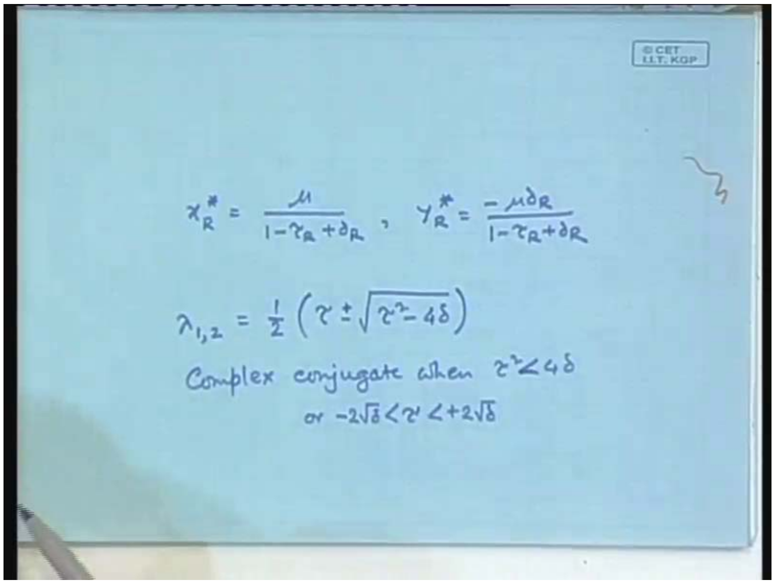

Similarly when this x_R star, this quantity is positive this fellow exists, it is real. If comes to be negative then it is positive. That is one important character of the map. The second important character is given by, here is a fixed point. Natural question is will that be stable and that stability is given by the eigenvalues of the fixed point. For a map of this form, the eigenvalues can easily be written. Then the eigenvalues are $\text{lambda}_{1,2}$ will be half tau plus minus root over tau square minus where this tau and delta could be of either side. That's why I have not given the subscript. This is the expression for the eigenvalues and when this eigenvalues are less than unity this fellow is stable. If this eigenvalues are greater than unity this fellow is unstable.

One condition is when this is plus thing, when this is substituted by the plus 1, only you take the larger one that equals plus 1 that is one point where it can become unstable or the lower one when it becomes equal to minus 1 that it can become unstable or if there is a pair of complex conjugate eigenvalues and you can easily find out the condition when the eigenvalues will be complex conjugate. Easily you can see that the complex conjugate eigenvalues happen when this is greater than this. So tau square less than 4 delta. Complex conjugate less than or tau lies in between minus 2 root delta and plus 2 root delta, same thing. It becomes complex conjugate only when the relationship between the trace and determinant is like so. Let us now find out what is the condition for the real eigenvalues stability. Can you just work it out? Take the plus and make the whole thing equal to one, take the minus and make the whole thing is equal to minus 1. Do it.

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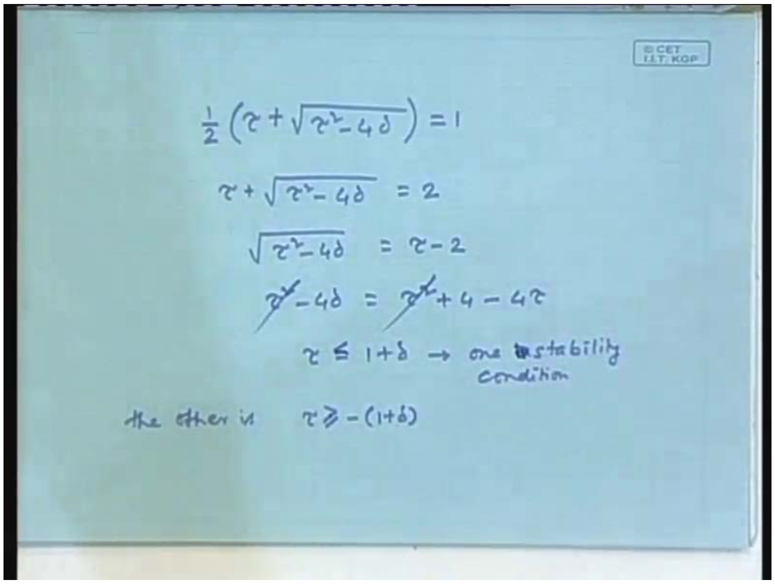

The half tau plus root over tau square minus 4 delta is equal to 1 is the condition. It is critical condition so tau plus root over tau square minus 4 delta is equal to 2. So root over tau square minus 4 delta is equal to tau minus 2, so tau square minus 4 delta is equal to tau square plus 4 minus 4 tau. You will get tau equal to 1+ delta. That is the critical condition and the fixed point will be stable so long as tau is less than 1 plus delta and it will be unstable if it is greater than 1 plus delta.

Similarly you can work it out for putting negative here, minus here and equating it to minus 1 that is another instability condition. Can you do that and give me the condition. This is one instability and the other is delta plus tau is equal to minus 1 but I would like to express it as tau as related to delta. It will be tau greater than minus and the stability condition is this. There are two conditions. I am talking about stability condition that happens when it is less than 1 plus delta and greater than minus 1 plus delta then they are stable. Now you know that the fixed points are classified depending on the character of the eigenvalues and you know that if there is one eigenvalue let’s just recapitulate that.

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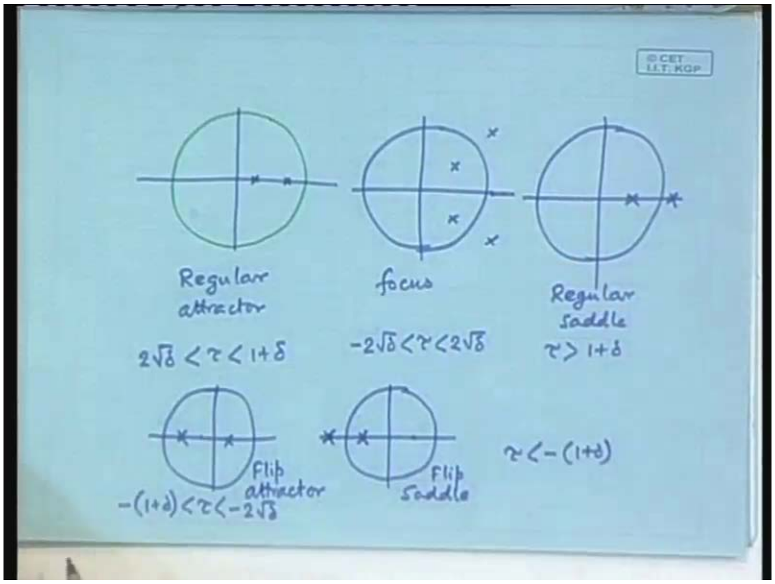

Say I am drawing the complex plane and the position of the eigenvalues as related to the complex plane. This is the unit circle. What if you have two fixed points like this, what it is called? It is a regular attractor. From whatever already derived when will that happen, regular attractor? Firstly you should not satisfy the complexity condition and secondly it should be stable. The stability condition is that it will be between 2 root delta less than tau less than 1 plus delta. This is where it becomes complex and this is where it will be stable. In between it should lie. Similarly if you have, when will that happen (Refer Slide Time: 19:00)? This fixed point is called a focus and that happens when it is between minus 2 root delta less than tau less than 2 root delta. When will this condition occur? What it is called? A fixed point outside the unit circle, a fixed point inside the unit circle both real it's a regular saddle. When will that happen? When one stability condition has been violated. Which one? This one, so then this is tau greater than 1 plus delta.

Similarly let me draw here so that we can finish within this page. This can also be here. It is an attractor because it is inside the unit circle but there is one eigenvalue negative which means that there is a direction in which it flips. It is a flip attractor. When will that happen? Flip attractor that is… (Refer Slide Time: 20:50). Next what is it called? It could be here also that means I am saying that one eigenvalue is outside the minus one point. That means it has gone out and the other one could be this side or that side. Then it is a flip saddle and when will that happen? Yes, that is tau less than minus 1 plus delta. Here I am again not writing the subscripts because that pertains to both the sides. When we are talking about the left hand side fixed point, we will simply put L in the subscript. When we are talking about the right hand side fixed point, we will put an R in the subscript but these are the basic types of fixed point that can be. Can it be like this? Is it possible? Yes, it is possible but you would notice that the magnitude of the fixed point, magnitude of the eigenvalue; just obtain its magnitude you will find it is nothing but delta. So obtain this fellow's magnitude, you will find this is nothing but delta. Can you do that, can you check? Magnitude of this means you have to square and then simplify it.

When the determinant is exactly 1, it should lie on the unit circle but if the determinant is less than 1 it is actually a dissipating system and the eigenvalue will be inside and if it is greater than 1 it will be outside. If it is greater than 1, it is an unstable focus where the orbit will be spirally outwards. While if it is inside, it will be spirally inwards. The fixed point that we are talking about, we have decided about their existence, when will they exist and when will they not and we have also decided about the type. These are the types that are available. Now you would notice that then depending on the relationship of the trace with the determinant we can sort of tabulate the character of the fixed point.

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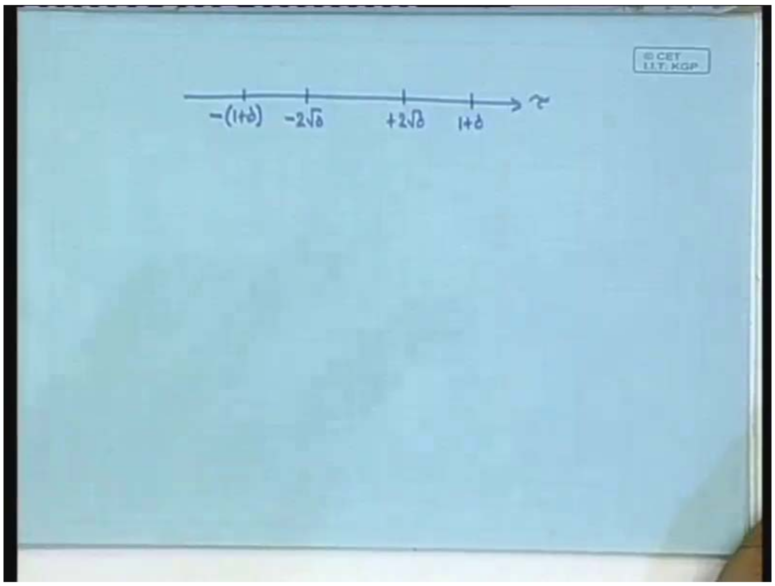

If this is the trace then if it is between minus 2 root delta to plus 2 root delta, it has one character. What is it? Complex conjugate. If it is 1 plus delta, if it is between 2 root delta to 1 plus delta it is a regular attractor. If it is outside that, a regular saddle. Another here is minus 1 plus delta. If it is within this 2 it is a flip attractor, outside this flip saddle. On the real line, I am now able to categorize this fixed point depending on the relationship with the trace, with the determinant.

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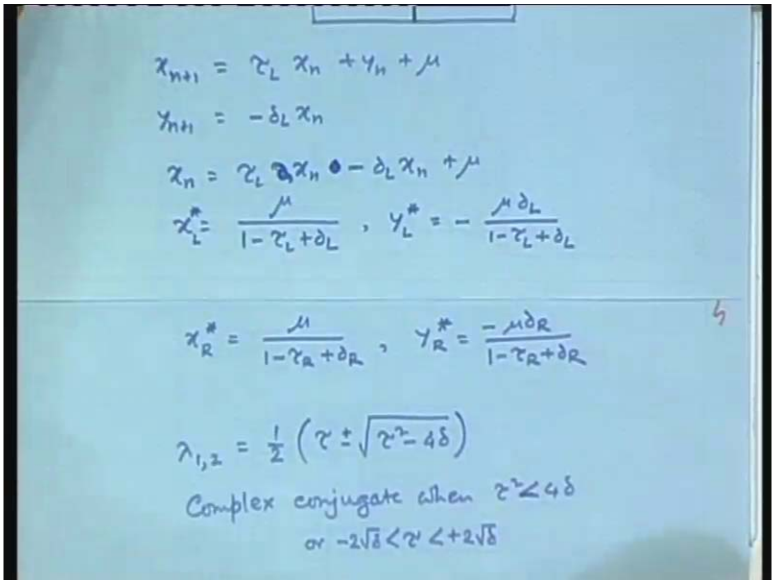

Now in this system what happens? As you vary the parameter mu, where was the tau_L? It was written here. Let us keep this side by side. If mu is negative then this x_L star, the fixed points position is also negative provided depending on the denominator but in general you can consider that this fellow is positive. So that mu is negative means this fellow is also negative. This is positive means this will also be positive.

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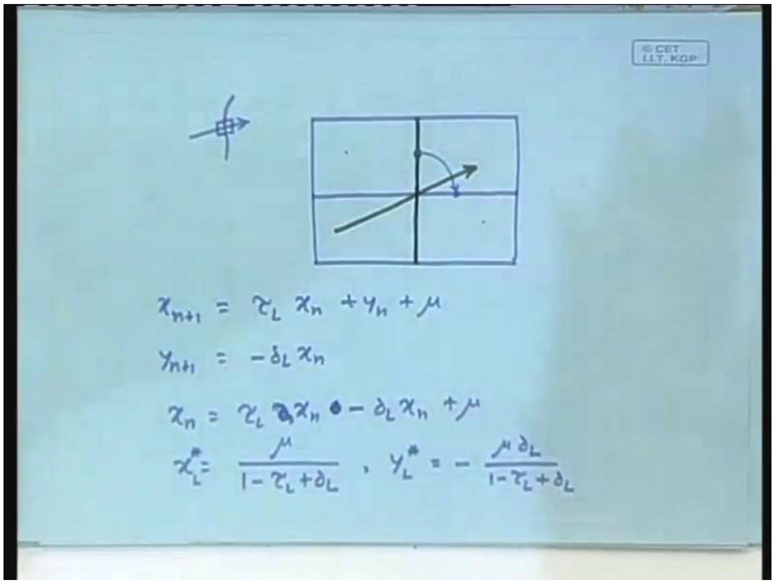

So which means that as the parameter is varied, you have a fixed point crossing like this and at mu is equal to 0 it is on the origin.

From this side it crosses the origin or it may not cross also. But nevertheless in general in order to understand, you can say that at mu is equal to 0 the border collision occurs. I am not saying in the general sense. I am saying that mu with respect to, as related to this quantity it will be either existing or not existing. So always you cannot say that fixed point comes and hits the border but always you can say that the border collision happens at mu is equal to 0. That is definitely correct because mu zero means this fellow is 0. The fixed point is on the border means it has hit the border.

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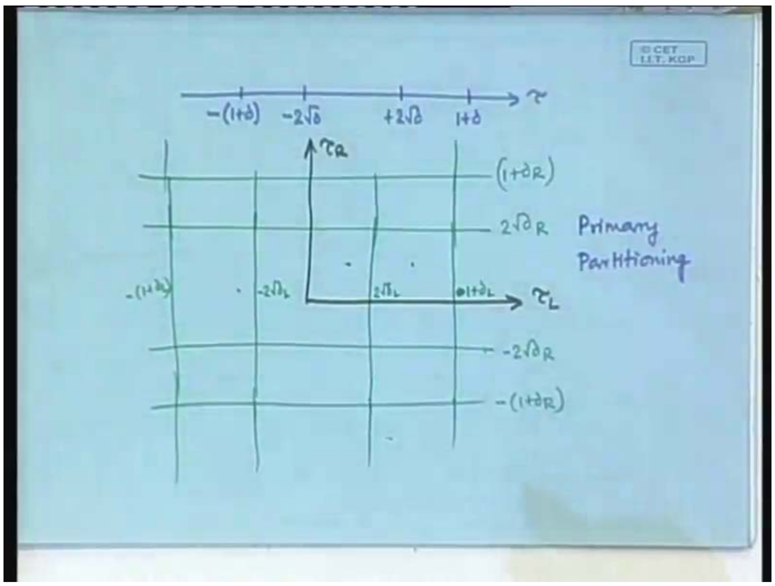

While it has not yet hit the border for mu less than 0, there is a fixed point and its type as we have already shown. After it has hit the border there is a fixed point and its type. So for mu less than 0 we can say that there is a type of the fixed point, for mu greater than 0 there is another type of fixed point. A type of fixed point changes to another type of fixed point. How will we represent that? There is a nice way representing that, we can say that I will make a parameter space of tau_L tau_R. Tau $_L$ tau $_R$ parameter space and then I know that the tau_L represents or tau_L as related to $delta_L$ represent the type of the fixed point while it is on the left hand side. We will say that I will now subdivide and this is my 2 root delta, this is my minus 2 root delta, this is 1 plus delta and this is L. Here it is minus (Refer Slide Time: 28:00). The tau_L as related to this is just like here.

Similarly tau_R can be related to the determinant of the right hand side which will be like this. We will say that I will divide like this also and this is 2 root delta minus 2 root delta. This is minus 1 plus delta, these are all R. We have essentially said which type of fixed point gets converted to which other type as it moves across the border. A border collision means a type of fixed point hits the border and changes to some other type and that we have essentially defined. For example if say I place my point here. What does it mean? It means that while it is in the left hand side, its eigenvalue satisfied the condition that the trace is between 2 root delta to 1 plus delta.

If it is in 2 root delta to 1 plus delta what is its type? It is a regular attractor and when it crossed over to the right hand side then its character is that its eigenvalues are such that the trace is below 1 plus delta which means it is a flip saddle. You can then say that here is a situation where a regular attractor hits the border and changes into a flip saddle. Just by placing a point in this parameter space, you are conveying that. This will represent our primary partitioning. The parameter space is now partitioned depending on which type of fixed points gets converted to which other type. Primary partitioning is pretty simple stuff. This is the primary partitioning.

Now you would realize that there is a similarity with what we did in one D. Similarity in the sense there also we had, in case of one D the determinant is 0. Imagine the determinant is slowly changed and it is approaching 0. What will happen to this parameter space? This will become 1, this will become minus 1. We had done exactly that and what will happen to these two fellows? It will shrink to 0. So as you have the determinant shrinking to 0, the range for which you have a spiral attractor that also shrinks to zero size. Ultimately you don’t have a spiraling behavior at all.

The next job obviously is to consider what happens in this parameter space, every point of this parameter space. First let us consider some trivial situations in order to get used to what will happen. Say if I have initial condition somewhere here, what do you expect? Here a spiral attractor because it was between minus 2 root delta or plus 2 root delta converse into a spiral attractor because it is still within that or spiral attractor remains a spiral attractor. In this part it is a regular attractor, this converted into a spiral attractor but either case it becomes the attractor.

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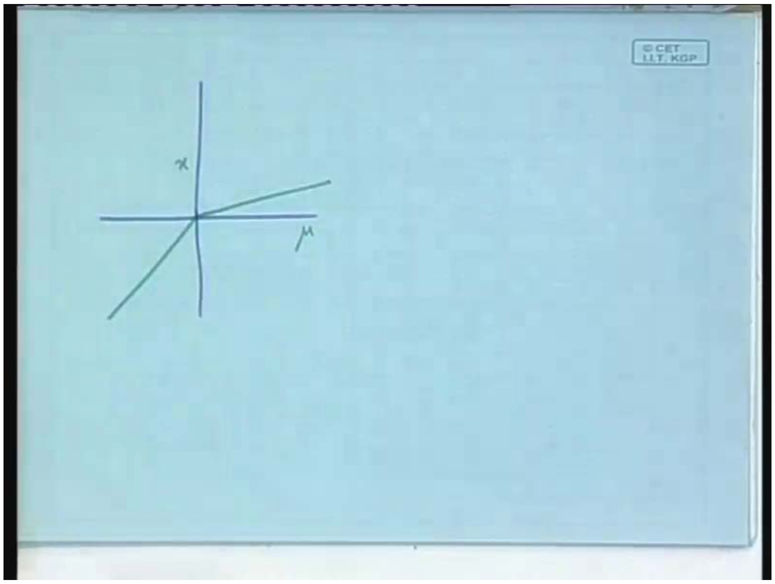

You can say that the behavior will be like this. This is the mu and this is the x, this is a bifurcation diagram. There you will only see a bend. In the root of the orbit you will see a bend but nothing more. You are not expected to see much more because both were stable. Yes, this is the simplest situation and you can say that in this situation some people would like to say that there is no bifurcation.

Some people would like to say let's call it a bifurcation, there was a bend and we need to define that. Either way, you can say that it is a situation where the stability status of the fixed point does not change. A similar situation will also occur say here, what has happened? Here it was a regular saddle that changes to regular saddle. I mean it doesn't change. Here a flip saddle remains a flip saddle. In these three compartments, the stability status of the fixed point did not change and while it was regular saddle, it was unstable. While it is flip saddle it is unstable and the unstable thing remains an unstable thing, so you don't expect any periodic orbit to occur. It will actually go to infinity. These three are somewhat simple situations. I will mark them like this and this whole block I shall mark or not because I will need to do something about it later.

Just remember this also falls into this box. Also falls under the situation where this stability status of the fixed point did not change. But there will be complexities which I will come to later, not at this stage. First let us understand the basic character of this. What happens here in this part? Can you work that out, what happens in this part? In this part means what is the character? Your trace of the left hand side is greater than 1 plus delta L and in the right hand side it is less than 1 plus delta R. Now you already have the equations, you look at these equations of the positions of the fixed points. What do they tell you? Tau_L greater than 1 plus delta L, tau L greater than 1 plus delta L means this term is negative. This term is negative means this left fixed point will exist only if this fellow is positive.

Now let's look at the right fixed point. Tau_R was let us look at it again. Tau_R was less than 1 plus delta R, means this fellow is positive. The denominator is positive. This fixed point will exists so long as positive which means that for mu negative, none of these fixed points exists and both the fixed points exists for mu positive. This means that this is the situation where there is a birth of a pair of fixed point at mu is equal to 0. At the border collision a pair of fixed points are born. If you look at it from the other side that means mu is being reduced by positive value then you would see a pair of fixed point approaching each other and then colliding at the origin, on this state space and disappearing. Now it is not true that you have never come across this situation. You have, in fact in a smooth system, smooth map you have come across these situations where a pair of fixed points are born, here also you have a situation like that.

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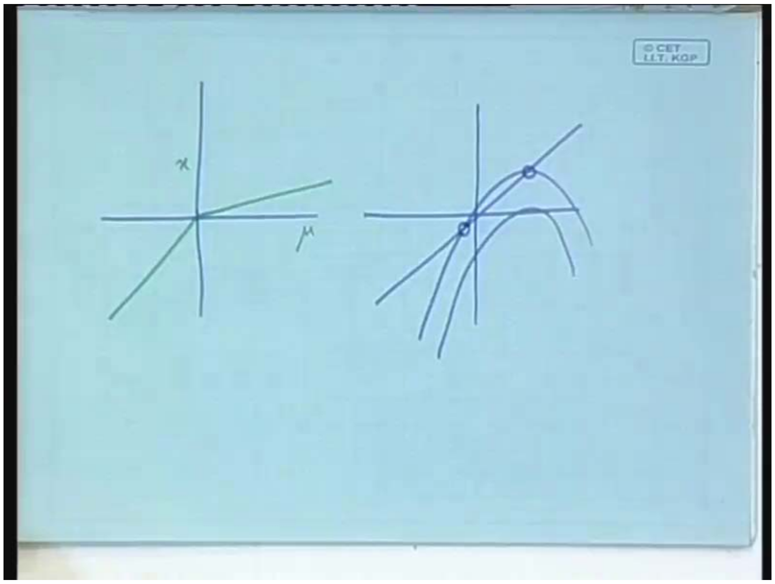

It is actually the non-smooth analog of a saddle node bifurcation. It is a non-smooth analog of a saddle node bifurcation that is fold bifurcation. This is also called a non-smooth fold or non-smooth saddle node bifurcation but the only difference is as you have already noted while we are talking about the one dimensional system is that in a smooth map, one is unstable the other must be stable. One fixed point is unstable, the other fixed point must be stable because it is a smooth line. You cannot avoid it. While in this case there is a possibility that both could be unstable. Could both be stable? No, that's not possible because you have taken tau_L to be greater than 1 plus $delta_L$ that's where you have doomed it. One is definitely unstable but the other yes, that could also be unstable.

Obviously if let's look at the fixed point of the right hand side, if the eigenvalues of the right hand side is such that the tau_R is between the 1 plus $delta_R$ and minus 1 plus $delta_R$ then the fixed point there is stable but here, where it is below minus 1 plus delta is obviously unstable. If you place the point here in the parameter space, it means that you have chosen a parameter condition for which there is a birth of a pair of fixed points both of which are unstable, one a regular saddle, another a flip saddle. A pair of unstable fixed points are born, one a regular saddle, another is flip saddle. You have also seen in case of one D map under that condition when both the fixed points are unstable there is a possibility of existence of the chaotic orbit. You have seen that. Here also that can happen but we will take a closer and a more detailed look at that a little later. As I have said that here are some details which I will come to later. There are some details also we will come later but let's realize what happens here.

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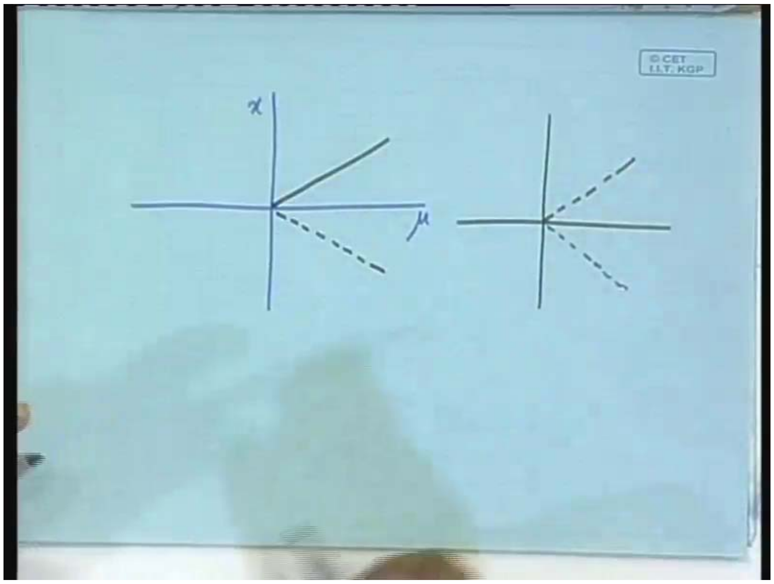

Here what happens is on the bifurcation diagram, here is the mu axis and here is the x. What will you see? You will see the birth of a pair of fixed point one of them is stable, the other is unstable. That is what happens here in this part and in this part you will expect something like this to happen. This is not a period two orbit, this is a pair of unstable orbits out of which one is a regular saddle, the other is a fixed saddle.

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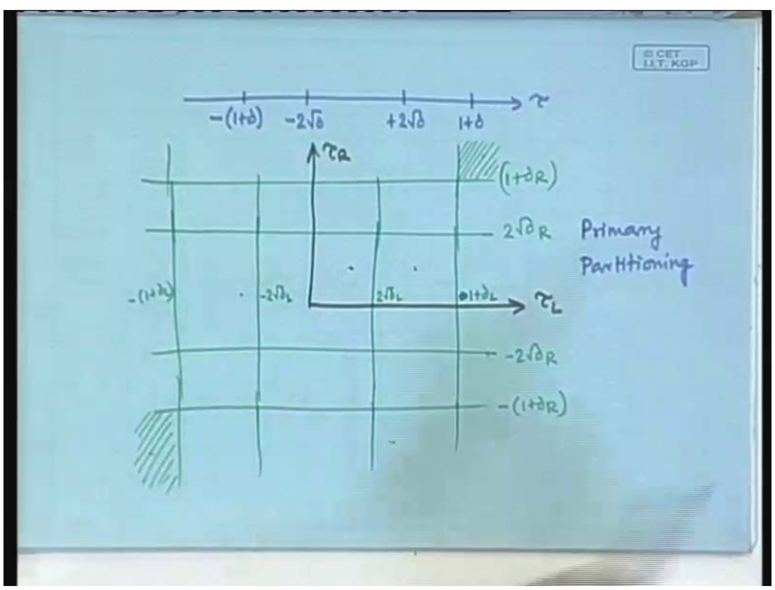

Now if there is a question of a regular saddle and a flip saddle, obviously we have to find out what other orbits can be existing. Is it possible that a period two orbit will exist? I wouldn't say that is impossible, we will consider that condition later.

Let us for now come to this side here. Let's understand the primary partitioning first in clear terms. What happens here? Your trace of the left hand side is between minus 1 plus delta to plus 1 plus delta. Trace of the left hand side between these two means that before colliding with the fixed point, it was stable. After colliding with the fixed point what happen to it? It became R and it is less than minus 1 plus $delta_R$ means it becomes a flip saddle. You can subdivide this. If you place the point here, you can say that a spiral attractor gets converted into a fixed saddle. Here a regular attractor gets converted into a flip saddle. A flip attractor is converted into a flip saddle but either cases, an attractor gets converted into a saddle.

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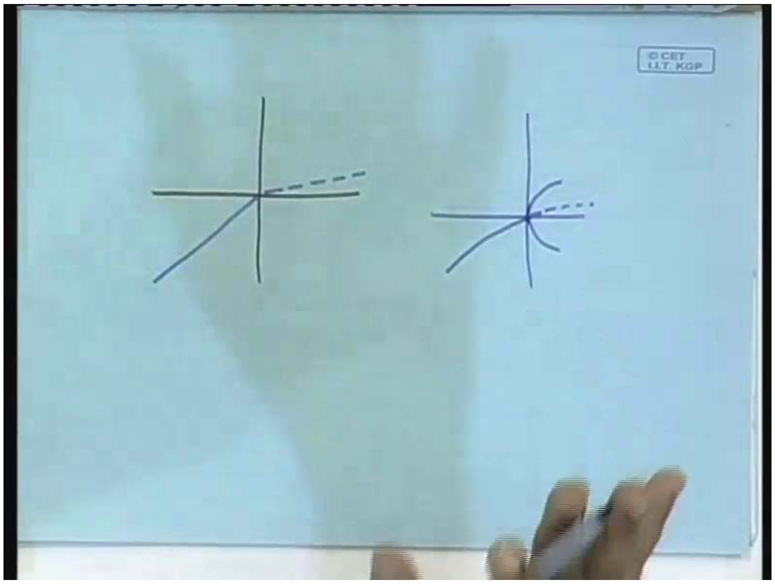

In that case what will happen? I can expect the behavior is something like this in the bifurcation diagram. A fixed point comes, hits the border and becomes saddle. Obviously the question is what happens then? Will there be any stable orbit? Whenever this kind of thing happens, do you notice that this is similar to the situation in the smooth map where you had the fixed point that loss stability and as it losses stability, you had a period two orbit being born. The obvious line of thought would be that here I find that it is losing stability, here as it loss stability it become a flip saddle. In case of a smooth system it becomes a flip flip saddle. Here also it is becoming a flip saddle. The natural line of thought would be let us check if the period two orbit will exist. How will you check if the period two orbit will exist? Simple.

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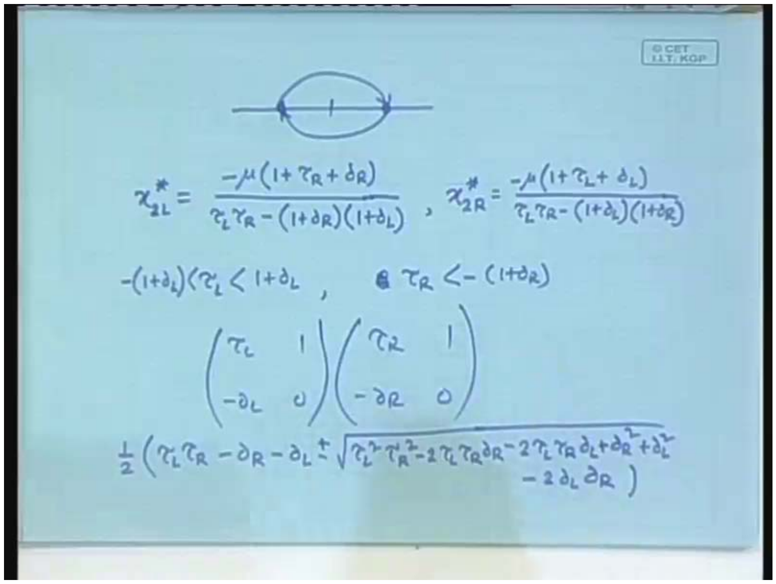

You have got the real line. This is the 0, period two orbit means something like this. So start from a point in the left, apply the left hand map, come to the right, apply the right hand map, come to the left and then equate x_n to x_{n+2}, y_n to y_{n+2}. Then you get an algebraic equation solving which you should get the position of the fixed point. That is reasonably a simple affair. I will not trouble you right now with the task of actually deriving but you can see that it can be very easily derived. Again what will you do? You will start from here, assume the x_n y_n are here in the left hand side, apply this, obtain x_n plus y_{n+1} and then it has come here. You will have to apply the right hand map which will be these two subscripts replaced by R. Apply it again, come back here, you have obtained x_{n+2} y_{n+2} and then equate that, you will get a algebraic equation.

If you solve you will get $x_{2\,L}$ star, the one that is in the left hand side. X_2 that means it is a fixed point of the second iterate map. It comes to minus mu_1 plus tau_R plus $delta_R$ by tau_L tau_R minus 1 plus $delta_R$ and the other one $x_{2\,R}$ star is equal to minus mu. This should be by symmetric because if you start from here, come here and come back here, what you get? If you start from here, come here and come back here if you get the same thing only with R's replaced by L's and L's replaced by R's. I can blindly write 1 plus tau_L plus $delta_L$ tau_L tau_R minus 1 plus $delta_L$. The denominators are the same, only the numerators are different.

Now I said that the question that I am addressing now is when this fellow has become unstable, will the period two orbit exist? What is your answer by looking at this, notice. In order for the period two orbit to exist, this term should be negative and this term should be positive. Let's first consider mu, I am now considering mu to be positive. If mu is positive then the existence will depend on this and I am considering the situation here which means my tau_L is between minus 1 plus $delta_L$ to… and tau_R is less than minus 1 + $delta_R$, on this basis take your decision. Will this fixed point exist. This period two fixed point or not? What is your answer? It will exist or it will not? It will not exist. Let us clarify your argument. Let's look at the situation here tau_R which appears here.

Tau_R less than minus 1 plus $delta_R$ so take it to this side, you have negative. This term negative and this term is negative and then this fellow will be, I want this to be negative and this term is negative, here is the minus so it will be existing for… also there is this term (Refer Slide Time: 49:50). Here is this term, this term and this term, we have made a hard conclusion that because of this, this term is negative. This term times this minus is all positive so forget about this. Then ultimately your decision should depend on the relative magnitude of mu and this. Now this we don't know yet but we have concluded that the existence will depend on mu and this in such a way that this division mu by this should be negative.

Now let us look at the stability condition of this period two orbit. What is stability condition? When will this period two orbit be stable? You have to compute its eigenvalue and how you would you obtain its eigenvalue? By multiplying these two matrices, you multiply these two and obtain its eigenvalues. Now if you do that then the eigenvalues come to be, let me again not trouble you with the integrity details. The eigenvalues of the second iterate are half tau_L tau_R minus $delta_R$ minus $delta_L$ plus minus root over tau_L square tau_R square minus twice tau_L tau_R $delta_R$ minus twice tau_L tau_R $delta_L$ plus $delta_R$ square plus $delta_L$ square minus twice $delta_L$ $delta_R$. You might say this is too big or complicated.

It is not this easily manipulateable and so I wouldn't say that this is an unhandleable situation. I will give you the task, find out when this thing will have… this is an eigenvalue, the stability condition is that it has to either be plus 1 or minus 1. So equate it to plus 1 and simplify this, you will find that can be done in just two steps to obtain in a very simple expression. Similarly for this equal to minus 1. Here we are talking about real matrices with real numbers, so the period two orbit will be unstable in this two ways. Can the period two orbit become unstable as a complex conjugate? No, so long as you are considering the determinant to be less than 1, meaning you are considering a dissipative system, the determinant in the two sides multiplied together will also give a determinant that is less than 1 and therefore that cannot go out. So going out like that is out of question here.

These two possibilities we have to consider that it can either become equal to plus 1 or it can become equal to minus 1. Before you come to the next class, you just work out yourself that from this expression what will be the condition for which, notice this and clarify your logic that here is three terms really. One term mu, another term here, another term here (Refer Slide Time: 54:36). Here also mu, this, and the stability will depend on something but the existence will depend on their relative magnitudes. Ultimately their magnitude should be such that this fellow is negative and this fellow is positive then only it will exist. So that decision you take, I will leave you to take that decision on the basis of your own logic where here is a clue this is the eigenvalue and you can work it out yourself. Can you see that? This will get simplified out, take that out. I will stop now and we will continue with the next class where we will go on to further partitioning of this parameter space to see what are the possibilities existing in this case.