Chaos Fractals and Dynamical Systems
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Lecture No # 31
Non-Smooth Bifurcations

So far we were talking about systems that are smooth everywhere, smooth means everywhere differentiable. That means we have been considering equations of the form x dot is equal to f of x where x is a vector and this function f is continuously differentiable everywhere. We were considering that kind of systems. Now in nature as well as in engineering, there exist a large number of systems where there is some kind of a switch over action. For example you might imagine that the state space is divided into some kind of compartment so that if the initial condition is here, it will be guided by some equations of the form x dot is equal to f_1 of x. While doing that if it somehow intersects this then the system equations that govern the evolution of this curve. This curve is the solution of the differential equation, the state space trajectory that equation it will change to something else for example, so that it will then go in some other directions, doing some other thing. Then again when it comes back, again it undergoes a switching and it goes in some other directions.

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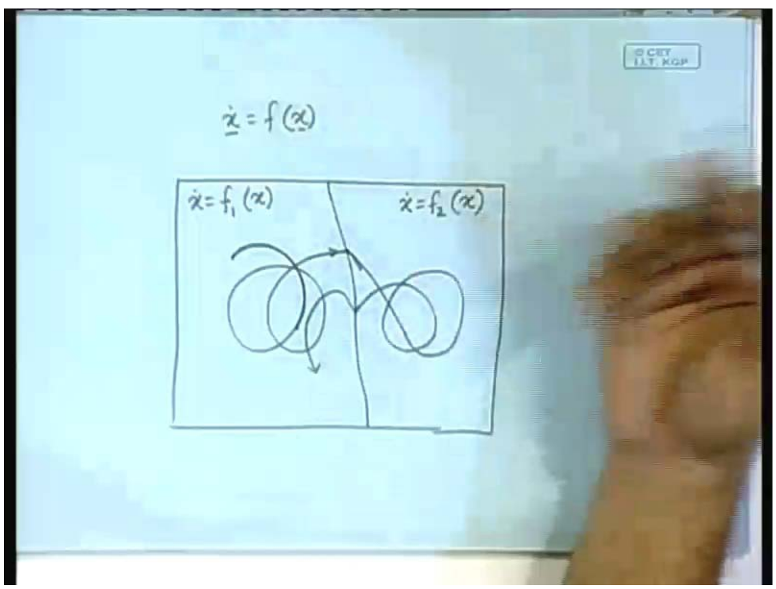

You might imagine situations like that where there is some kind of a switching. Such systems are called switching dynamical systems also called hybrid dynamical systems. The word hybrid comes in the context of where the continuous time evolution, there is also interplay between some discrete events in this case discrete switching events. What happens in such cases? When we try to understand what happens in such cases, try to picture that such systems will also have to be discretized. That means you need to obtain some kind of a discreet time description of the system and we know that discrete time description of the system is obtained by the method of

Poincare section. If there is a system like this, the normal description would be something like this.

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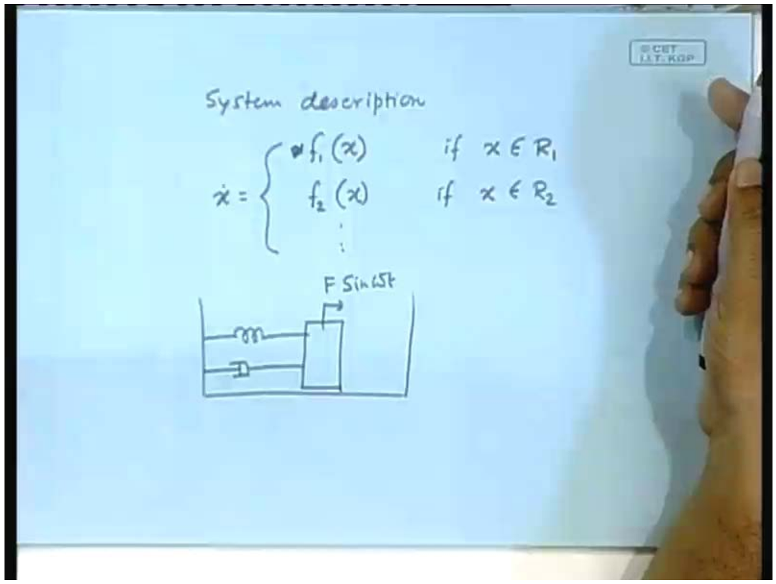

System description would be x dot is equal to, it will be f_1 of x if x is in the compartment say R_1, f_2 x another function if a state x is in the compartment R_2 and so on and so forth. This is the general system description that we are talking about. But first in order to understand what happens in such systems, let us consider only two such compartments in this state space. You might imagine what is a practical implication of this? For example there are large number of switching circuits, power electronics circuits or switching circuits where there is an on state of the switch and there is an off state of the switch. During the on state it follows one set of differential equations. If it is in the off state of the switch, it follows another set of differential equations and obviously this is the description that we need but there are also many other practical examples when this happens.

For example imagine the bouncing ball. Imagine the mechanical systems where there is some kind impact. When there is an impact it goes a bit to the other side depending on the flexibility of the impacting surface. Then so long as it has not impacted, it is governed by one set of differential equations and if the impact is governed by another set of differential equations. Schematically drawing, we would draw it like this. There is one spring and there is one damper connected to a mass and there is some kind of a force acting which makes it oscillate back and forth and there is a wall. It could hit the wall and come back. If the wall is rigid then that will result in instantaneous reversal of the velocity. While if this surface is somewhat soft then it will go into that and while it goes into that surface obviously the system equations change. This exactly brings in this kind of description. You might also imagine the situations of hydraulics systems where there is some kind of a closing and opening of valves.

When the valve is open it will be one set of differential equations, if valve is closed it will be another set of differential equations and the good example is the human heart in which there are of course valve openings and closings. The heart action is started by a triggering action. That's also a non-smooth behavior. The smoothness relates to the differentiability.

Now here when it goes from within this compartment, at every point it's differentiable, this curve is differentiable but the moment it hits and goes in another direction, at this point it is not differentiable. At this switching surface the behavior is called non-smooth. This is what non-smooth should be understood, it means that is not differentiable on a specific surface or a switching plane. If the original system is n dimensional, the switching plane can be imagined to be an n - 1 dimensional hyper surface. So that whenever this state comes and makes a contact with that hyper surface, it goes into another set of differential equations. How do we analyze such systems? The moment we try to do that, try to picture this kind of an orbit.

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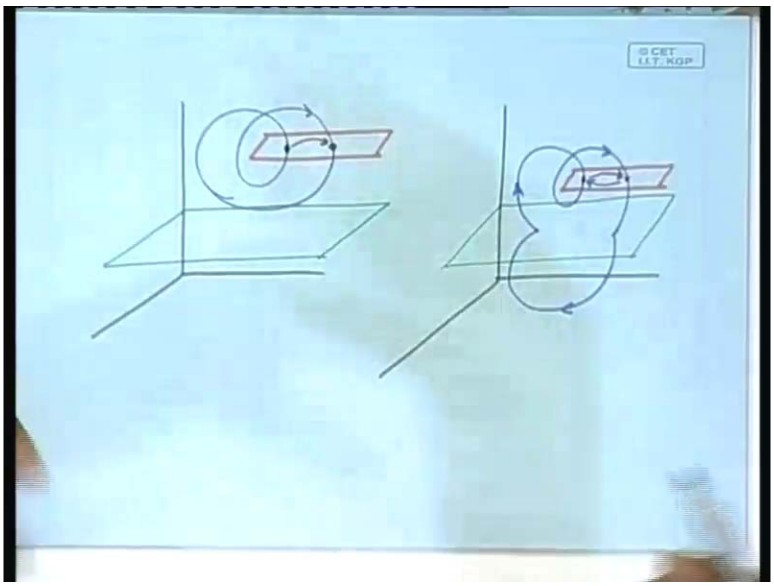

Suppose it is a three dimensional system and suppose here there is a switching surface and I am drawing some kind of a switching surface. It could be nothing else but I am just drawing. Now suppose there is an orbit something like this. Now this orbit does not intersect the switching surface, it always remains in one side of the switching surface and therefore all points of the orbit would be smooth but you might also imagine a situation where the orbit is something like this but intersects, goes in the other side and comes back and then goes.

Now what is the difference between these two? Obviously the difference is that it has intersected the switching surface, gone to the other side and come back and that's how it completes. In both cases you might place a Poincare section. For example if you place a Poincare section here then you will see the mapping as this point mapping to this point and you would get some kind of a functional form of this mapping. You might not be able to obtain the functional form in close form but nevertheless you know that there is some function that maps from this point to that point. If you do the same exercise in this system, you would place the Poincare section here and you would now talk about the map from this point to this point. Obviously just try to understand logically. If you think that there is some kind of an expressions for the map, will the two expressions be the same? Obviously not, because of, in going from here to here, what happens? It went through this and it landed up here. While going from here to here to there it is like this, it goes like this and it comes here. So all through, it goes to one set of differential equations (Refer Slide Time: 11:43).

While in this case, in going from here to here you have to go this way. It comes here, goes across the switching surface and there is another set of differential equation that describes its evolution and finally it comes here, again finally it comes. Obviously in order to find out the map from this point to this point, you would need a much complicated things to be performed and obviously the functional form will be different. Now these two different functional forms will have a critical value where you might imagine, let me draw in another piece of paper.

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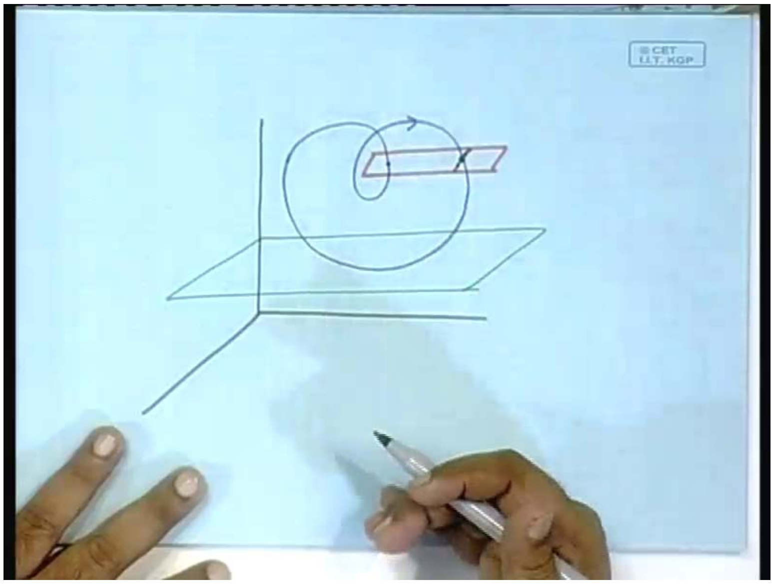

You might imagine that's starting from a point and then it just graces and goes. It does not go to the other side, it just touches but does not really go to the other side. This orbit will be a critical behavior between these two types of behaviors and if you place the Poincare section now, it is not difficult to realize that starting from this point it maps to this point. Means it goes like this and comes back here.

If you started slightly this way then it would have not graced, if you started slightly that way it would go to the other side and come back. This point is sort of a critical behavior between these two different types of behavior and it is not difficult to see that there can be many such points. That means there can be another point here which will also have the property that starting from here, it will grace and come back here. Starting from another point it will grace and come back here.

Therefore there will be a line that will divide these two types of behavior. In this part it will go to the other side that means it will undergo the switching. Starting from this side it will not undergo the switching and the functional form in this part will be one type like this and the functional form in this part will be another type like this.

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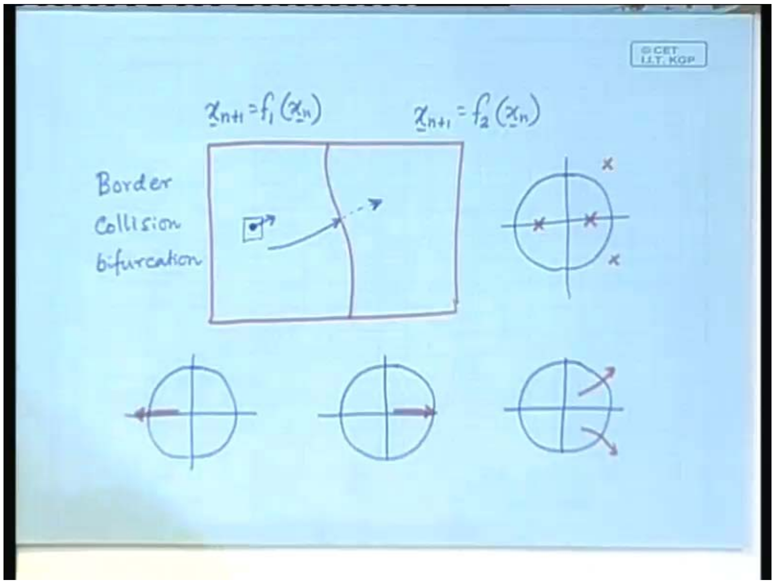

What does it all mean? It means that at end of the day, you have a discrete time state space if I now blow up this in the next page. It will be some kind of a description like this where in this compartment it will be x_{n+1} is equal to some function x_n and in this compartment it is x_{n+1} another function of x_n where x are vectors. What is so holy about it? We know that in such a system, as you change a parameter the location of the fixed point will change. The local linearization will also change, the Eigen value will also change so on and so forth. We know that when the Eigen values exit the unit circle, cross the unit circle you have bifurcations.

It might happen that the fixed point is here and as you are changing the parameter, you are always keeping track of the local linearization and keeping track of the Eigen values and as you change the parameter, the position on the fixed point moves and it crosses the unit circle. One of the Eigen values crosses the unity circle. When that happens obviously we can explain what happens in terms of what we already learnt. Why, because we have already learnt that if this is the unit circle and if one Eigen value goes this way then it is the period doubling bifurcation. If this is the unit circle, circle of radius 1 and Eigen value goes and hits this way. What is it? It is a saddle node bifurcation.

If you have a unit circle like this and Eigen values move like that what is it? It is a half bifurcation or nine marks sacker bifurcation giving rise to the birth of a quasi-periodic behavior, these are all known. If similar things happen while the fixed point is in one compartment, we know what happens. But there is also the possibility that with the change of the parameter, a fixed point may come and hit the border. Maybe it goes to the other side. If it goes to the other side then obviously so long as it is here, its evolution is defined by this.

So long as it is here, it is evolution is defined by this one (Refer Slide Time: 18:01). They will have different Jacobians, they will have different Eigen values. In other words what will happen is that, the moment it hits, the Eigen values will discretely jump from one place to another. It may be so that the Eigen values were, so long as it was here in this side, suppose the Eigen value were say here and here. What is it? An attractor, a flip attractor because one Eigen value is negative. But when it landed here, when it crosses the border it is possible that they discretely jump to this location which means now it has become unstable, it has been complex conjugate.
It will lead to a spiral outward orbit and so on and so forth. There is a possibility of a sudden discrete and abrupt change in the behavior caused by this kind of phenomenon. That kind of phenomenon that class of phenomenon that are caused by the collision of a fixed point with such a border line is called border collision bifurcation.

Let me give an example. This is a two D description that means here we are considering, since I have drawn it in two dimensions it means that x is a two dimensional vector x and y. Here you would say x, here you would say y and so on and so forth. But there may also be situations where it is just one dimensional. In that case what does this picture look like? In one dimension you can draw the graph of the map.

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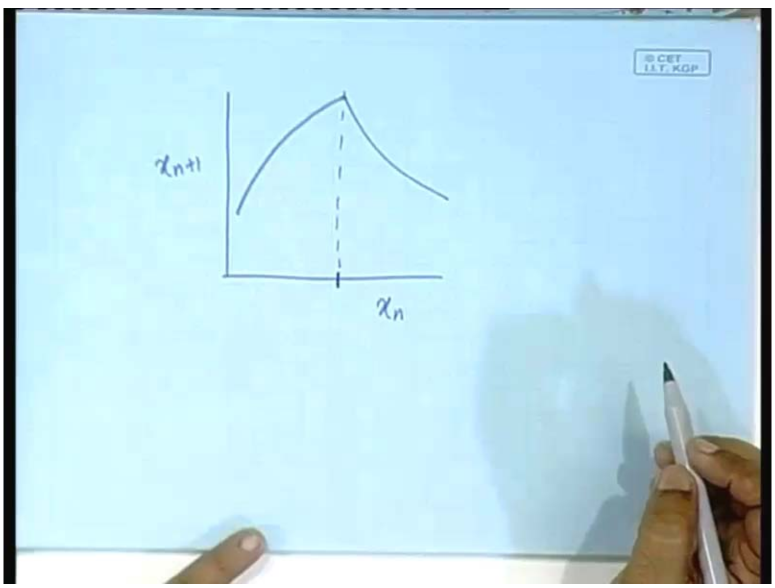

In that case you might end up in situations something like this, say in one side say this is the border line x_n and this is x_{n+1}. Suppose this is the border that means for these values of x_n, it will have one functional form, for those values of x_n it will have another functional form and it is divided like this. Suppose so long as x_n is less than this value, the behavior is something like this. A smooth but at this point there has to be non-smoothness, the character has been changed. It might be like so. It is not difficult to see then that in one D also such things can happen and let us start to work on this kind of idea because one D, one dimension is easier to understand because you can draw a graph. But first let me give an example, very simple example. An example that probably I have already done but never nevertheless for this context let us repeat.

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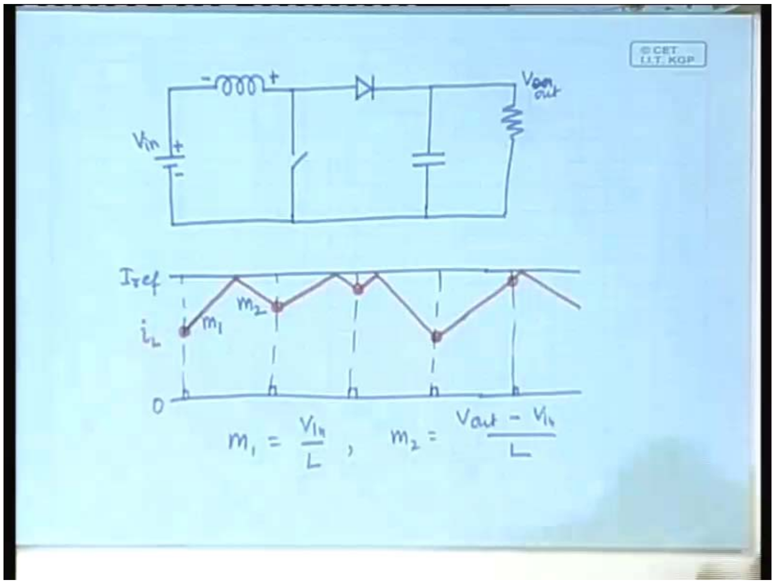

Let us consider the power electronic boost converter where you have got a battery, you have got an inductor here and you have got a switch here. Here is a diode, here is a capacitor and here is a load resistance, this is a load. Now this explanation you should recall. What happens when the switch is on and off? When the switch is on then the current through this loop builds up because it is a voltage current across an inductor so the current builds up linearly and when the switch is off then the whole circuit is connected, the diode is forward biased. The current will flow like this and the energy that has been stored in the inductor will then be going to this RC network. The energy shifts from here to here, the energy in the inductor then drops. The inductor current as a result falls and if the inductor current falls then the L di dt will have a different sense, if this is positive and this is negative. This will become positive and this negative. As a result this voltage and that voltage will add up which appears across the load. The load sees a voltage that is bigger than the input voltage that is the concept of the boost converter.

Often the control of the switch is done by what is known as the current mode control. I am not showing the circuit but let me just illustrate the principles in terms of a diagram. Here suppose I am drawing the current waveform and this is the zero level and there is some kind of a reference value so this is the $I_{reference}$. There is also periodic clocks so I am drawing the clocks. You might say that okay let me draw the clock instance all through, so that it’s clear. Now at a clock instant, the switch is turned on so the inductor current goes up and when it hits the $I_{reference}$ when it becomes equal, this switch is turned off which means it goes like this. Again it turns on, again it turns off, again it turns on again it turns off. Let me draw some more. Again it goes but now imagine that it does not reach the $I_{reference}$ before the next clock comes, in that case it goes on and then it turns off so on and so forth. That is the kind of switching logic that is followed and this red line would be the current through the inductor.

Now suppose I want to obtain the discrete time description of such a system. How will you do it? Obviously in order to do that we will have to sample it because this is a non-autonomous system. There is some kind of an external clock and we have learnt that whenever there is an external periodic input it is a non-autonomous system and then the discretization has to be done by placing a Poincare section in synchronizing with the external periodic input. That is what we do. We observe from this point to this point to this point to this point to this point and so on and so forth.

Now it is not difficult to see that if the capacitor is large then the voltage here would be held all most constant. If we assume that means if we proceed under the assumption that the capacitor is large and therefore this voltage, V_{output} and this is the v_{input}. This is V_{out} and is more or less constant then this will be really a straight line because it is only this part and the straight line will have a slope. How much? V_{in} by L so here the slope is m_1 where m_1 is equal to V_{in} by L. Here when it drops, it is V_{out} minus V_{in} by L so here this slope is m_2 where m_2 is equal to $V_{out} - V_{in}$ by L. In that case if you try to obtain the discrete time description, just recall we have done that already. How will you proceed? We will first have to obtain this time.

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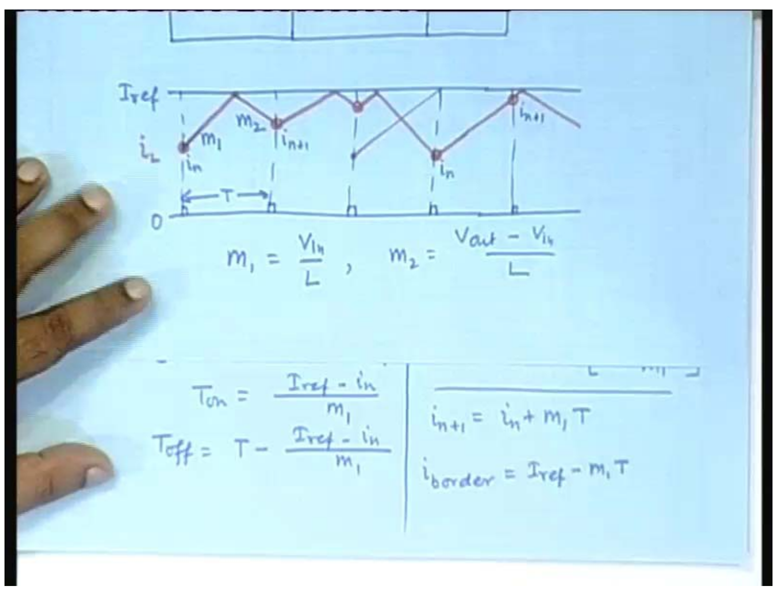

So is that visible? Yes. So what will do is we will say i_L this is the nth time instance so I will say i_n and I am trying to find out i_{n+1} is i_n plus m_1 T_{on} is equal to I_{ref} which gives me T_{on}. How much is the off time? That is the total clock period T - T_{on}, so T_{off} is equal to T - I_{ref} minus i_n by m_1. Then what is the final value I_{ref} – m_2 times T_{off}? So i_{n+1} is equal to I_{ref} – m_2 T_{off} which is T - I_{ref} minus i_n by m_1. But this is only half of the story because there is also possibility of evolution like this which means that it does not reach the $I_{reference}$ before the next clock. If that is happening then this one, in that case if I take this as my i_n and this as my i_{n+1} then what will be the i_{n+1}? In that case i_{n+1} can be straight forwardly written as i_n plus m_1 capital T. You can see that there are two possibilities. One, where the map will be given by this. Two, where the map will be given by this and the critical distinguishing behavior would be one where starting from a point here it reaches exactly at the next clock. Then I will say i_{border} is… can you say what will be this value? It is essentially that value with which I will add m_1 capital T, I will reach $I_{reference}$ so this is $I_{ref} - m_1$ T. Here you have the final description of the system. These are the two compartments and this is the border line condition.

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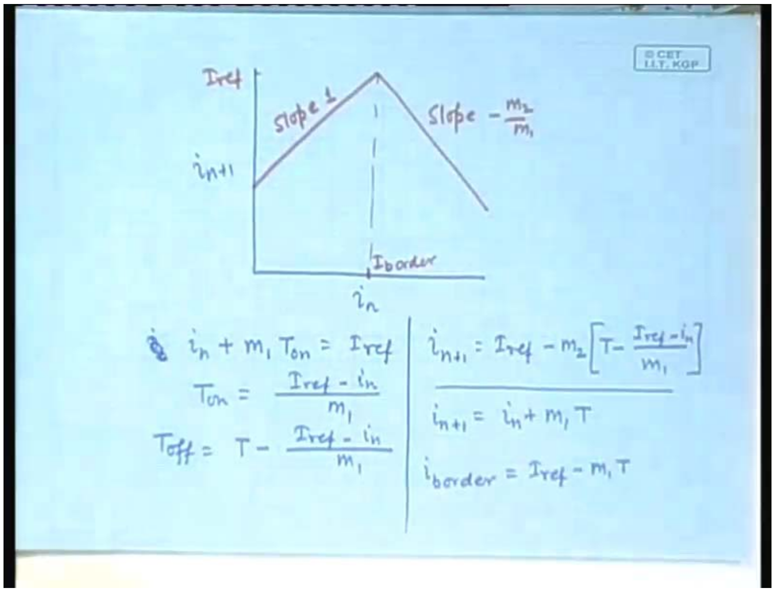

Let us draw this graph i_n verses i_{n+1}. Somewhere there would be the border line case, say the border line case is here. This is i_{border}. Below that value, below i_{border} that means if the initial condition is below that, obviously it reaches like this. You have to take this equation, i_n plus m_1 T. What will be the graph like? The slope of the graph will be 1 and plus m_1 T. It will go something like this, here the slope is one and it will reach $I_{reference}$. This is the border line situation where it reaches so if you put i_{border} here, in this case you see that m_1 T cancels off, you get $I_{reference}$. Finally you get $I_{reference}$ after that we have to take this equation. This equation is you can see that this is a constant, this will be a constant. All parts are constant, only this one is a variable part where the slope is m_2 by m_1 times minus m_2 by m_1 so here the slope is minus m_2 by m_1 times i_n. It will be something like this where the slope minus m_2 m_1. This is the graph of the map for this practical system.

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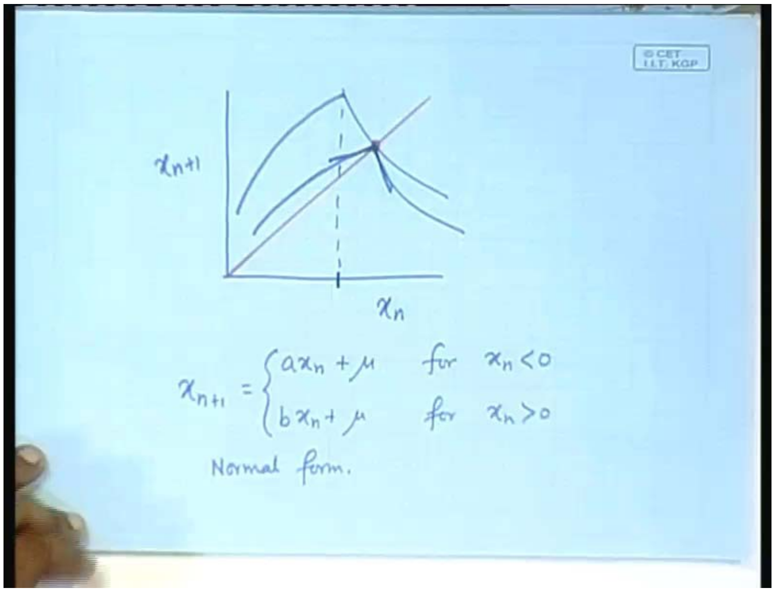

See the similarity that we had said that a normal piecewise smooth map would be something like this. Only in this case after the simplifying assumption that the output voltage is kept constant
We see that both the sides are linear but nevertheless it satisfies the definition of piecewise smooth. We need some kind of a theory for the bifurcation that will happen when the fixed point will hit the border. That means when the border collision will take place.

Let's see when will the border collision take place in this case? Fixed point. Where is the fixed point? Simple, draw the 45 degree line so here is the fixed point. Now with the change of the graph of the map with the change of the parameter, if it so happens that at some point it changes and becomes like this. Yes, then as you change the parameter you can see that there is a border collision occurring. We will try to understand the character of that. Now it is not difficult to see that the event that will occur exactly at that point is no way related to this character of the graphs in this part. It is only related to the slopes here. That means the local linearization in this chunk and the local linearization in this chunk.

In fact that is what is normally used. It is to our luck that it is already linearized but in most different situations it might not be linearized. In order to understand what really happens, I do not need to consider the whole graph of the map. I only need to consider the linearized chunks. If you make some coordinate transformation, this piecewise linearized map can be expressed as X_{n+1} is equal to ax_n plus mu for x_n less than 0 and bx_n plus mu for x_n greater than 0. That is why this form is called the normal form because just by using this, you can exactly understand the character of the border collision bifurcation which orbit remains, which orbit exists, which orbit would be stabled all that can be understood just by considering this simple map. Let us from now onwards consider this simple map. Can you see?

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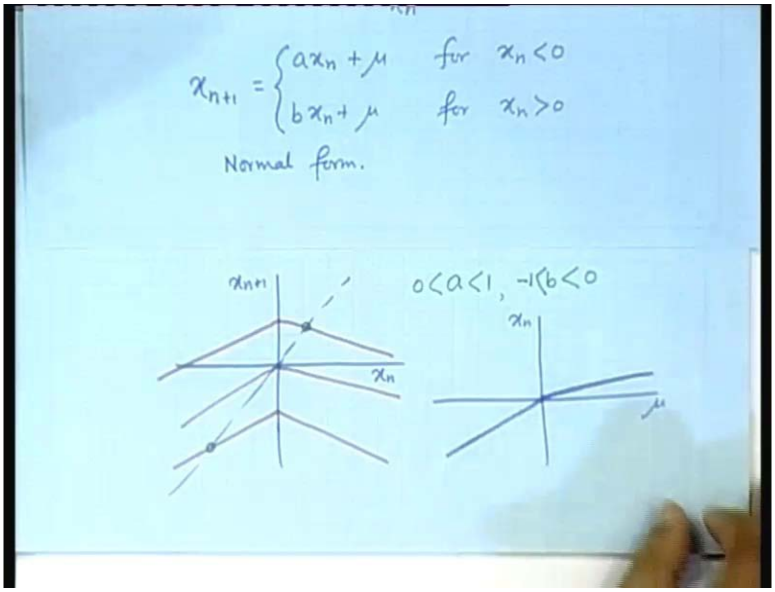

What will be the graph of this map? Obviously that will be depend on a and b but it's not difficult to see that it will be… the graph of the map. What is the result of variation of mu? Mu is the parameter and that the whole graph of the map goes up and down. You might say depending on the value of a and b, the graph of the map is like this here and like that here. As you change mu, it goes here and here. This point represents the critical value of mu, if we draw the 45 degree line it will be clearer at which the border collision takes place. Just look at the result of the variation of mu. Initially when the mu was negative, the fixed point was here. As mu was increased this graph goes up, it's not difficult to see that the intersection with a 45 degree line will go closer and closer to the y axis and at this point when mu is equal to 0 then this is the border line, the fixed point collides with the border line. Border collision occurs at mu is equal to 0 and after that a fixed point moves here.

Depending on the value of a and b, we can then identify a different bifurcation that will take place. For example in this case what will happen? In this case while mu was negative, the fixed point was given by this slope a and when mu was positive, the fixed point is given by the slope b and in this case, the way I have drawn a is positive so between 0 and 1 and b is negative. In this case we have drawn it as b is greater than minus 1 but less than 0. That is how we have drawn it and in that case in the bifurcation diagram what will we see? It is a stable fixed point and in the other side also it is a stable fixed point.

If you draw the bifurcation diagram you will see something like this that this is the bifurcation diagram that I am drawing, not a picture like this. Here it is x_n verses x_{n+1} but here I am drawing mu as a parameter which is being varied and here the x_n. You will see that there is a stable fixed point existing for negative values of x_n, so it will go like this, hit it and then there will be a stable fixed point existing in the other side also. It will be like so but notice that the slope will change.
Stable fixed point remaining a stable fixed point but the slope undergoing a change. This kind of events are very often seen, there is a bend in the bifurcation diagram. These are often caused by border collision but this is not danger because these are situations where the fixed points remains

stable. It's not difficult to see that such situation which will occur where the a lies between minus 1 to 1 and b also lies between minus 1 to 1.

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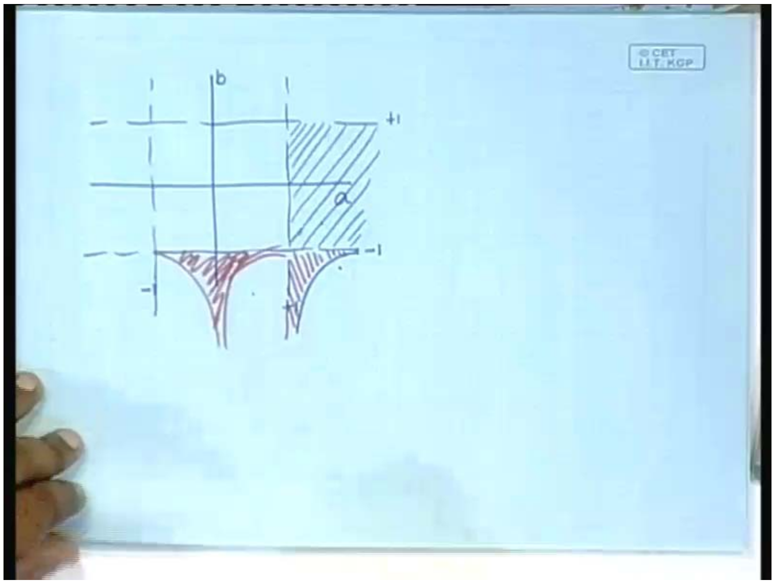

We might draw a parameter space with a and b as the axis and we would say that if the parameters are chosen within the box given by minus 1 to plus 1 and minus 1 to plus 1 in this box then the fixed point will remain as a fixed point. What if I choose a parameter say here, notice where the parameter is. The a value will be greater than plus 1 and b value will be less than 0 that means it is negative but greater than minus 1. What will be the behavior? Let us try to figure it out.

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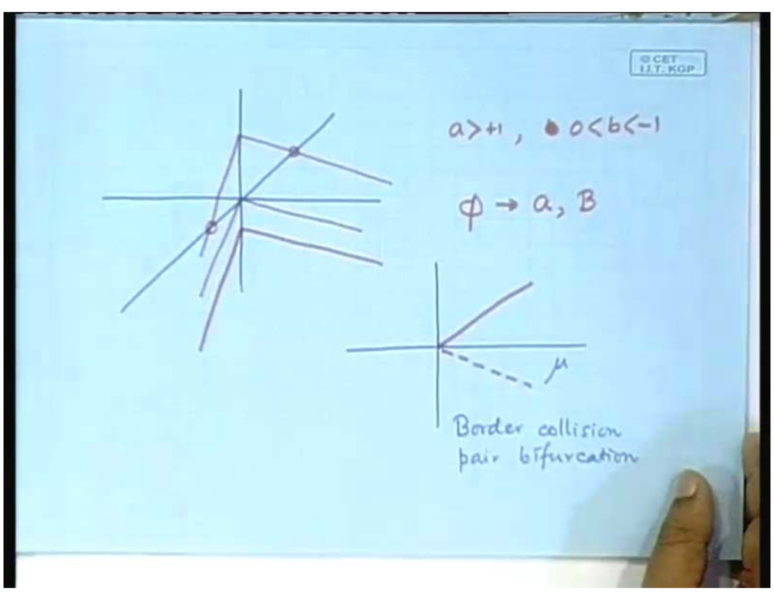

The a value greater than plus 1 so b value between 0 and minus 1. Now you see at this situation there is no fixed point. As it goes up, there will be a time when the graph of the map will look like this. A fixed point critically starts to exit but when it has crossed 0, notice what has happened. There was no fixed point here but at this point, two fixed points have come into existence, one here another here. This fixed point is unstable because the slope is greater than 1, this fixed point is stable because the slope is less than 1. It is similar to a saddle node bifurcation or a tangent bifurcation.

Only thing is that in this point you cannot really call it a tangent but it has similar characters. This a border collision bifurcation so I have taken a greater than plus 1 and b between 0 and minus 1. In that situation we are likely to observe no fixed point, first I will designate it with the symbol null to two fixed points. There is a fixed point here, it is unstable so let us call it small a and this fixed point is stable let us call it capital B. Capital for the stable once in this side, small a for an unstable one in this side so this is like a saddle node bifurcation.

In the bifurcation diagram what will you observe? You will observe that in this side where mu is less than 0 there is nothing, no fixed point but at this critical juncture two fixed points have born one stable, another unstable. This has been called a border collision pair bifurcation because a pair is born. In this part you expect a situation. In this part you expect a situation where a pair of fixed point are born out of that one is stable so that is what you will actually be able to see. Now imagine that you change b to a value somewhere here. What is the situation? The a is greater than plus 1 and b is less than minus 1. Let’s see what happens then.

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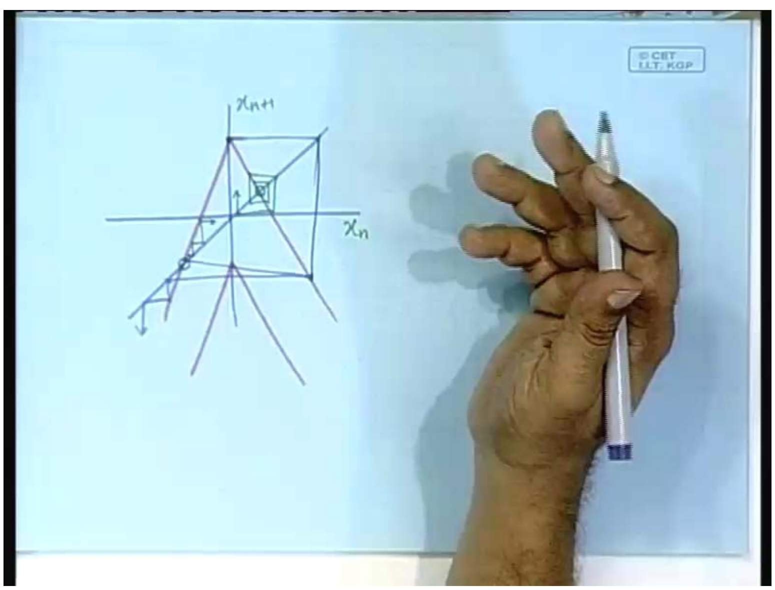

Now the situation is a is greater than plus one, b is less than minus 1, when we have crossed it will be like this. Notice now two fixed points are born. In this situation there was no fixed point, there was no intersection with the 45 degree line but the moment it has crossed mu is equal to 0, you will have two fixed points but both unstable.

Will any periodic orbit occur? No, because if any periodic orbit had to occur then it will have some iterates falling here and some iterates falling there. For the iterates falling here there will be local stretching because the slope is greater than 1. This neighborhood when it maps to here also the slope is greater than 1. All the neighborhoods will always expand and therefore you cannot have any high periodic orbit that is stable, impossible. Can there be chaotic orbit? A chaotic orbit will occur only if the orbit is bounded. Will it be bounded? Notice that any initial condition here will tend to go like this. It will go away from here, any initial condition here will tend to go like this. It will go towards that therefore there is a possibility of the orbit being bounded because from this side it goes to the right. From this side it goes to the left. Obviously it has a possibility of remaining bounded. If it is bounded, the behavior would be chaotic. But over the whole range there is also the possibility that this point if any iterate falls here, it will in the next iterate go here and will come here and in the next iterate if it lands up here in this side then there is no problem but if it lands in this side than it goes out. There is a critical value. As you keep on changing the values of a and b, there is a critical situation where the orbit may become unstable.

Notice what is the kind of instability. This is a boundary crisis, this was the extremity that means the extremity of the attractor and the extremity of the basin boundary they collide and whenever we have that the attractor no longer is existing. That means any iterate in this zone will keep on oscillating. Finally it will map here at some point and then it will go to infinity that is an unstable chaotic orbit. The critical difference between the stable chaotic orbit and a unstable chaotic orbit will be where this point maps to a point outside, it will become unstable. If it maps to a point inside it will remains stable. I will leave it up to you, to find out the condition under which the chaotic orbit will remain stable. You do it before you come to the next class. It's a very trivial trigonometry to be done. We were plotting it here.

In this part we are considering, under some conditions the chaotic orbit will be stable. There will be chaotic orbit and under some conditions the chaotic orbit will become unstable and as a result there will be no orbit that is stable. I can tell you that you obtain the condition, the condition is something like this. It comes to be something like this that in this region you have a stable chaotic orbit, in this part there is nothing stable. Let us consider the situation here say I have chosen a point here which means that a is between 0 and plus 1 while b is less than minus 1.

The a is between 0 and plus 1 like this and b is less than minus 1 like so. When it goes to the other side notice that for mu less than 0, the fixed point exists, when it goes to the other side. What is the result? That the fixed point as this mu changes, this whole graph goes up. This point moves this way and finally at this point, at mu is equal to 0 this one moves to the other side there is a fixed point but this fellow is now unstable.

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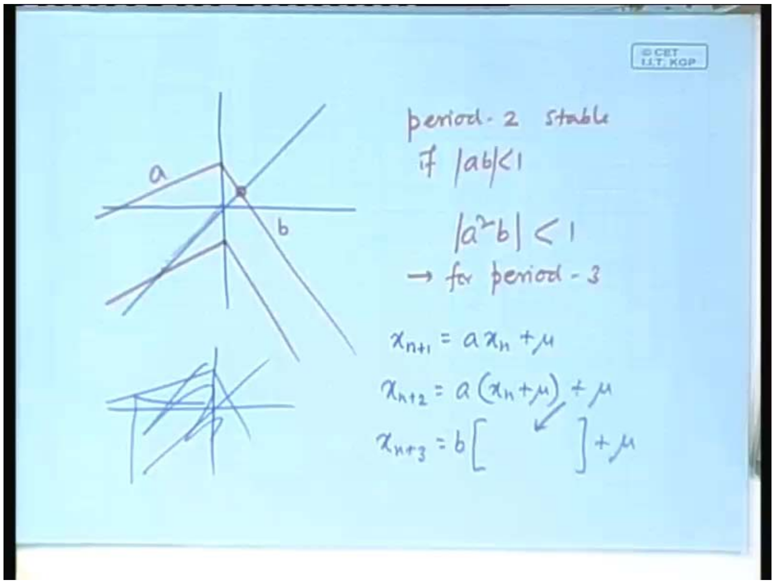

At this border collision the fixed point loses stability, it is no longer stable but here since the slope is greater than minus 1, even though it did not go exactly through the minus one point, it jumped across the minus 1 value but nevertheless you might imagine that this is something similar to the period doubling bifurcation. But the period two orbit whether it would be stable or not, if it is a period two orbit there will be one iterate here, another iterate here. Here the slope is a, here the slope is b so period two will be stable if ab is less than 1 or you can say mod of a b is less than 1.

Now this means that over this whole range, you will not see a period doubling really. You will see a period doubling only if the mod of ab is less than 1 and that yields a range something like this. In this range it will be period doubling but in this range it will not be, period two orbit will be unstable. If the period two orbit is unstable, is it possible to have a period three orbit? Not impossible though because there may be two iterates in this side and one iterate bring it back here possible. But in that case how would you obtain its condition of stability? The condition of stability will be two points to the left a square and one point to the right b, this magnitude has to be less than 1. This is for period three. Not only that you also have to find out, you also have to worry about whether the period three orbit exists.

In order to find that all you need to do is to, if you are imagining a orbit something like this 1 2, I am not going to draw but nevertheless two points to the left and one point to the right. You will have to write down the equations as x_{n+1} is equal to ax_n plus mu, x_{n+2} is equal to again ax_n plus mu plus mu, x_{n+3} equal to now it is in the right side b over this plus mu and this has to be same as x_n then only the period three conditions satisfied.
Solve this and obtain the condition for x_n that will be the condition of existence of period three orbit and this will be the condition of stability of period three orbit. When these two conditions are satisfied, then the orbit will actually occur so on and so forth. You can work out the conditions of existence of all possible orbits, I will come to that in more details in the next class.
Thank you.