Chaos, Fractals and Dynamical Systems
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Lecture No # 33
Normal from for Piecewise; Smooth 2D Maps

In the last class we proceeded up to the point where we divided the ab parameter space as a and b. This parameter space we have divided into boxes. We said that this box which is between minus 1 and plus 1 this way and minus 1 and plus 1 this way. This box essentially represents a situation where in both the sides the map is stable. That is in the left hand side and the right hand side the map is stable. You have got an evolution from period one orbit to period one orbit. In this part we decided that in this part, there will be a situation something similar to the fold bifurcation or saddle node bifurcation but due to border collision where a pair of fixed points are born and out of that one would be stable and another would be unstable. In this part we had decided that the situations will be similar but what will be born are two unstable periodic orbits.

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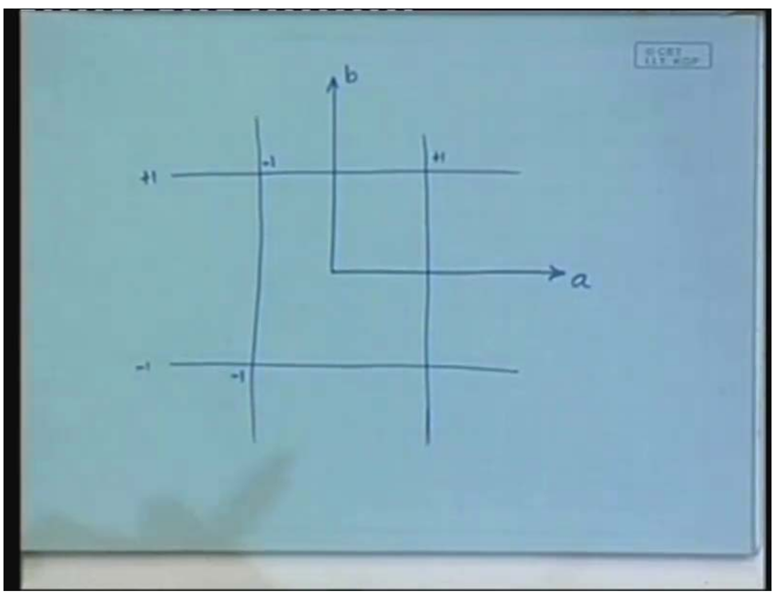

Here it is a situation similar to the saddle node bifurcation but there is no node, both are saddles and then we went to this part where we said that the situation is where a fixed point that was stable moves across the border and becomes unstable. So a being between minus 1 and plus 1 means that the slope in the left hand side is less than 1 and that in the right hand side is less than minus 1. You have got a slope that is less than minus 1, a situation something like this.

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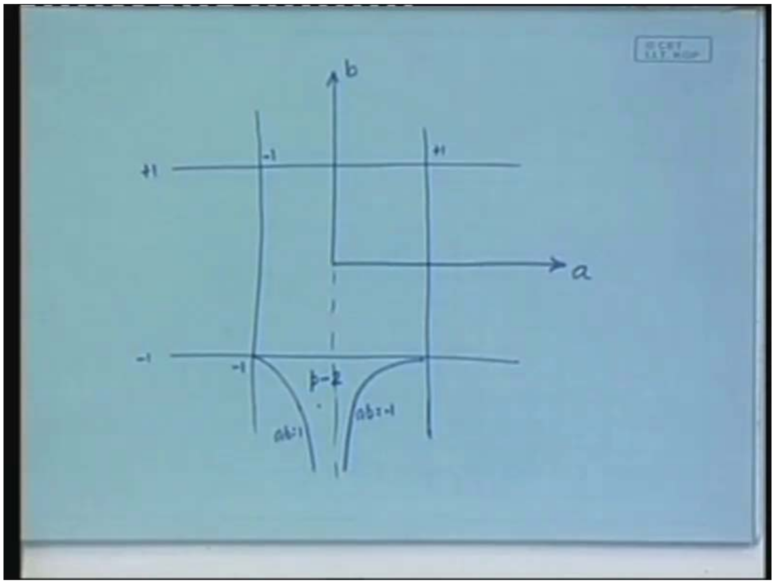

In this side the slope is less than 1 and this side it is less than minus 1. As it moves you will have a situation like this. This was a fixed point that was earlier stable. Now it has moved across the border and has become unstable and so there can be a large number of possibilities. We had exploded the possibility of the existence of period two orbit, up to this point we had come. We have decided that this part will be divided into cases, this was the range of occurrence. I should not say existence because the period two orbit exist over the whole range but this is the range of occurrence of period two orbit.

Now there are two boundaries we had decided. This boundary represents the situation ab is equal to minus 1 and this boundary represents situation ab is equal to plus 1. How did we arrive at that? We simply looked at the condition of existence of the period two orbit and then we said that this orbit will be stable if the product of the two slopes ab is less than 1 in magnitude and that valuation happens in these two places. But we also saw that at this point there are two things that are happening not only that the ab becoming equal to plus 1 means that this fellow is now becoming unstable. Not only that we decided that in this part, the period two orbit will exist for mu greater than 0 and in this part the period two orbit will exists for mu less than 0.

If say we take a point here that means we take some value of ab which is within this zone means a is taken between these two parts and b is taken less than minus 1 suppose. Then what is the behavior going to be? As mu is varied from the negative value to positive value, we should understand that. See in the left hand side when mu is less than 0 then the slope was between minus 1 and plus 1, value of a represents that it was between minus 1 and plus 1. So for negative value of mu it would be stable. For positive value of mu it becomes unstable and that is when the period two orbit exists. The behavior is simply a period doubling, a period doubling bifurcation. If you now look at the computer screen, we can show that bifurcation for a specific set of parameter values.

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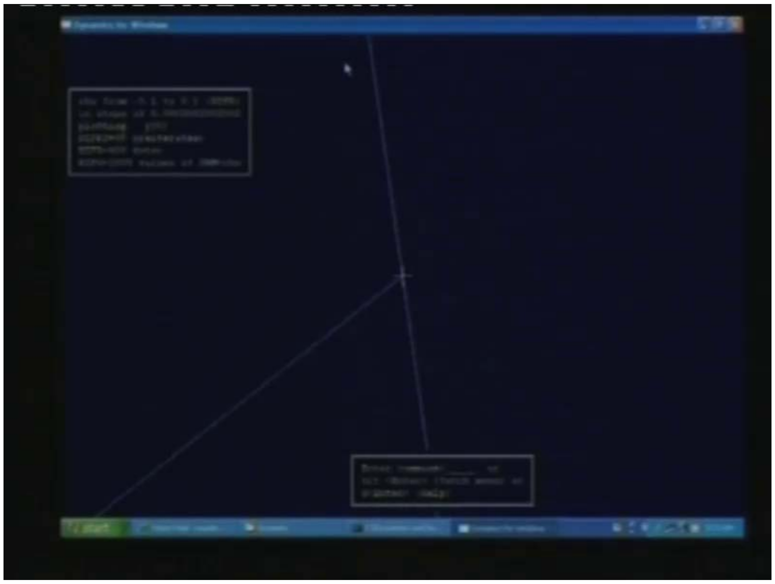

You see this is a bifurcation diagram drawn from top to bottom, where the period one orbit comes at this point, hits the border and then a period two orbit emerges. Now one important distinction between the standard period doubling bifurcation that you have seen so far is where you have a period doubling like this.

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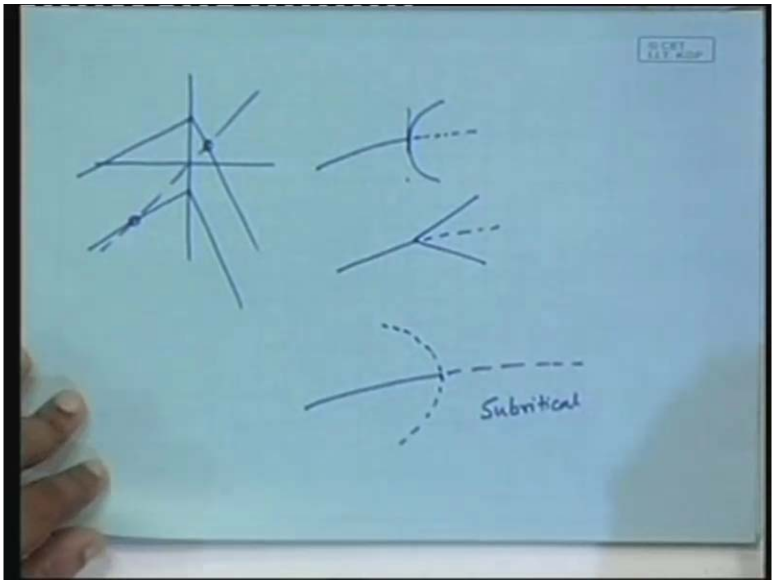

There these two orbits emerge orthogonally from the original orbit like this and in this case it goes like this. So they got acute angle but remember in both the cases this unstable period one orbit continues, here also the unstable period one orbit continues. We have a typical period doubling induced by border collision here.

What happens if you take a combination of parameter that is here. If you take a combination of parameter that is here then a peculiar situation is there so that let me draw in a fresh page.

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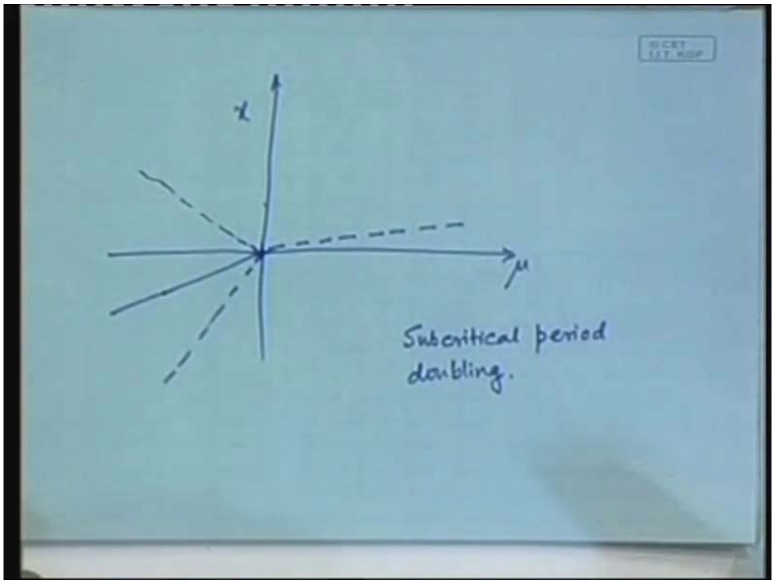

We are talking about the bifurcation diagram so say this is my mu axis and this my x. What will you see? Suppose this is mu is equal to 0. So before mu is equal to 0 situation reaches, there will be a stable periodic orbit. That becomes x is equal to 0 at mu is equal to 0, for this map so it goes and hits. After that the fixed point becomes unstable and it emerges and the period two orbit exists but it exists in the same side as mu negative. This is a period two orbit which means the period two orbit exists in the same side where the period one orbit was stable and in this side there is nothing stable. This is a typical subcritical period doubling, subcritical means where the period two orbit exists in a same side where the period one orbit was. If you figure out how will it look in the typical smooth situations it is like this and at this point it becomes unstable and the period two orbit, it actually happens in many smooth maps. This is the subcritical.

Here is the situation that is similar to that but in a non-smooth situation. We have understood what happens here. The period two orbit actually become unstable. If you go across this period two orbit becomes unstable then what? When the period two orbit still exists but becomes unstable and ab becoming equal to minus 1 means, period two orbit under goes a period doubling bifurcation. There is a period doubling bifurcation. This fellow becomes unstable, period two orbit no longer exists. Then we have to face the next question. Will a period three orbit exists, will a period four orbit exists, will all these periodicities exists, will chaotic orbit exists will have to face that question. How do you face the question, will a period three orbit exists?

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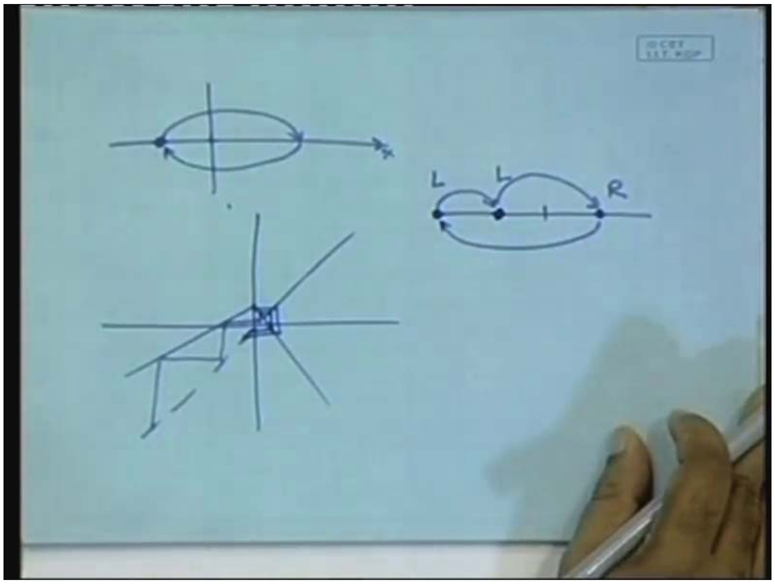

The way we tackle the situation of the period two orbit was that we had say this is the zero point, this is the x coordinate and there was one point in the left hand side that map to the right hand side and this point again mapped back. We had worked on this condition and then said that this orbit will exist so long as this point is in the left hand side and this point in the right hand side. We are talking about this kind of a situation, let me draw again. So a situation where you have got to apply this and here it is because see if you start from here, there can be a large number of iterates going like this and finally somewhere it will come back.

Now you notice that here there is nothing that prevents a more than one iterate in the left hand side. There can be two iterates in the left hand side but once it falls here it comes back. It stands to listen that we should actually consider, if you consider period three orbit then there are two possibilities. It could be two points to the left and one point to the right or one point to the left and two points to the right. It will be more logical to consider two point to the left condition. What will you consider? We will say that here is a zero point, one point is here, another point here and the third point here. It goes like this, it goes like that and it comes back like this, LLR (Refer Slide Time: 12:25). How we will work out the existence of this orbit? Suppose we start from this point. I would prefer to start from this point because that actually determines the condition. This point maps to the R then comes here then comes here.

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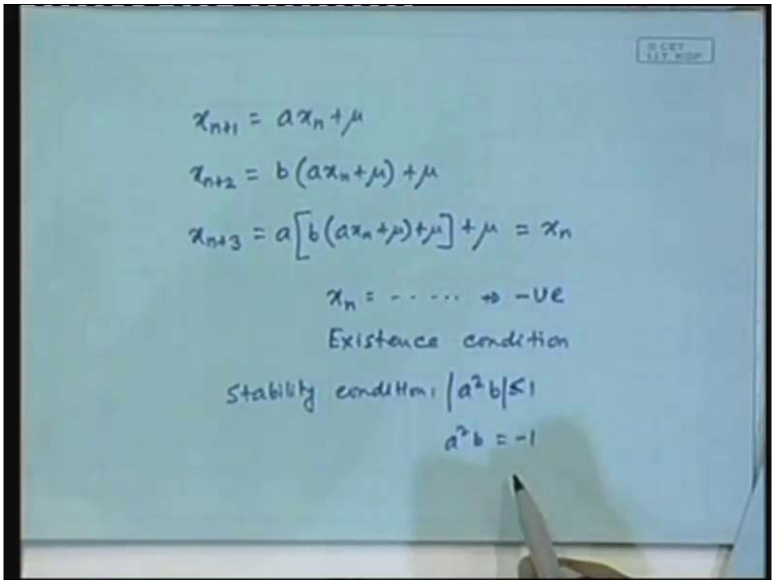

We will say starting point x_n so x_{n+1} is equal to ax_n plus mu because it is in the left hand side then x_{n+2} is equal to, now this fellow is in the right hand side so it will be mapped by the right hand side equation, b ax_n plus mu plus mu. Now it has come here, now again with the help of the left hand map, x_{n+3} is equal to ab ax_n plus mu plus mu. Now this fellow must be equal to x_n in order for this orbit to occur. Just solve this equation and you get a value of x_n is equal to something. Some expression you will get. Now that expression is the expression for this point, remember it is not this point or this point. Why? Because in order to find out for this point, you will have to map twice in the left and then on the right. While if you want to find this point then you have to map first with the right map and then twice in the left map that we have not done. We have done for this map.

Now notice that when this point is stable, this particular orbit is stable this point must be to the left because that is the condition we assume. The condition for existence of the periodic three orbit should be this number that you get here should be negative. Just consider the conditions of violation of this. This can be violated the moment it hits the border. Can this hit the border, this point? No, not before this fellow hits the border. There is no point considering this point. Can this point go and hit the border? If it does then all the points will be in the left hand side. If all the points are in the left hand side, they cannot be a periodic orbit. Why? Because in each side it is linear. If it is linear, it could be only a period one orbit but you cannot have a high periodic orbit in a linear map. The only condition is this fellow should be negative. That is the existence condition. Can you find it? It is not difficult, doable by hand.

Now what is the condition for stability? This particular orbit will be stable so long as there are two points to the left and there is one point in the right. The stability condition is a square b is equal to this magnitude that should be less than 1. Then it’s stable. Obviously this can also be violated in two possible ways. If you consider the period doubling condition then a square b is equal to minus 1 is one violation condition.

We obtain two possible conditions in which this orbit either ceases to exists or it ceases to be stable. Now if we super impose that on this you will get a range that is something like this. Here is the condition for existence and here is the condition for stability that we are just talking about.

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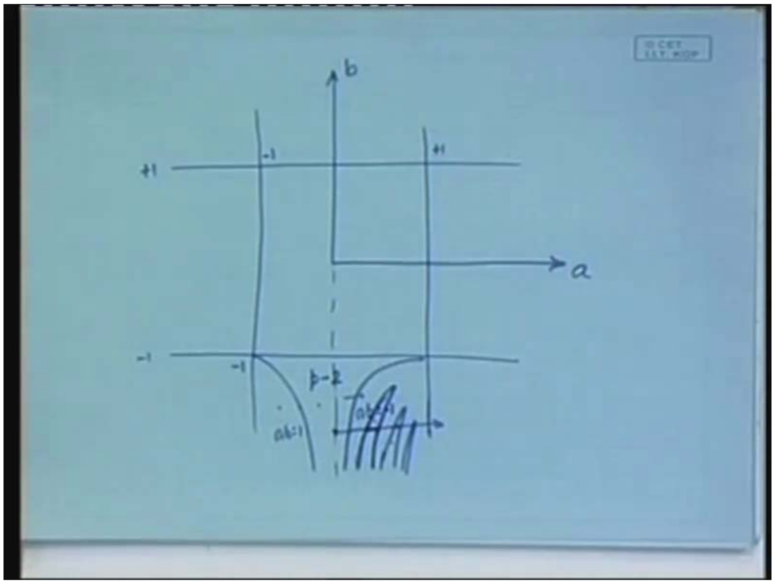

Similarly you can find the condition for existence stability for the period four orbit, it will be like this. There will be all sorts of orbits like this and then let’s check out, if this is really true. Suppose I take a point here which means I take this particular value of a and b, particular combination of a and b. If you do that then as mu varies from negative value to the positive value what will be the behavior of the bifurcation? What we will see? For mu less than 0 it is still stable because a is between minus 1 plus 1 and as mu goes beyond 0 then what happens? It’s a period three orbit, so it’s a direct transition from period one to period three. Let us see.

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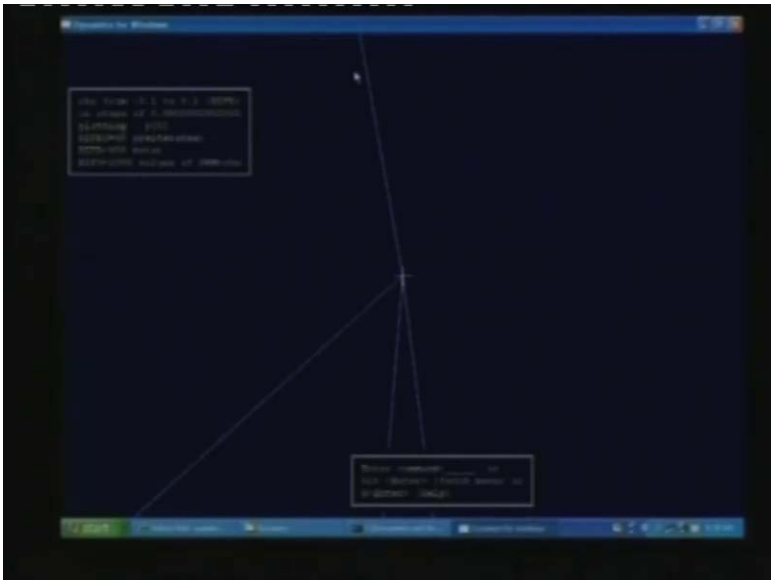

I have taken a parameter in that range. It is a period one to period three, direct transition from period one to period three where you see the bifurcation diagram looking like this.

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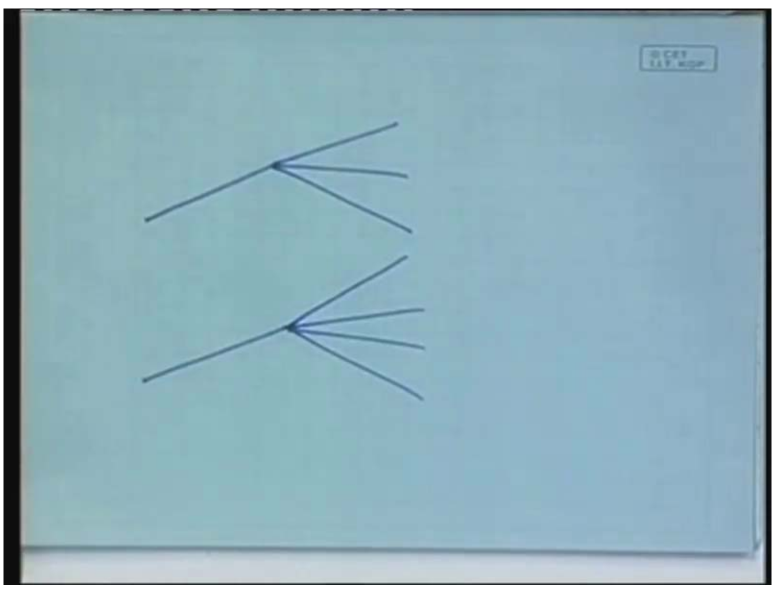

It is very peculiar, cannot ever happen in a smooth map but can happen in a piecewise smooth map. Similarly if you take a point in the period four tong, these are called tongs. If you take a point in the period four tong then you will have period one directly to period four orbit. Can you see? Now if now I take the parameter mu to be positive and I vary the bifurcation parameter a like this, what will you see?

Here it will be period two orbit then here it will be a period three orbit, here will be a four orbit and so on and so forth and in between one, a period two orbit is not there and period three orbit has not yet been born. Any other periodic orbit will not be there but still the orbit will be content bounded and therefore the only thing that can happen is a chaotic orbit.

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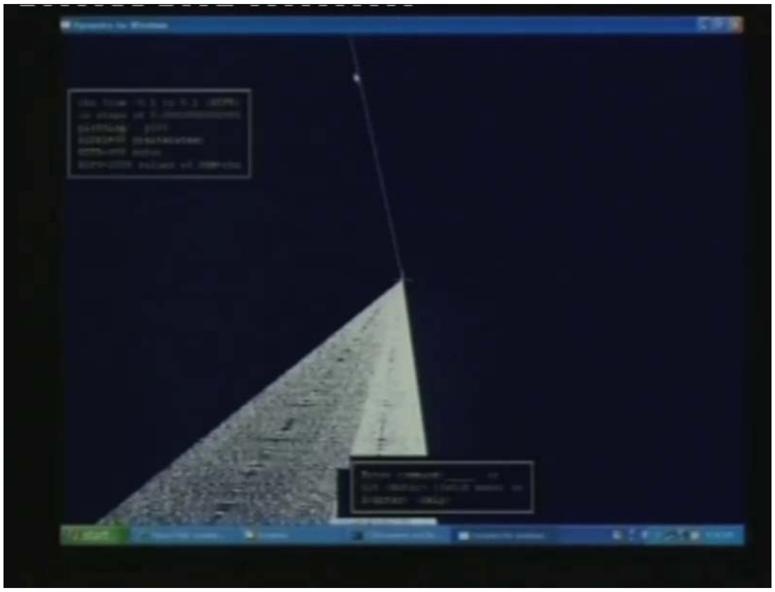

Let me show an example in between these two. Between the period two and period three, period two tong and the period three tong I am taking a parameter value and then we will see. This is a situation where there is a direct transition from a periodic orbit to a chaotic orbit. That is another peculiar thing that can happen in non-smooth maps, it cannot happen in smooth maps. As I told you if you instead take a as the bifurcation parameter with the mu value kept at a positive thing then what will happen? We will not have the period one two whatever, that kind of bifurcation. Why? Because it is already mu, it is already positive. Let’s do that and you will be surprised. Let us move it from 0 to 1.

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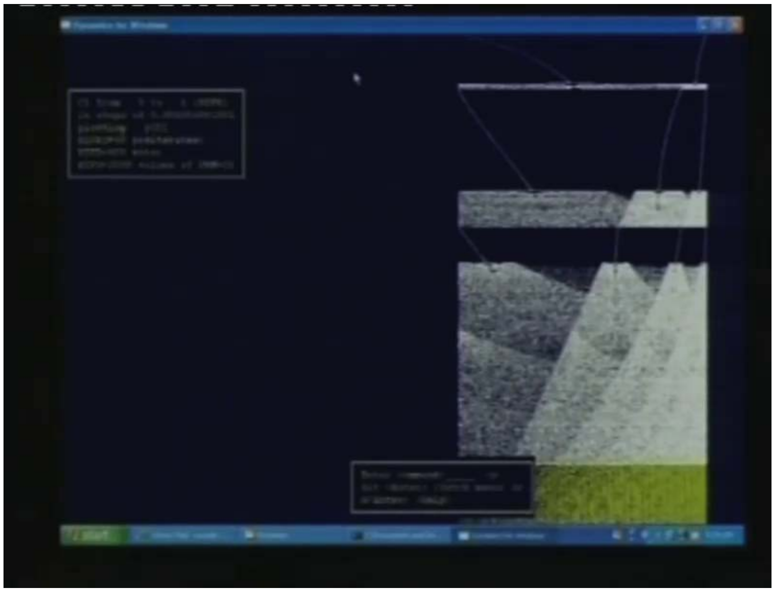

Initially in this part it was period two then a narrow bend of chaos then you had period three then another bend of chaos then you had period four and again you have a chaotic orbit. What does it mean? It means that it is a period adding cascade. Under such situations you are likely to see period adding cascades. In fact in many experimental situations people have observed period adding cascade and this forms one of the explanation of what mechanism can give rise to the period adding cascade. We have understood the whole character of the parameter space and as I told you, the whole thing is symmetrical about this 45 degree line. Whatever happens in this part, we have to explode that the something happens in this part. If mu is varied from a positive value to a negative value so there is nothing new in this part I don't really need to cover that.

Now whatever I have said that was a simplest possible case, simplest possible in the sense that it is a one dimensional map, it is a piecewise linear map and therefore everything can be worked out by hand. Every situations of the combination of a and b, you can draw the graph of the map and work out by probable diagrams how this behavior is going to be. This is rather simple but these forms the ground on which more studies on the behavior of non-smooth maps actually are done, because this can be easily worked out. Now let us go to a relatively more complicated situation where we are considering two dimensional maps, so one D is simple. Let us go to the next level of complex.

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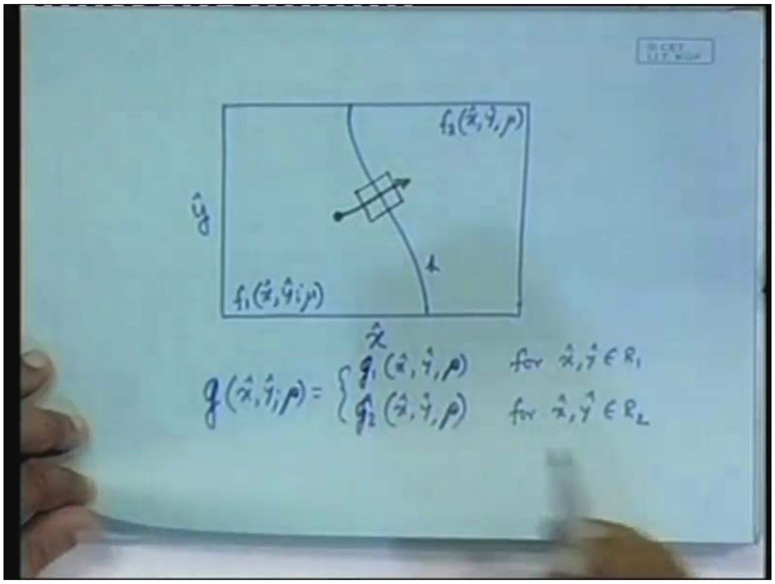

Here what kind of situation are you considering? In a one D map we are considering situation where there was two sides each with a different functional form. Here also you will consider that. Suppose this is my state space but in this case we will have to consider x and y because it is two D. In this two dimensional state space, there will be some kind of a border that divides the state space and in this part, there will be one functional form say f_1 and this is f_2. Initially let us consider the situation with hats that means a state space is x hat y hat state space. Why, I will come to that later. This was necessary so that we can later deal with relatively small number of notations.

It is actually a map were f of x hat y hat mu is given as f_1 x hat y hat mu, f_2 x hat y hat mu. This is for x hat y hat in say region one and this is for x hat y hat in region two. That is the situation that we start from where as I told you that in practical situations of piecewise smooth dynamical systems, smoothing systems you are likely to come across this kind of maps; discrete and dynamical systems. What we will do? We are considering a situation where a fixed point say it was here for some value of the parameter mu and then as you change the parameter, it hits the border.

Now in order to do that obtain the normal form, it will be logical. The logical course of action should be that we will take only the local linear neighborhood of the border crossing fixed point. We will only consider the local linear neighborhood of the border crossing fixed point. This we will do through a series of coordinate transformations. Now what is the first transformation that comes to your mind? First thing that we will do is your x coordinate, y coordinate, the axis are anywhere. First thing I would like to do is to get the y axis here along the border that is the first thing that we will like to do. Get the y axis here so that the fixed point crosses the y axis. What kind of parameter or coordinate transformation will be necessary?

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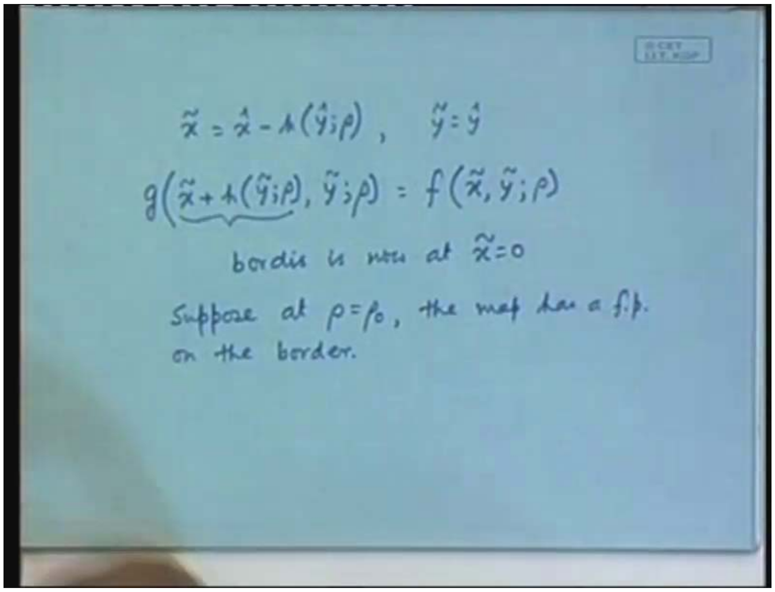

We will do it by defining a new coordinate x tilde which is x hat minus h of y tilde say let the parameter be rho because I will come back to the new parameter later, this parameter is a rho. If you define it this way then your x tilde because I have already subtracted the position of the border and therefore my y axis always is my border line, border line has to move to the y axis. This and y tilde is equal to y hat, let that be the same.

Let us start with g so that we will later come to f, this is g_1 g_2. We have then the map as g of, it was x hat y hat x; hat is now written in terms of x tilde. It is x tilde plus h of y tilde rho comma y tilde rho, this is the map then. I have just substituted for x. Now let us call it f that is why I sort of kept f and started with g because I wanted to later do with f. This is function of x tilde y tilde and rho. That is a first coordinate transformation that you have done. The border is now the… (Refer Slide Time: 30:45). We need to do something because now the border collision can occur for any value of rho, these are rho. For any value of rho now we would like to normalize that so there will have to be a coordinate transformation so that things happen for the parameter value of zero. So that is what we are trying to do now.

Suppose at rho is equal to rho_0, the map has a fixed point on the border. What happens? As you change rho at some point, it hits the border and suppose that parameter value is rho_0 and we will now make some change so that this thing happens at 0. Now we need to do that coordinate transformation. But how the coordinate transformation is done, let me illustrate that with the next figure.

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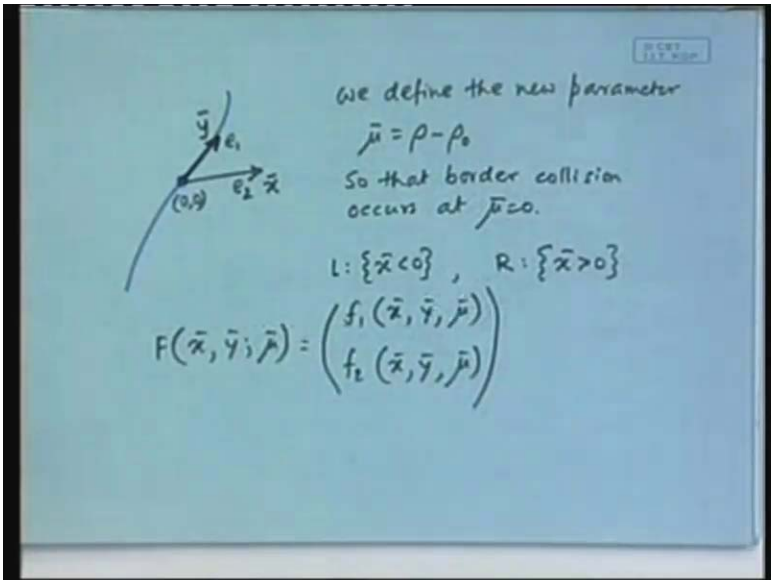

Where did we start? We started with a border something like this and at this point, it was here. At rho is equal to rho_0 and the position was like this. We define that the tangent along this direction let that be called my say new y coordinate. Now if you have this and you have the map already then you can find out where this vector maps. Suppose this vector maps to some place like this. Let that be called the x. What have we done? We said that at rho_0 my position is here then I will take a vector along the border and we will say let that be my y vector. The y vector maps somewhere and that vector we will call it the x vector, that will require a coordinate transformation.

We are actually making that coordinate transformation so that if there is a unit vector e_1 in this direction and that maps to the unit vector e_2 then this direction we will call y and this direction we will call x. It is actually considering the unit vector. We consider unit vector and where it maps. Is the new coordination understood? New x is this, new y is this and how have we defined it? We know that my fixed point is now on the border at rho is equal to rho_0. We make a tangent and take a unit vector e_1 and find out where the unit vector maps in this next iterate, that's e_2. We say that now my y direction is along e_1 and x direction is along e_2. That is a coordinate transformation and this is my new fixed point, new origin.

If this is my new origin, this is x axis, this is y axis then we can now write down the actual transformation that we have done. We define parameter mu bar is equal to rho minus rho_0. What does it mean? It means that border collision happens at mu bar is equal to 0. Why mu bar, I will come to that later, we will later correct it. There are few things that we have done. We have changed the coordinate in terms of x and y and we have also changed the coordinate in terms of the mu, the parameter. Now what is state space? Our state space is now divided into two halves, one half this here another half here left and right. It is divided into two parts. For x bar less than 0, it is left half. For x bar greater than 0, it is right half. We can now define it, earlier we couldn't.

So L is x bar less than 0, R is x bar greater than 0. They are different I am now defining this bar quantities. We started with tilde, we started with hat, came to tilde and then we are doing another thing, a new thing which is the bar. After having reached the stage of tilde, we only moved the origin to the border but then also we have to move it so that we get the right coordinates. That's what we are doing and for that we have defined the coordination in a specific way. The moment it is defined that way, something will happen you will see that.

Now we have the map as F_x bar y bar and mu bar, this can be written as, this is f_1 of x bar y bar mu bar and f_2 of x bar y bar in the two halves. This is in this half and this is in this half. These were nonlinear functions, initially when we started these functions were nonlinear functions. We only changed the coordinates but in order to obtain a local linear representation we have to locally linearize that. So we take the next step of locally linearize these two.

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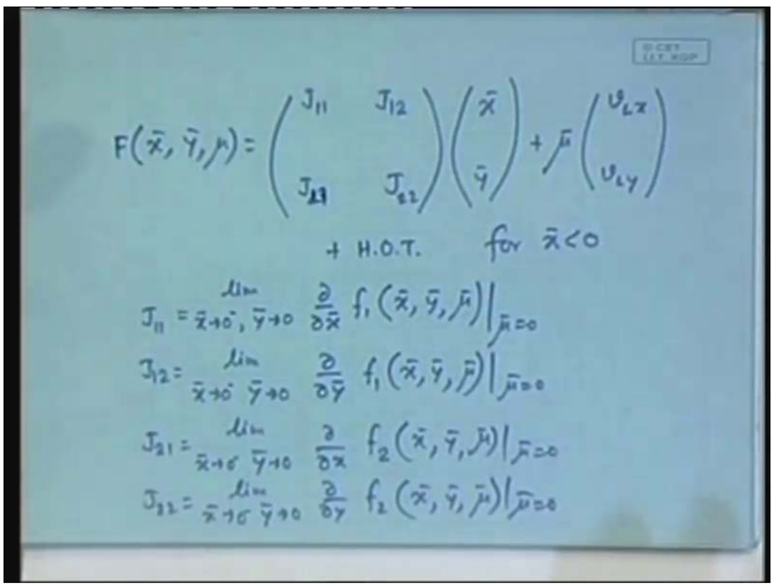

As a result we will say this F_x bar y bar mu is, it will be consisting of J_{11} the Jacobians, J_{12}, J_{21} and J_{22} times x bar y bar plus mu bar times… Here also the partial derivative with respect to mu have to be taken which we will take v_{Lx} and v_{Ly} plus the higher order terms for x bar less than 0 and similarly there will be similar terms for x bar greater than 0. Let us first consider only the left half and then we will do the same thing in the right half, there is nothing special about it. Let's now write down what these guys are? For example J_{11} is you have taken the partial derivative with respect to x bar whose partial derivative f_1 which is a function of x bar y bar mu bar evaluated at mu bar equal to 0. Now this fellow we are considering the limit x approaches 0 minus and y approaches 0.

Similarly all these terms will have to be written, J_{12} is limit x bar approaches 0 minus y bar approaches 0 partial derivative with respect to y bar whose f_1 x bar y bar mu bar. Similarly J_{21} is equal to limit x bar… (Refer Slide Time: 41:20). Additionally we have to talk about these two fellows, these are the partial derivatives with respect to mu.

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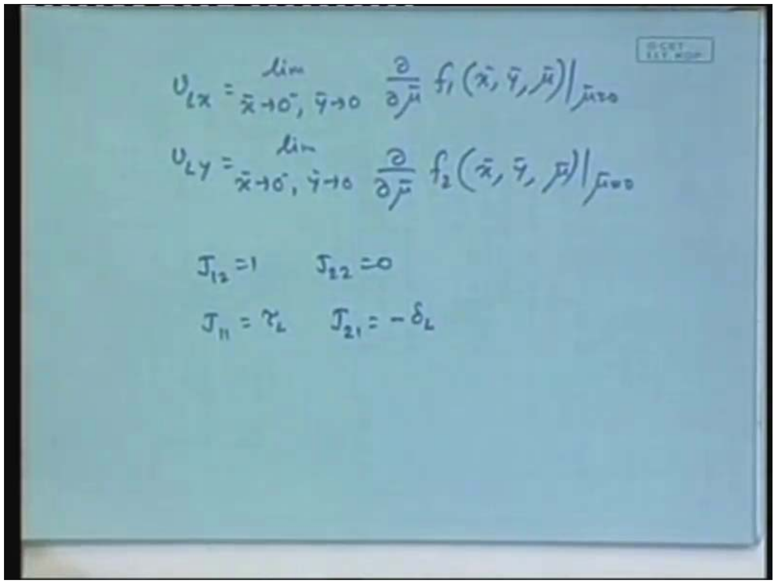

So v_{Lx} is again the limit, same limits x bar going to 0 minus and y bar going to 0. This is now derivative with respect to mu bar of the first one similarly v_{Ly} is limit x bar. We have defined all these fellows and similarly if in the right hand side for x bar greater than 0 we will have similar quantities. Let us first consider on the left hand side, we will come back. Now notice that the moment we have considered this particular thing, we have considered that unit vector in the y direction maps to the unit vector in the x direction. The moment you say that what happens to these quantities? The moment you say that notice unit vector along y direction means x equal to 0, y is something that has to map to x direction. No, J_{12} is 1 and J_{22} has to be 0.

The moment we have imposed that condition, we have actually imposed a condition J_{12} is equal to 1 and J_{22} is equal to 0. This fellow is 1 and this fellow is 0. What is this now? You would notice that this fellow is nothing but the trace because these two additions is the trace. This is 0 therefore this must be the trace. So J_{11} must be the trace of the left hand side. What remains is this fellow J_{21}. If this is 1, your determinant is these two minus these two and these two fellows product is 0 because this is 0, this is 1 and therefore this is the negative of the determinant. So J_{21} is equal to minus… we have now arrived at this equation for the left hand side only.

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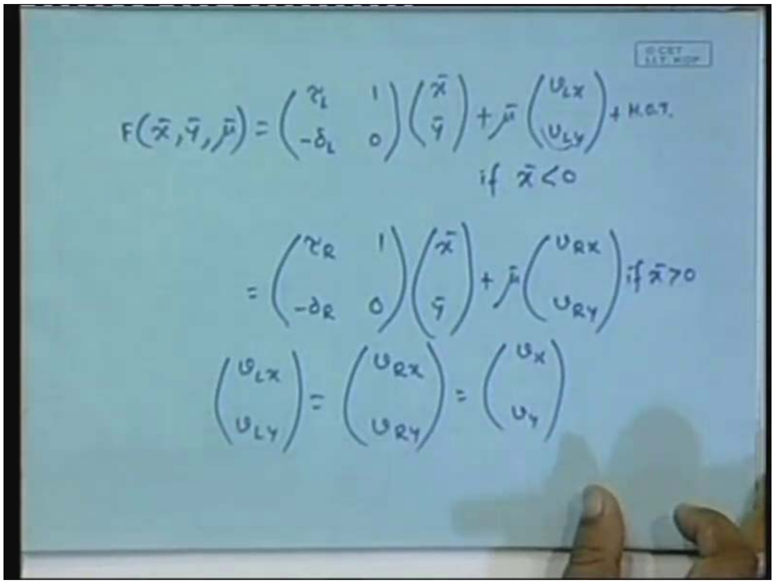

F_x bar y bar mu bar is equal to… plus the higher order terms which we can ignore for now. This is if x bar less than 0. In the right hand side you will arrive at the similar thing, so this is equal to tau_R 1 minus $delta_R$ 0 x bar y bar plus mu bar v_{Rx} v_{Ry} if x bar greater than 0. I am ignoring the higher order terms. Because we are obtaining the local linear representation. Now let's consider these two fellows, these two and these two. Notice that we are considering continuous maps. If the map is continuous then that would immediately demand that this vector is equal to this vector, else the map cannot remain continuous. As mu changes, the map will become discontinuous unless this fellows are equal, so v_{Lx} v_{Ly} must be equal to v_{Rx} v_{Ry} which we can say v_x and v_y. We have arrived at that conclusion.

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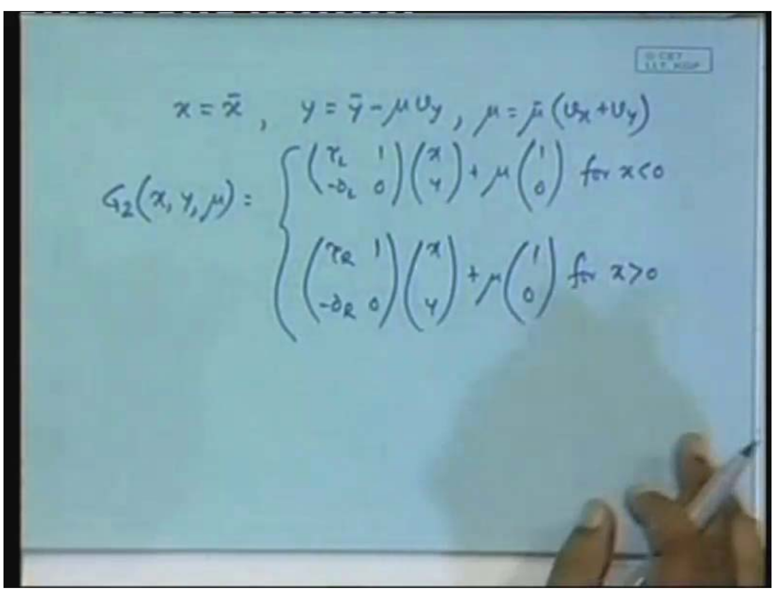

Now we make the final coordinate transformation where we say x. Now this is where we arrive at x, so far you are using tilde, hat, bar but now we arrive at x. The final choice of the coordinate for our convenience is then x bar but y has to be changed to y bar minus mu v_y and mu is mu bar times v_x v_y. We assume that v_x plus v_y is not equal to 0 then only this works. If you make this coordinate transformation then we have x y and mu. In that case your map will finally be written as x y mu, x y as the coordinates and mu is the parameter is equal to, in this side it is tau_L 1 minus $delta_L$ 0, x y plus mu. Then this is 1 0 for x less than 0. Tau_R 1 minus $delta_R$ 0, x y plus mu for x greater than 0. What I have actually done? We have actually made a change of coordinate, so far your choice of y bar was parameter dependent but the moment you have done this coordinate transformation, y is no longer parameter dependent. We have freed the choice of coordinates from the parameter. Parameter is an independent thing that only meets the fixed point… (Refer Slide Time: 00:49:43). This is the two D normal form.

Just contrast with it the one D normal form. One D normal form was g_1 x mu ax plus mu v_x plus mu. This was for x less than 0 and this is for x greater than 0. This was the one D normal form, this is one D and two D normal form is this. How are they related? Derivation of the one D normal form was far easier but the two D normal form is as you put C, we have to go through a few steps. But the link is that if you consider in this two D normal form, the determinant is 0. Then it becomes the one D normal form. If the determinant is 0 it becomes a one dimensional normal form. Any two dimensional system, if you make the determinant 0 it actually becomes a one dimensional system. Here this line goes, this line goes and then the tau L becomes equivalent of a and the tau R becomes equivalent to b. In your thought you should remember that the determinant is something that makes the distinction between one D and two D but the equivalent of the slope in one D is here, the trace. Trace represents the equivalent of the slope. If you make the determinant is 0, the trace is nothing but the slope but so long as the determinant is there, trace is not really the slope. It is actually a kind of derivative quantity addition of the two cross terms.

In the next class I will continue with this particular map, two dimensional map and from there we will try to obtain. Now I will give you a clue to handle all this coordinate transformation. Now see we have done a series of coordinate transformations and through that we have tried to sort of simplify the mater, whatever may be the complication involved in the coordinate transformation ultimately we wanted to arrive at something simple and that is what we have done.

Now you could argue that in the left hand side you could locally linearize and get a matrix. In the right hand side also, you could locally linearize and get a matrix. But these two matrixes will have four tongs here and the four tongs there. Naturally this representation with two matrixes each tong containing non-zero and non-one terms that will be more complicated than this one, you could argue that I have a matrix in the left hand side, couldn’t tie simply by matrix manipulation, obtain this. Yes, you can do that. You have a matrix, you know how to do matrix manipulation and matrix manipulation is nothing but coordinate transformation which is basically the same thing as coordinate transformation.

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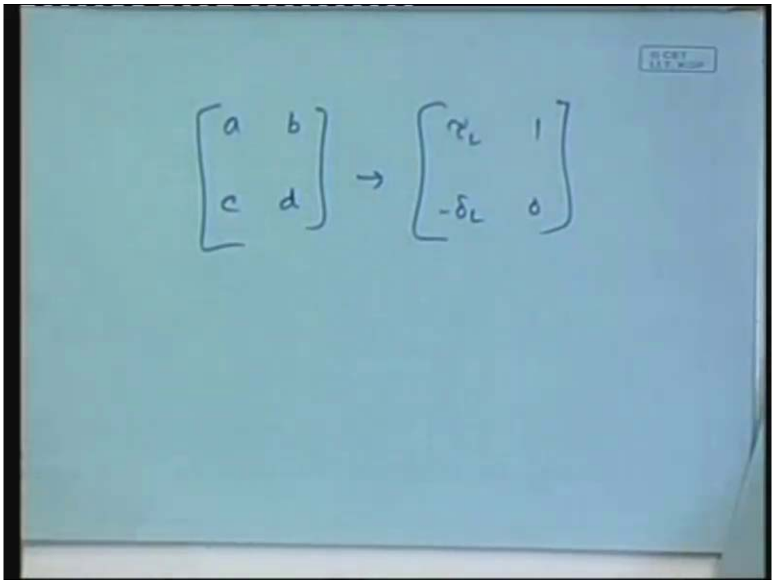

So you could argue that no, I don't want to look at all this coordinate transformation. I simply want to do matrix manipulation by which I want to arrive at this. Can you do that? You should be able to do that. I did all this procedure because that gives a geometrical intuition because the fact that y axis maps to the x axis is important. Those things will not be clear if you simply do the matrix manipulation but you should know how to do the matrix manipulation of a matrix that has the shape abcd to tau_L 1 minus del_L 0. You should be able to do that. That's all for today, we will continue in the next class.