Chaos, Fractals and Dynamical Systems
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Lecture No. # 32
Non-Smooth Bifurcations (Contd.)

In the last class we have seen that we were talking about the non-smooth or border collision bifurcations. You have seen that these happen when the slope or Jacobian matrix, slope in case of one dimensional system or the Jacobian in case of higher dimensional system that changes abruptly. For example if there is a map whose graph would be of this form x_n versus x_{n+1} something like this. Here the slope changes abruptly.

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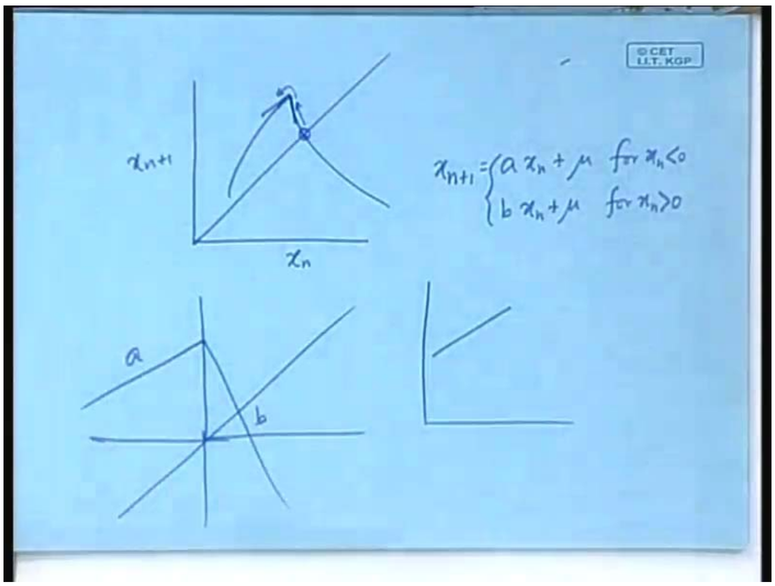

In case the situation is that this is the fixed point, as you change the graph of the map with the change of some parameter. If it so happens that the fixed point is now here. The point of intersection with a 45 degree line is now here. Suppose with a change of the parameter, it moves like this and at some point it goes like this. Then obviously at this point, the slope abruptly changes. Therefore the stability status will also change abruptly and the resulting bifurcations were called the border collision bifurcations. Then we said that in order to probe the quality of this bifurcations, there is no need to look at all these areas of the map. Rather it suffices to look at only the local linear neighborhood. Then we said that let the map be then given by x_{n+1} is equal to ax_n plus mu for x_n less than 0 and bx_n plus mu for x_n greater than 0. This was the map that we had considered. As a result the a side is the slope. These are piece wise linear map. So a side represents the negative side, I will have to put it differently, the 45 degree line. The a for x_n less than 0 so represents this kind of a slope and b represent this kind of a slope. It is a piece wise linear map.

Probably you are now able to see the link between this map and the map that we considered while talking about the Frobenius perron operator that was a piecewise linear map. So that occurs in many physical system. We are ought to understand what happens for different values of a and b.

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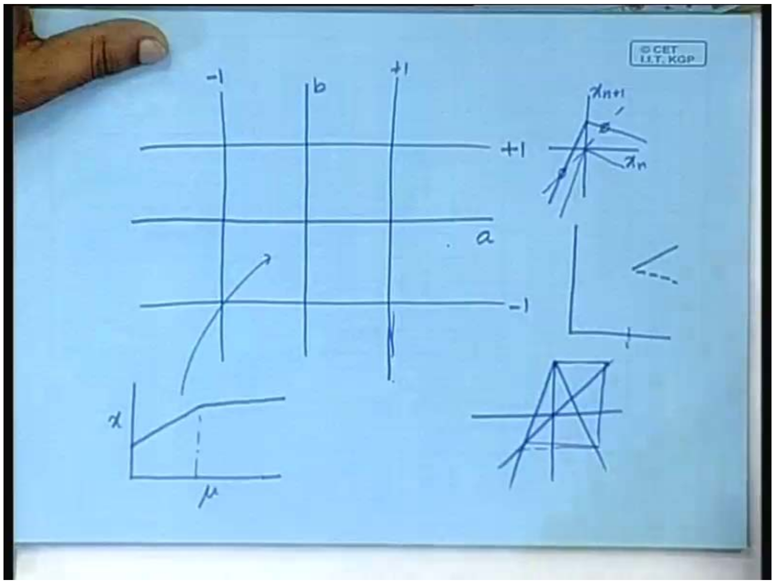

Then we said that we can draw a picture of the parameter space a versus b parameter space and for a less than 1, over this range it will be one type of behavior in the left hand side and for b less than 1 and greater than minus 1, there will be another type of behavior. In this box where this is minus 1, this is plus 1 and this is plus 1 and this is minus 1. In this box which means a ranges between minus 1 to plus 1 and b ranges between minus 1 to plus 1, you expect a stable behavior as the fixed point moves across the border. This point would be called the border because that represents the border line between two different types of behavior. That is where we were. We had explored this part, we had explored this part. Just to recall what happens in this part; it was if you draw the bifurcation diagram it would be something like this.

I am now drawing the bifurcation diagram, it would be a stable fixed point approaching and then at this point hitting the border. Then again going off as a stable fixed point because in both the sides, in the left hand side as well as in the right hand side, the slope is less than 1 and therefore the fixed point will be stable. These are reasonably easy situations to handle. This is your mu and this is your x, so this would be the behavior in this part. What will be the behavior in this part? As we had concluded, the graph of the map would look something like this. For example if you have somewhere here, the graph of the map would look like a value greater than plus 1 so it will be something like this and b value like this. The b value is negative but greater than minus 1. As you can see this graph of the map so this is x_n versus x_{n+1}, as you change the parameter mu it goes up.

At a specific parameter value it would go into a situation like this before that there would be no fixed point and at this point, two fixed point will be born, this one unstable and this one stable.
If you draw the bifurcation diagram you would expect there to be no fixed point before the border collision occurs but at this point a pair of fixed points will be born like this. This is similar to the saddle node bifurcation that you have already come across in all senses except for that, earlier we have seen that this would be a smooth curve. Now this is a king that's all not a big deal though. We have also seen that in this part both the fixed points are unstable. Even if both the fixed points are unstable there can be stable orbits not a periodic orbit but a chaotic orbit is possible because there would be a trapping zone in the graph of the map so that any point inside that trapping zone maps inside the trapping zone. So that any initial condition cannot escape. We had come up to this stage.

We decided that a situation like this, both the size will have slope greater than unity. This will become stable so long as this point maps like this and it maps to a point that is inside but it will become unstable where this point maps to a point that is outside. That is what we have decided. From this condition it is very simple to obtain the condition for which the chaotic orbit will remains stable, have you obtained that. Have you obtained that? The position of this point is what? Notice that this is the value of x_{n+1} for x_n equal to 0 so if you put x_n equal to 0 both give mu. At this point, the height is mu. Next what are you asking? Where does mu map?

If in some iterate x_n equal to mu where does it map? Just put it here, it will map to this side so put it here b mu plus mu, so you put it here and you find out where it maps. If you do that, you will find that this will have a curve like this where this curve is given by minus a by a minus 1. I am leaving this proof to you, just I am writing the value. We have come up to this stage. What happens for this part? Now let us take up this part. What does this represent? The a value between minus 1 and plus 1 and b value less than minus 1. What will the graph look like?

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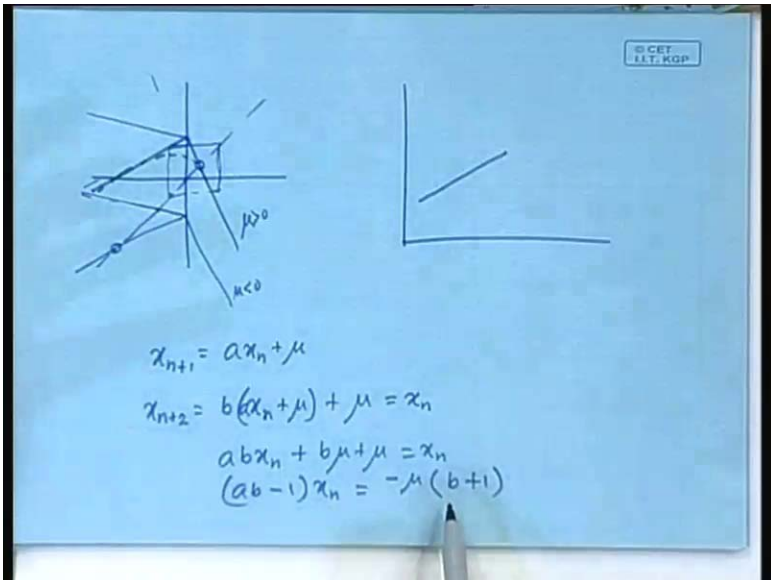

The graph will look like a value between minus 1 and plus 1, so let us say like this and b value greater than or less than minus 1, so something like this. Now here you notice that before or for mu less than 0, the graph would be something like this. At this point it will hit and then it will go off. When it is less than 0 that means, when mu is less than 0 and mu greater than 0 then this is the fixed point and the slope is less than unity. Therefore this fixed point is stable. You might notice that if you take this as a pair of sticks and if you raise it, this point will move progressively to the origin. In the bifurcation diagram you would expect that the fixed point will move in position and at that point, the border collision takes place. What will be the result after that? You would notice after that the fixed point becomes this one and at this one the slope is less than minus 1. It will be unstable. If it is unstable what will be the behavior like? If the fixed point is unstable, from our idea about the smooth systems that means where the behavior was something like this.

We have studied that like a logistic map and when this slope becomes less than minus 1 what did we expect? A period doubling. Here also it is natural intuitive to expect that but we will need to check whether it really happens or not. How do you check? The first question is does the period two orbit occur at all that means does it exist. In order to check that we will need to do this that we will say if there is a period two orbit, how will it look? It will have one point here and then it will go like this. So one point in the left side another point in the right side. Suppose you start from here that means x_{n+1} is equal to ax_n plus mu that brought me to the right hand side. Then you say x_{n+2} is equal to b xa x_n plus mu, I am substituting x_{n+1} plus mu.

In order for the fixed point to exist, this must be equal to… that is the condition for the fixed point. I started from x_n went to x_{n+1}, went to x_{n+2} and then said if this is equal to x_n then that is a condition for the period two orbit. We have got an expression that we can solve. What do you get? abx_n plus b mu plus mu is equal to x_n. Bring it to this side. Is it right? ab minus 1 x_n is equal to minus mu b plus 1. We have the position x_n given and let's see what is the condition we are getting. This is the condition under which it will occur. You might simplify it later but I am just showing you the way to obtain the condition for the occurrence of this fellow. The a is between 0 and minus 1 then the graph will look like this (Refer Slide Time: 14:50). That means in this case it is like this, in this case it will be like this. No important change will really occur because here also this will be the fixed point and its slope will be stable.

There will be no important difference though that's why I am not going into that. But now notice that we had started from a point in the left hand side, went to the right hand side and came back to the left hand side. Therefore x_n must be negative. This fixed point will exist if the x_n value obtained from here is negative. Similarly you can start from point in the right, go to the left and come back to the right, obtain a similar condition. That will be positive. That will be the condition of existence. If you do that you will find that over the whole range, the period two orbit really exists. But then will it be stable? What will be the condition for stability of the period two orbit? There is one iterate to the left side another iterate to the right hand side. Here the slope is a, here the slope is b therefore the slope of the period two orbit will be ab. In order for that to be stable, the ab magnitude has to be less than 1.

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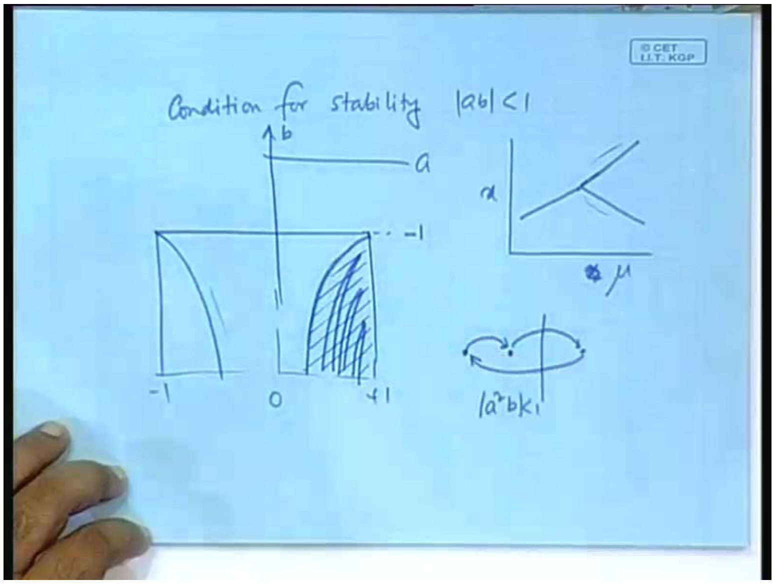

Condition for stability is ab. The ab less than 1 actually gives two conditions, ab equal to plus 1, ab equal to minus 1. Naturally we can expect if I draw this part here a fresh. We are now drawing only this part so that here this point is a and this side is b and here it is minus 1, here it is 0, plus 1, minus 1. This is the box that we are really drawing. The ab less than minus 1 will have two curves there. One will be like this, another will be like that. This will be the curve for ab equal to minus one and this will be the curve for ab equal to plus 1. In order for ab to be plus 1, both a and b have to be negative. This is the side where a is negative and therefore this is the side where b is negative both are negative. This is the curve for plus 1. What happens here in this part? In this part we understand that the period two orbit will occur.

If we now draw the bifurcation diagram, how will it look? Mu here, x here. It will be stable so long as the point is moving towards this point and then at this point, it becomes unstable and the period two orbit then becomes stable. But its behavior would be a bit different, it will be like this. It will not be the typical this shape that we have seen. It's a period doubling but caused by a border collision. This part we understand what will happen for these values of a and b. In this part what will happen? The period two orbit will exist but has become unstable. Now if period two orbit becomes unstable, what is the next intuitive reasoning? Period four but in that case we need to justify why period three orbit will not be possible.

Now in case there is a fundamental difference between the smooth map and non-smooth map that incase of non-smooth map that is also possible, we can easily check. What will you have to do in order to find out the region of existence of period three orbit? All we need to do is that there will be one point to the left, another point to the left and another point to the right. You will take this map, you will take this map and you will come back here. You can easily find out the algebraic condition for which this will occur. What will be the stability condition? a square b less than 1 because there are two point to the left and one point to the right.

If these conditions are satisfied, this fellow will exist. One will be the stability condition and another fellow will be the existence conditions. You will get again a range for which this condition will occur. There will be another range for which the period four orbit will occur, another range for the period five will orbit will occur. As a result you would notice, let me show you the actual pictures.

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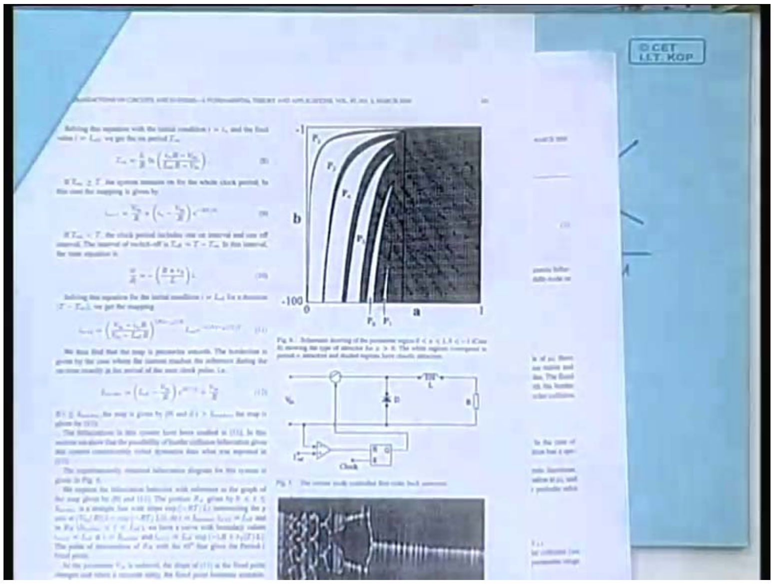

Is this visible now? Yes. In the ab parameter space in this part, all these different orbits will occur like this. The white regions are the regions of occurrence of different periodic orbits. Here it is period two, here it is period three, here it is period four, here it is period five, here it is period six and so on and so forth. But in between no periodic orbit really can occur. What can occur in between? Chaotic orbit. As a result here we come across a typical situation that is often encountered in engineering system, a period adding cascade. That means as you increase the parameter this way, what do you observe? A period two then a thin range of chaos, period three again the thin range of chaos, again a period four again another range of chaos, period five again another range of chaos and so on and so forth. That’s a period adding cascade not a period doubling cascade. This is one of the typical thing that can happen in non-smooth systems. Let me illustrate this on the computer.

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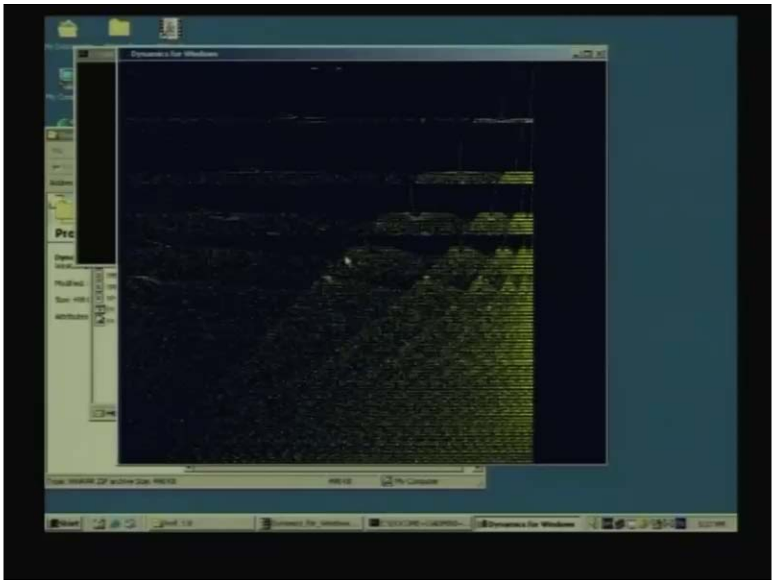

Here you notice the behavior. Can you see? There is a period two orbit. No, here this is another. That means period three orbit, there was a period two orbit. At the top there is a period two orbit, here there is a period three orbit. Again a bit of chaos period four orbit, again a bit of chaos period five orbit, again a bit of chaos period six orbit so on and so forth. This is called the period adding cascade that can happen in this kind of systems. In fact they have been observed in many electronics circuits where there is some kind of a switching phenomenon. We are more or less worked out the whole situation here.

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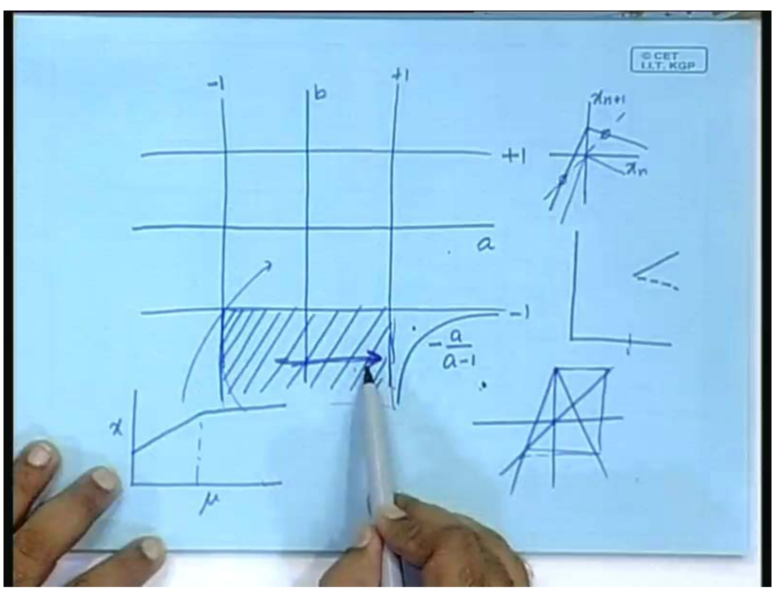

We understand. Again let me recapitulate what happens here, period one orbit remains stable. What happens here? Like a saddle node bifurcation a period one orbit stable and another period one orbit unstable are born. What happens here? Two orbits are born both are unstable but that can be chaotic orbit here. In this part what can happen? Two orbits are born, two periodic orbits are born both unstable but the resulting chaotic orbit is also unstable. This will happen here and in this part there can be a transition from period one orbit to period two orbit or a period three orbit or period four orbit, all these are possible. That means here you see I have moved it like this. Let's come back, I have moved the parameter like this with mu held positive. That's why you are getting this diagram. Now if say I choose the range here for example in the period three range then what will be the bifurcation diagram like?

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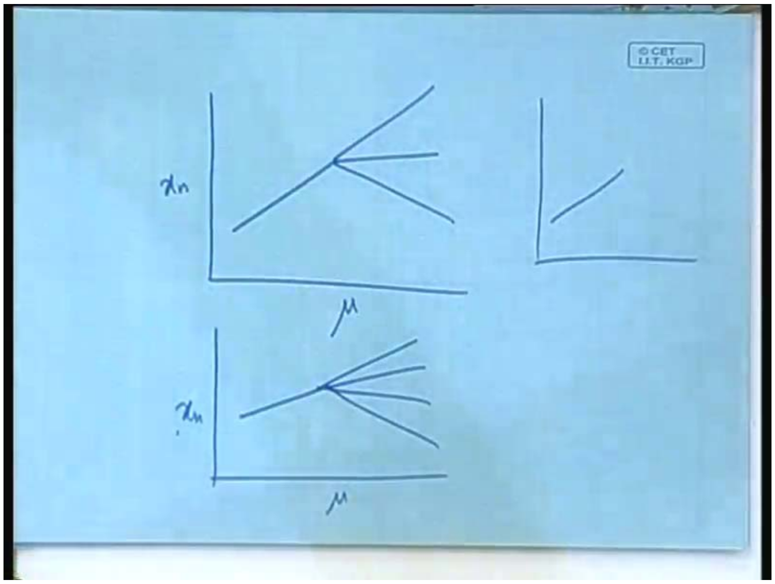

It will be period one, two, period three. As mu is varied, if I chose a parameter in the period four range what will be the bifurcation diagram like? 1 to 1 2 3 4, so all these are then possible. So far you had the understanding that these are very, I mean one would not expect but in non-smooth system these are also expected. Yes period two can also happen. No, here we are talking about a single border collision event as mu is varied. Since both the sides are linear then nothing can happen after this. For mu less than 0, there will be one type of behavior. For mu greater than 0, there will be another type of behavior. It cannot be anything that happen after this because here we are not considering the nonlinearity in each side but actually the system will be nonlinear in each side. There can be things happening in the other parts but you cannot say there will be succession of this.

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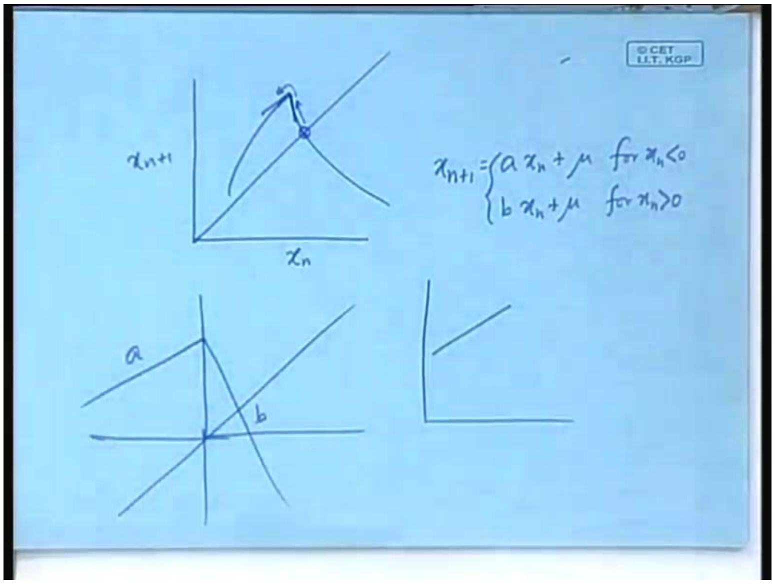

But in a realistic systems, see here we are considering system with this specific algebraic representation. Here we have modeled it in such a way that ab are parameters and mu is such a thing that when it is varied then you get a border collision. But in the real system nothing is really modeled like this. It may so happen that you have a parameter, if you change that only b changes not mu. It is possible. If that happens for example here in this diagram that I have just shown on the computer, I have changed b. As a result you do see this period adding cascade and that is exactly why a period adding cascades are reasonably common place in many physical systems and engineering systems. Is that understood, why period adding cascades occur? By the way this is one of the mechanisms of period adding cascade there are others. But I will not get into those complications. Now so far we have been considering one dimensional systems. There is no reason to believe that every system is one dimensional. We also need to encounter the situation where there is a two D system and as you change the parameter, there is a border collision. How can that happen?

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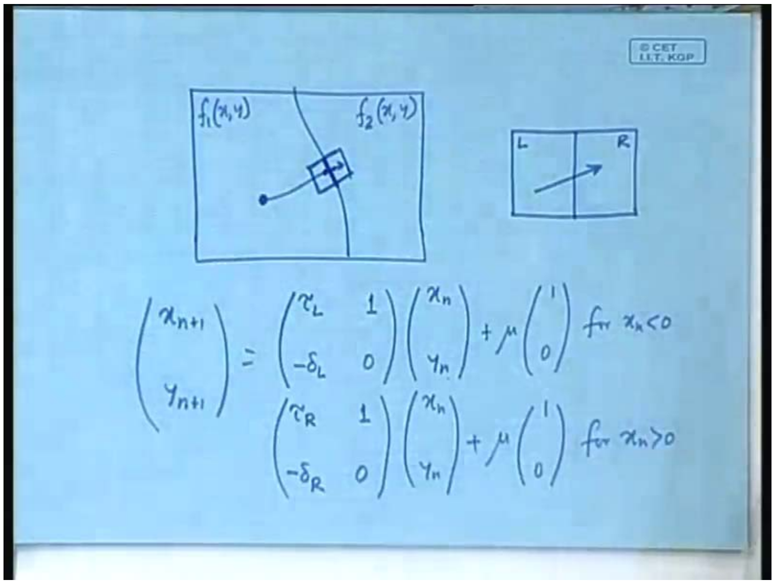

It happens like this. Supposing this is the state space and in this state space there is some kind of a border line like this. So that in this part there is one type of description and in this part there is another type of description. If here is a fixed point, if you change the parameter it moves and say hits the border. Then at this point if you obtain the Jacobian here and the Jacobian here, you will find the Jacobian has abruptly changed as it hit the border. That is the position of the border collision and here also in order to understand it, we will need to do a local linearization here and a local linearization there. It has been shown that this local linearization, I mean it will be very convenient to do some coordinate transformations. So that this border line becomes orthogonal. So that let us look at only this kind of a reoriented states space that only comes from a bit of coordinate transformation. So no big deal.

As a result the map itself can be expressed in the very convenient form where x_{n+1}, y_{n+1}, this will be expressed as… here there will be matrix x_n, y_n and there will be a parameter like this for x_n less than 0. There will be another matrix x_n y_n plus mu 1 0 for x_n greater than 0. What are these matrices? Now these matrices in the very convenient normal form are expressed only in terms of the trace and the determinate of this Jacobian matrix. Normally what would you do? You calculate the Jacobian matrix here and from the Jacobian matrix even I extract the Eigen values. The additional of the two Eigen values is the trace. The product of the two Eigen values is determinate and the matrix can be suitably modified only in terms of these two things. This is the left hand side, we will say tau L, L for the left hand side, one. Notice that in this form what is the trace? Tau L. what is a determinant? Delta. It is very convenient to express in terms of just two numbers and here also it will be the right hand side.

Even though the representation in two D looks, I mean when you just look at the picture and try to figure out how will it look, it may look a bit complicated but actually it is not all that complicated. We can express in this very convenient form. Here there are a few terms parameters.

In case of one D there were a b and mu, three parameters but here you have two parameters on the left hand side, two parameters on the right hand side and mu parameter which when varied it goes across the border line. But now the state space for this is very conveniently expressed as this. The border line is just vertical along the y axis. This is the left hand side and this is the right hand side. You would notice that as you change the mu, vary the mu from a negative value to the positive value a fixed point crossing the border like this. You can calculate the fixed point and you can easily do this. For example how you will calculate the fixed point? Simple, if you have the fixed point in the left hand side then you would have to calculate from this equation x_{n+1} y_{n+1} is equal to this. How will you calculate? Simply by making x_n here and y_n here. You can solve it and you can obtain it.

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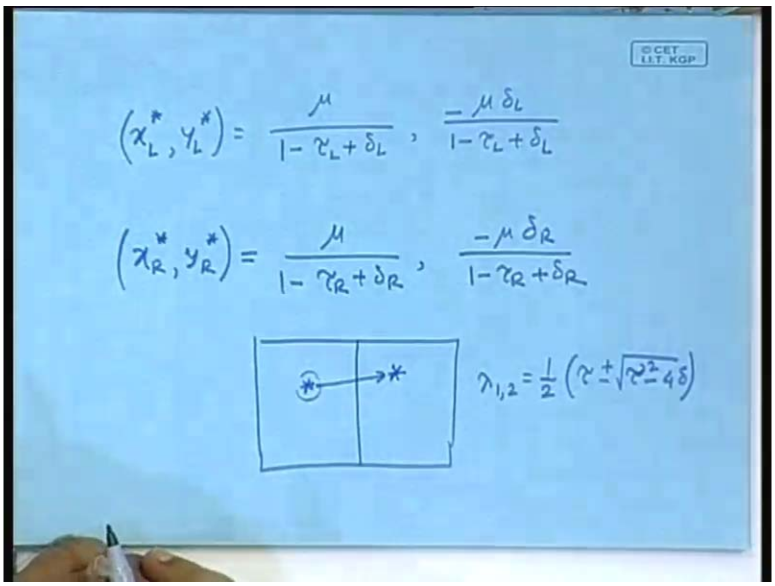

That way if you do, you will see that I will represent it with star because representation of fixed point is generally a star. For the left hand side it will be… this fixed point is given by mu by 1 minus… and this is really trivial I am just saving time by not asking you to obtain it because we have only a very little time left for this semester. That's why I am sort of saving the bother of calculating. This is the fixed point. When will it occur? See this has been calculated using the expression in the left hand side, therefore if the x coordinate is negative then only it will occur. Similarly the one that is calculated from the right hand side equation is like this because of symmetry I can blindly write it as… this fellow will also occur if the x component is positive.

Seen from this angle so in the left hand side there is a fixed point say here there is a fixed point. What is the character of this fixed point? We have learnt that the character of the fixed point is given by the Eigen values and in this case the Eigen values are… can you see? Yes. If you calculate from this equation, from this Eigen values they obtain simply like this. If these are real and between minus 1 and plus 1, you know that it is a stable thing. If these are real and less than minus 1, one of them is less than minus 1 you know it is a flip saddle. If it is greater than plus 1, you know it is a regular saddle and so on and so forth.

This fixed point you can identify the character of the fixed one depending on the relative positions of the trace and the determinant. Now this fellow as you change the parameter moves and becomes another fixed point here. When it becomes another fixed point here that means the character must have changed because it has gone from one side to the other. Again you can calculate the character of this fixed point. It is again categorized as either at the regular attractor or a flip attractor or a regular saddle or a fixed point saddle or a spiraling orbit whatever it is but it has a specific character.

Suppose we want to find out which type of fixed point goes and changes into which type of fixed point. We need to categories this and since the Eigen values are given by this, a little bit manipulation will tell you that… we need to find out under what condition will it be stable? Under what condition will it be real? Can you see under what condition will it be real? This term should be positive, in other words the trace should be between minus 2 root delta to plus 2 root delta, towards this. In this range it will be complex. Outside distance it will be real.

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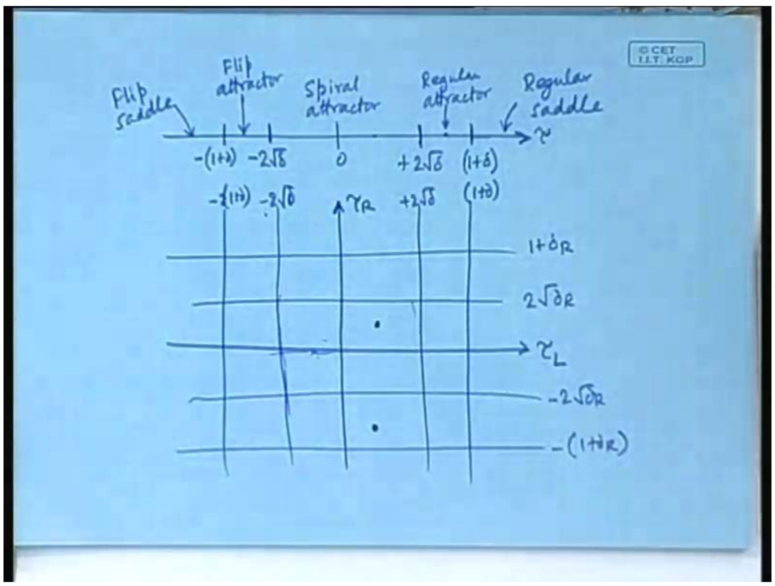

Say suppose this is the trace, this is the 0 and I am sort of comparing with the determinant. Then we can say that there are two ranges minus 2 root delta and here the plus 2 root delta in between this range it is complex. When is it stable? Stable means its value has to be between minus 1 and plus 1. All you need to do is to equate this to minus 1 and obtain this. Again you have to equate this to plus 1 and obtain this condition. If you do that, you would notice that there are two conditions. This is minus 1 plus delta and here is 1 plus delta. From here we can get a picture representation. If I place my point here, what is a character? Stable and spiraling inwards. If a place my point here, it is stable but it is between two root delta and 1 plus delta that means it is no longer spiral but so long as it is below 1 plus delta it is stable. What is the character in this range? What are the Eigen values? The Eigen values would be in this case real but less than unity means regular attractor.

In this part I will say regular attractor. In this part I would say spiral attractor. In this part I would say Eigen value would be negative which means it would be flip attractor. In this part it is positive but the value will be greater than 1, so it is regular saddle (Refer Slide Time: 39:50). Is this visible? I will have to write larger and in this part it is flip saddle. When you write down the equations like this then in each side, the character of the fixed point will be determined by the magnitude of the trace as compared with the magnitude of the determinant on this line. But we are considering a specific type of fixed point turning into another specific type. This thing has to be done in two directions so we can now shape it like this.

This is the trace in the left hand side and suppose this is the trace in the right hand side. The trace in the left hand side will be compared with these lines, so this is minus 2 (1 plus delta), minus 2 root delta, plus 2 root delta, 1 plus delta and so on and so forth. Similarly the trace in the right hand side will be again compared with similar things. This is 1 plus delta R, this is 2 root delta R, this is minus 2 root delta R, minus 1 plus delta R which means now if I place the pen anywhere say here, it will mean a specific relationship between the trace and the determinant in the left hand side as here. A specific relationship between the trace and the determinant in the right hand side like here which means a specific type of fixed point turning into another specific type.

If I place my pen here, what does it mean? The trace in the left hand side tau L is between minus 2 root delta to plus 2 root delta. That means while in the left hand side whatever its character, a spiral attractor. While it has gone to the right hand side what is its character? It is here as compared to this. It's again minus 2 root delta plus 2 root delta that means a spiral attractor remaining a spiral attractor.

Similarly if I say put my pen here. What would it mean? As far as the trace in the left hand side is constant, it is between minus 2 root delta to plus 2 root delta means it is a spiral attractor. Here it is less than minus 2 root delta between minus 2 root delta and minus 1 plus delta which means a flip attractor. Which means if the parameter is placed here, a spiral attractor hits the border and turns into a flip attractor. Now this picture sort of more or less resembles the picture that we have drawn earlier. Only in this case these were simple plus 1, minus 1, a, b. Here you have a little bit of complication but it is not difficult to see that as delta becomes 0 these becomes plus 1, these becomes minus 1, these become 0.

So as you change the delta to 0, these ranges shrink to 0. That means the range for which spiral attractors occur shrink to 0 and you get exactly the same picture. That is why while comparing these two you should remember that a one dimensional system can be represented in 2 D with determinant equal to 0, same thing really. Trace is similar to the slope a and b in this case are same as the trace. Often by looking at the matrix, we cannot figure out what would it be if it is really representing a 1 D system. Yes, you can do that. In that case determinant would be 0 and trace would be similar equivalent to the slope. Now I will not get into the details of all that happens in this part because there is no need to. We have more or less understood that there can be three distinct different types of behavior as understood from the one dimensional case which is simple. Here also the same basic types occur.

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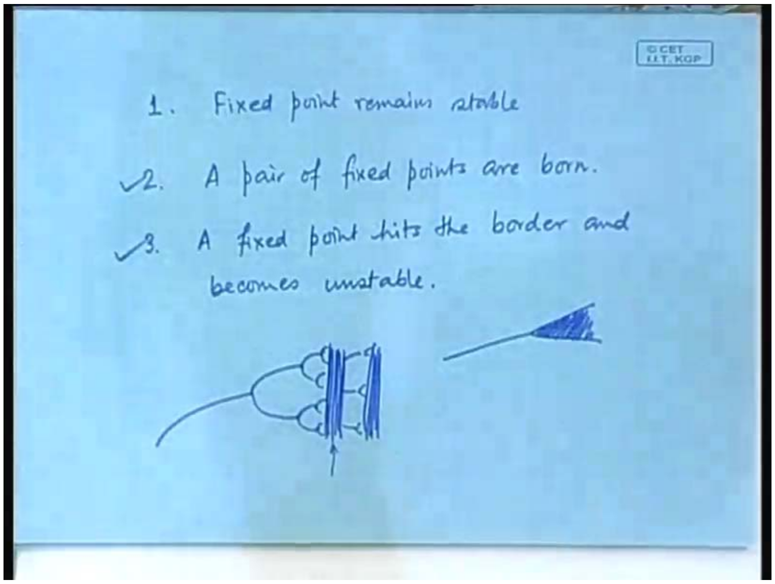

Basic type means one, fixed point remains stable. Two and three these are the three possibilities that can happen. Now in comparison or in contrast to what we have learnt in 1 D systems, all these have further sub cases which I will not go into but just let me give you a glimpse of what can happen. For example when a fixed point hits the border and becomes unstable, something else becomes stable. Now what can become stable? We have learnt that a period two orbit can become stable, a period three orbit can become stable or a period six orbit can become stable or a chaotic orbit can become stable. So that you can have the transition from a periodic one orbit to may be a chaotic orbit like this. This we have seen. That means it could be a period one to period two, it could be period one to period five, it could be period one to chaos. It directs transition from period one to chaos is then possible.

We have seen that when a pair of fixed points are born, one can be unstable another can be stable. A case which has reasonably well understood both could be unstable. If both are unstable there is a possibility of having a chaotic orbit, here also that happens. The only thing is that normally we have seen, when we are talking about the smooth systems for example the logistic map, we have seen that the periodic orbit becomes period two becomes period four becomes period eight is the standard picture for the logistic map. Then there is a chaotic orbit. But then the chaotic orbit is punctuated by period three window which again goes to like this and again a chaotic orbit and so on and so forth. In these chaotic orbits, you have seen that there are tiny periodic windows, we have seen that. If you choose any parameter here, there is a theorem that says that in arbitrarily close neighborhood there could be a periodic orbit. But here in such systems, no. In such system there would be a large range of parameters over which a chaotic orbit will be stable. That means there is no periodic orbit around. Such orbits are called robust chaotic orbits.

In fact wherever we consider practical application of chaos, for example nowadays there are various practical applications of chaos that are coming up including areas in which we look at spreading the spectrum. Spreading the spectrum is useful in electronic circuits, power electronic circuits cell phones, CDMA kind of transmission. There also the same things are useful but remember if you are aiming at an engineering application of chaos, it has to be robust. I mean you cannot really have a situation where tiny undesired perturbation in the parameter will get it out of chaos. No, it cannot happen. It is necessary to have robustness in the chaotic behavior and that is possible in these kinds of non-smooth systems. This is something we have understood in sort of very sketchily nevertheless but nevertheless we try to understand what is the essential issue. Here a fixed point remains stable.

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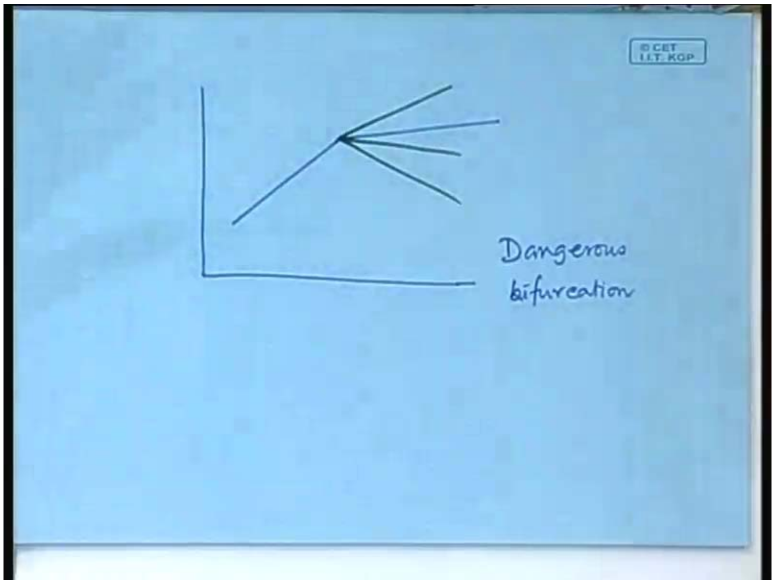

That means if you draw the bifurcation diagram, you would expect something like this to happen. A fixed point was there, hit the border and remain stable. Initially when the theory developed, people thought that this is the desirable situation in any given engineering system. That means in an engineering system you expect border collision to happen. This is the kind of situation you should aim at, so that the system does not get destabilized. But we have found then that there are situations where other periodic orbits are born at the same time. That means while this fellow was stable, another periodic orbit was born at this parameter value. You might ask so what.

Just imagine a practical situation which is undergoing this kind of a transition. You are approaching this critical parameter value, suppose you are changing the parameter. Just following this critical parameter value, the bifurcation point what will happen? There will be two attractors existing and they arbitrarily close to each other. Can you see that? Now in any real system there is some kind of a noise, so that due to the noise there will be some oscillation and there are two orbits that are arbitrarily close to each other. What does it mean? It will keep on jumping between the two orbits and as you change the parameter, their distance increases and finally the fellow gets locked to one of them.

Which one? There is no guarantee and it's not difficult to see that there is a fundamental source of uncertainty as to which one it will lock to. This is a typical situation that can happen in non-smooth system that people have come across. There is another situation where the fixed point goes and remains stable but what is born along with it is not another periodic orbit but an unstable periodic orbit. That means if you have unstable periodic orbit then you know that its stable manifold causes a basin boundary. What will happen is that this orbit will have a basin of attraction. Here also this orbit will have a basin of attraction. Now what happens is that this orbit has had a basin of attraction, this orbit had a basin of attraction. Everybody would think that so what. What happens is that the basin of attraction shrinks to 0 at this point. That means the region of the state space from which points are attracted to this attractor shrinks to 0 at the point of bifurcation.

As a result even though the fixed point remain stable. If you calculate the Eigen values it is stable. Yet, any initial condition will go to infinity at this bifurcation point. This situation has been called the situation of dangerous bifurcation because there is no signal. You keep on monitoring the Eigen values of the system, it is stable. Always you will get a signal that it is stable. This system is going to going to remain stable, yet the system collapses. All these possibilities are there in non-smooth systems and as I told you, non-smooth system are very common in physical world and as well as in engineering.

For example power electronics circuits are non-smooth system where there is switching, any switching circuit for that matter is a non-smooth system. Human heart is a non-smooth system because of the opening and closing of the valves. That's also switch in hydraulic domain. Walking robots, these are also non-smooth systems. Why? Because one foot down and another foot up, another foot down and another foot up these will have to be represented in terms of some kind of set of differential equations and it's not difficult to see that this set, when this foot is down and another foot is up is different from the situation when both the feet are down. The critical parameter value is where it is just making contact so all these possibilities are there. In order to understand the behavior of such systems, we need to understand the theory that we just talked about. That's all for today, we will try to finish whatever remains in the next class.