Chaos Fractals and Dynamical Systems

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Lecture No. # 30

Control of Chaos

A Set of lectures, you have learnt that there are some very peculiar specialties of chaotic systems. Most important of them being that tiny perturbations in the state can lead to very different results. Tiny perturbations in the parameters can again lead to very different results. Things are very sensitive to perturbations in the state and perturbations in the parameter. Naturally these often give rise to the intuitive feelings that these guys are very difficult to control. In fact that has been the belief ever since the advent of this subject that here are some systems which are chaotic by themselves and if that happens then you have difficulty because it has certain characteristic features but nevertheless ultimately when you want to control it. It becomes very difficult because slight change here and there will lead to completely different results. But over the last 12 years or so, there have been some very important developments that have proved these beliefs to be wrong.

In order to illustrate what this is I will get into step by step. Another important specialty of a chaotic system we have learnt is that while the state moves in the attractor chaotically that means this state never repeats itself. There are infinite number of unstable periodic orbits embedded in that attractor. While going through this kind of erratic motion, the state often comes very close to one of those unstable periodic orbits. It cannot get locked there because it is unstable. Slight perturbations, slight difference from the unstable periodic orbits will lead it to elsewhere. But nevertheless the point is that there are an infinite number of unstable periodic orbits embedded in that attractor. It may so happen that one of those unstable periodic orbits represent a very desirable behavior of the system.

For example there is an engineering system where the desirable behavior is represented by one of the unstable periodic orbits, but it's unstable. Therefore if you really release it there, it will not remain there. It will go away elsewhere but if you can identify one of those unstable periodic orbits as representing a desirable behavior. For example there have been the situations where there was a laser and you see people want to improve the power throughput in the laser guard. That depends on which periodic orbit you are actually working in. They were shown to a situation where the power can be doubled, if you lock into one of those unstable periodic orbits. There are similar situations in other areas also. The problem then becomes, can we control the unstable periodic orbits embedded within a chaotic attractor. The reason that I showed that this is especially advantageous is that for a non-chaotic system, the kind of system that you have all come across in regular control theory courses are where if you want to bring about some change you have to put in some control action.

In order to get a desirable behavior which may be quite different from the presently operating behavior. That means if you want to get in to a large change in the resulting behavior, you have to put in a large control action. That means in the kind of systems that you have already come across, not in this course elsewhere. There large control action is required in order to bring for some kind of a large change in the character of the system. Small control action will only result in a small change. But here in the kind of system that we have been discussing, there exist a possibility that tiny change in the parameter or the state might lead to a very large change in the resulting behavior because there is sensitive dependence on initial conditions, there is sensitive dependence on parameters. There exists the possibility that a very tiny notch, a tiny perturbation, tiny very directed control action might lead to very different system behavior and that system behavior that very different system behavior might be the one that is desired.

You now have the possibility of enabling a control action by very tiny perturbations. How can we do that? That is the subject matter of today's topic. The point is this could be handled in the plane of continuous time dynamical system but as you know mostly such things are handled more easily. The mathematics becomes much simpler if we treat these problems in discrete time. We are essentially talking in terms of the Poincare section and what happens there. If you consider what happens there then you will find there is a chaotic orbit and if you place a Poincare section there are an infinite number of points through which this piercings happen. But out of that one represents an unstable periodic orbit and that on the Poincare plane is an unstable fixed point of the Poincare map.

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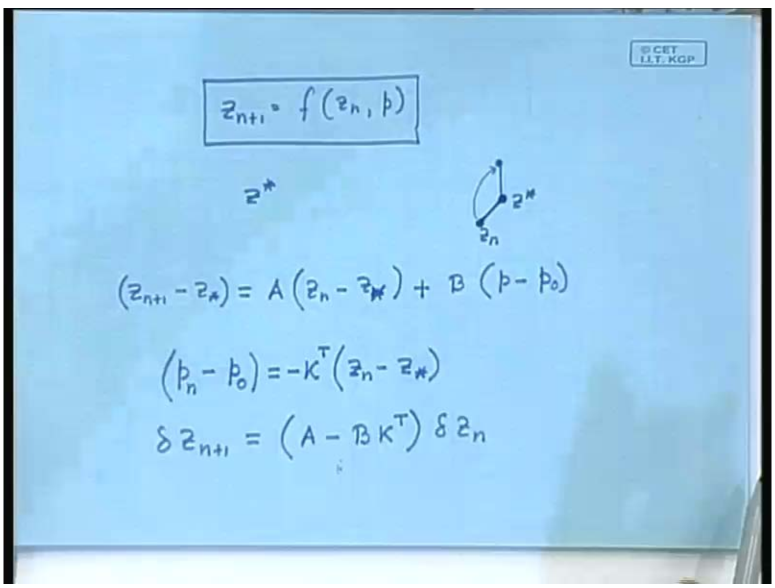

Suppose that Poincare map is represented as Z_n +1 is equal to some function of Z_n and the parameter. Z is the state. We are now bringing the problem down to the state of discrete time representation and this is the discrete time representation of the system. Then in the neighborhood of that unstable periodic orbit, the one that you are trying to stabilize suppose that unstable periodic orbit is represented by Z star. Then in the neighborhood of that suppose here is the unstable periodic orbit and this is the Z star and here is say Z_n. Presently the deviation is this much.

In the next iterate suppose it mapped to this point. See it went like this. As a result the deviation is this much. Then this deviation can be expressed as a function of this deviation. In fact in the local linear neighborhood this can be represented as a linear mapping. We can write that local linear approximation as Z_n + 1 minus Z star. That means the resulting deviation is equal to, we are now writing in terms of the linear. There would be a matrix multiplied, Z_n minus h Z star. This will be the relationship as dependent on the state. Now see the original thing was also dependent on the parameter. You can also write another parameter dependence factor as B (p minus p_0) where p_0 is a parameter for which you had obtained z star. This representation is nothing, very simple. It is essentially if you consider no change in the parameter then this part is zero. Then you are saying that the deviation in the next iterate is nothing but a matrix time the deviation in the previous iterate. That means you are essentially representing as a linear equation.

Similarly if you suppress this that means this remains constant then the change in this brought about by a change in the parameter is given by the matrix B. Here you have a local linear representation. Now what will you do? You will essentially observe how much is this deviation. That means you will observe in any particular iterate, how far have I fallen away from the state where I want to be. That means this is the state where I want to be and this is where I have fallen. I will measure this and depending on this, I will change the parameter by a small amount, very tiny amount but nevertheless I will change the parameter depending on this. We can write this again like a linear relationship as the P in the nth iteration. That means I am assuming that it is possible to change the parameter at every iterations by small amount but it is possible to change by a small amount. P, the parameter in the nth iterate minus the P nominal value that means how much perturbation I am giving in the parameter that should be dependent on some constant matrix times, the deviation in the state. If I have deviated so much, I will give so much parameter perturbation.

Here there has to be a transpose because obviously you need to bring it to a one dimensional state. Generally it is represented as minus k transpose that means if the deviation is positive, you give a negative deviation in the parameter. This is again a very linear way of looking at it. I will show that also works because in the neighborhood of that unstable fixed point, you can always locally linearize it and that behavior is essentially this behavior. So we can write it like this. What is the dimension on this? The dimension of Z that means if it is n dimensional then this would be n dimensional stuff.

Naturally this has to be an n dimensional into one dimension that their product gives a one dimensional material. Now if I substitute it here what do we have? We have Z_n + 1 minus Z_n, let us write it as delta Z_n + 1. See we are substituting it here so this Z_n minus Z_n star which is nothing but delta Z_n or deviation in the nth iterate that remains common. What do we have here? You have A minus BK transpose times delta Z_n. If the initial deviation were this much, this will be the final deviation after one iterate, if you are giving a parameter perturbation of this extent. Now do you see what is the condition for stability? Simple, this matrix here must have Eigen values all within the inverse… (Refer Slide Time: 00:14:00). That's it, as simple as that. Just ensure that this matrix has Eigen values inside the unit circle that immediately guarantees that in successive iterates, this deviation will die down to 0. But normally the way it is done is obviously if the Z_n is large, see this relationship. If the Z_n has large deviation from the Z_n star then what will happen?

This number is large, as a result multiply it with the K transpose it will give you a large perturbation. Mostly we don't want it because the main advantage of the chaotic system is that we can do it with a small perturbation. What we do is if the deviation is large we simply wait. No, this is not the right time to apply the perturbation. Because we know that if the system is chaotic it's also ergodic means that if we wait sometimes then sometime or other the state will fall in a close neighborhood of Z star. When it does, apply the perturbation. You simply wait till this term becomes tiny enough so that if your controller says that my control will be only this much and no further. Then simply wait till it comes within that range and then apply. You would normally see that there is a chaotic system going on, it moves chaotically and the moment it falls within that small neighborhood immediately you apply the control action and there it is. It immediately gets locked to the unstable periodic orbit.

Now this getting locked to the unstable periodic orbit or getting controlled into the unstable periodic orbit, you might visualize as something like this. Can you can make a stick, stand on your finger? So that's an inverted pendulum position. You know we have already said that it is a saddle, an unstable fixed point, unstable equilibrium point. You still can move it in tiny bits and keep it vertical. You don't really need to go around moving a large amount. You can do it by small amount. Can you or not?

The way you can, even here you are doing exactly that. Under work condition we will need to move it by large amount, if the position of the stick is like this. Then you will have to move it like this so that it becomes vertical. But if it is very close to the vertical position, you can always move it by very tiny amount to keep it vertical. That is exactly what we are doing. A is essentially the Jacobian matrix and B is this one as differentiated with respect to p. Here is the Z_n + 1 as function of Z_n and p. If you differentiate it with respect to Z_n, you get the Jacobian matrix which is A. If you differentiate with respect to p, you will get this B. But you might ask in that case this functional form has to be known. Otherwise how do I differentiate? I will come to that issue a little later.

This algorithm has been applied to many different situations but one of the most that goes into scientific focular is the paper where the scientist created artificial fibrillation in a frog heart. They dissected the heart that was still alive, heart of a frog and they induced artificial fibrillation, the way a man dies before that there is a fibrillation. That's why you put deep defibrillators and all that. They induced this and then they applied small tiny notches and they were able to stabilize the heart for a long time. That was a nature paper that has became a sort of a turning point in the application of this theory. It has been applied to lasers, it has been applied to many different areas. Let us try to understand this scheme somewhat intuitively. What are we doing? Suppose the fixed point Z star is a saddle fixed point.

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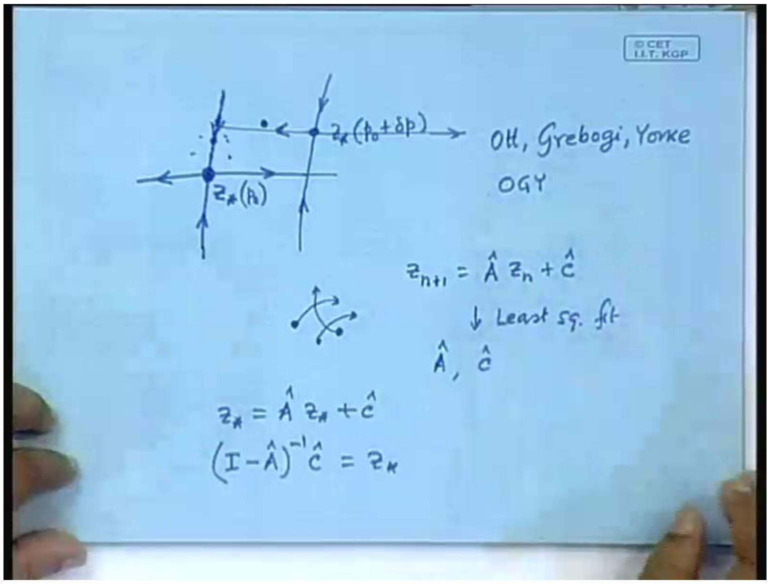

Here is your Z star in this particular position. A saddle means there would be the manifolds which in this case will be a stable manifold and unstable manifold. Suppose this is the unstable manifold and this is the stable manifold. Here is your Z star. Suppose at any particular point of time it lands here. What are you trying to do? You are trying to move it back here but notice here you have the advantage of having a stable manifold. Means that if you can somehow notch it to this point then you don't have to do anything. Automatically it will run into this. Essentially the point is to push it here. Now what are we doing? We are changing the parameter. A parameter change means for that changed parameter, if you now calculate the fixed point it will be a different fixed point. Fixed point position will be different.

Now suppose the fixed point's position is somewhere here, it has moved. That means this is Z star for P_0 and this is the Z star for P_0 plus delta p. It has moved here. As it is moved here, it will again have the stable manifold and the unstable manifold. This is the unstable manifold and this is the stable manifold. What will be the character of the unstable manifold? What will it do to a particular orbit sitting here? In the next iterate how will it move? It will move away more or less in the direction of the unstable manifold and it will move towards this in the direction of the stable manifold. As a result in the next iterate it will fall somewhere closer here because it will move towards this and towards that. By proper choice you might make it fall just here. That means exactly on the stable manifold of the earlier fixed point and then withdraw that perturbation. The fixed point comes back here.

Now you have the point exactly on the stable manifold. Just wait, it will automatically get there. What have you done? You have just applied a perturbation at a particular instant and then left it. Only once you have applied a perturbation and that's it and then we drew the perturbation not that you are keeping on the perturbation. But the action of this specific geometry of the system ensures that the iterate slowly converge on to the fixed point. We want that to one.

It’s not difficult to see that if you want to bring it exactly on the unstable manifold here. All that you need to do is to ensure that one of the Eigen values here is zero and the other one is what it was without the perturbation. You can easily calculate the K matrix because a and b are known here you have equation, one of the Eigen value should be 0. The other one should be as it was without the perturbation then it will simply be landing here. It’s extremely simple to obtain the K matrix. Of course all the time you might not need to set it exactly equal to 0 because here you are exactly putting it on to the stable manifold. That doesn’t always mean that it will remain there because the system has some noise.

Though I am saying that if you land it here, it will automatically come to this. It doesn’t really. This argument holds in absence of noise. If there is noise then obviously it again goes out. The moment it goes out, you wait for some time but again you apply the perturbation. So that this algorithm actually rests on repeated application of the perturbation depending on the amplitude of the noise. This method was invented by Ott Grebogi and Yorke and that is why this method is called OGY algorithm or OGY method.

Now he asked that question. What about A and B, how do you know them? Obviously if you have the system equations given like this, obtaining A and B are trivial but for a realistic system if experiment is running you don’t really know A and B. You would like to somehow estimate A and B. How to estimate A and B? What is happening? You have the system running, actually it is a continuous time system and you are placing a Poincare section, you are observing the points on the Poincare section. First what will be the character of the orbit when it comes in the neighborhood of that particular fixed point? Because it is an ergodic so it will go on moving everywhere and it will come arbitrarily close to the fixed point also. If it does what will be the character of the next iterate? It will again fall close to that because it’s close to a fixed point.

If you have a time series on the Poincare section that means this point to this point to this point to this point and if you have a file containing the time series, if you go on scanning the file you will find two lines that are very close to each other. If you do detect then you know that here, the point raised a close neighborhood of that unstable fixed point. Take these two values, again if you keep on scanning it again sometime later it will come very close to that. That means you will again get a pair that are very close to each other. Take those values.

Similarly by scanning the whole file containing the data of the system, you will get a large number of such pairs. The pairs that fell close to each other. That means you have got now a collection of points that fell close to each other. Now you are trying to estimate the values of A and B from there. What do you have? You have got a point that map to another point, you have got another point that map to another point, you have got third point that map to third point. All these data you have. Essentially you have Z_n mapping to $Z_n + 1$. As we have seen that you can represent it as a linear function like $Z_n + 1$ is equal to A plus c, it is like a affine transformation you can represent it like this. These hats I am putting because these are the values that we need to estimate. Essentially since we do not know, if we had known the fixed point then we would say that now let that fixed point be my origin and will count only the deviation from that origin, from that fixed point. But here we do not know the location of the fixed point.

We only know that this point map to this point but we only know intuitively that this fixed point must be somewhere in that neighborhood. We do not know the fixed point. If we do not know the fixed point, we are working on the original coordinate system. We are unable to move the origin to the fixed point and that is why we need to consider this C. This becomes an affine transformation. Once you have a large number of these values Z_n + 1 and Z_n, you can do a least square fit to obtain the values of A matrix and C matrix. You can do a least square fit to obtain these values A matrix and C matrix. Once you obtain so from here fit to A matrix and C matrix and you obtain. Now the question is can you locate the fixed point? Yes you can because once you have obtained it, you will say the Z star is equal to A hat Z star plus C.

How would you obtain Z star? I minus A into Z star so this has to come to this side, inverse so that will be the Z star. We have located the fixed point. Z star, the fixed point is we did not know where it is but simply from observing the data we can locate the fixed point. But for that we need to first obtain the A hat and C hat. Then just do this obtain the position of the fixed point. That means we have been able to estimate A matrix, this is the same as the matrix A that appears. Here estimated A matrix from the data. But what about B? Now B is obtained as you can see, B is where you are asking the question if I change the parameter how much will my fixed point move? That is the essential question you are asking. All you need to do is to give some perturbation to the parameter and redo this procedure as a result of which the Z star will change. Then you ask how much did my Z star change due to a unit change in my parameter that is the B.

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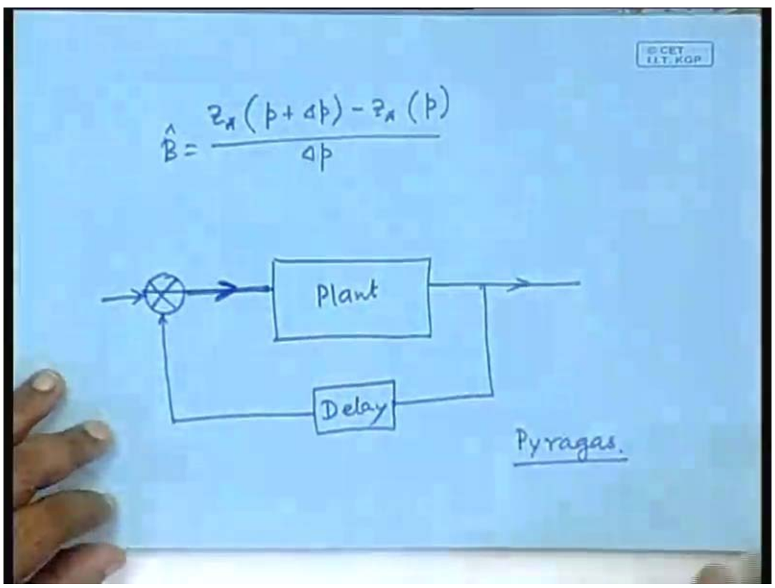

B is nothing but Z star of p plus delta p that means Z star has a function of p plus delta p minus Z star at p, this whole thing divided by delta p. Estimated B is this. You see you did not really require the system equations. You simply observed the evolution and from there, you could extract all these information's. Not only that, I have already told you that there are certain situations where you have only one train of data. That means some experiment is going on and you have access to only one state variable not every state variable.

You don't even know what is the complication of the system, how many state variables are there, you may not know. You have just one variable that has been measured. In that case what do you do? You do the delay coordinate embedding that means you create additional state variables by delay coordinate, you do the same thing. Then that delay coordinate system you place a Poincare section, you can do that. You can thereby obtain these points from which you will do the estimations. You can do that. From there you can estimate A and B and C. Yes, that is also possible. It has been demonstrated that this whole thing was even if you have access to just one state variable and then you can decide how much should the perturbation B that we apply in the parameter. You apply the perturbation in one instant and simply wait. You got the point? Still there is a problem.

The reason that people were so very excited about it is essentially the cardiac problem of humans. It is known that as a man goes close to death because of cardiac failure, essentially the dynamic which is a periodic dynamics that changes to various high periodic orbits and to chaos and naturally the problem becomes how to control that chaos. Presently do you know what is done? Presently they implant what is known as a defibrillator, this big device. Some of your fathers may have already, I know people who have and it is extremely painful when it really strikes because whenever that stability is lost, it is going into a high periodic orbit. That means you see erratic oscillation of the heart then the defibrillator works.

What it does? It gives an enormous shock that means it just gives a big nudge so that it gets into again the regular periodic orbit. It does and the man survives but there is due to that shock there is often death of cardiac tissues. That means the life is prolonged but not very long and also that particular event is very painful for the patient. I mean I have met people who have undergone that. It's like somebody dropping on the chest of the person, he feels like that. The idea is that can you then instead of giving one big shock, can you give tiny nudges to get the orbit back? It has been successful in non-human hearts. Yes, people as I told you that it has been done for the frog heart. Formation has not been obtained to do it on human heart as yet. That means it is done on by dissecting, opening the heart, keeping it alive and doing that. It has so far been successful.

That means you do have one train of data coming, from there you do delay coordinate embedding, from there you do estimate these values and depending on that you give very small amount of electrical pulse. Instead of any other thing, electrical pulse is the most convenient thing to apply here. Give a tiny electrical pulse and that does it. This has been shown and it does stabilize. But the question is when I said that if the state is far off from that equilibrium point, from the fixed point you simply wait till it comes back. For a patient will you wait? Will you wait long enough, let it come back it is ergodic. It will soon or later come back and the fellow might die before that.

Obviously the question comes that how can we quickly bring a state to a desirable state? That means instead of waiting, I want to bring a state quickly to a desirable state. Is it possible? Yes, it is possible only in chaotic systems because in a non-chaotic system you will again have to give a large change, large perturbation in order to move a state from one point to the other. While in a chaotic system, the advantage is that slight tiny perturbation can result in a large change. The question is how can we make a tiny perturbation so that within a very short time, I will get where I want. That is the problem then.

Let us illustrate that algorithm, that's what targeting. So far what I was discussing was it is known as control of chaos. By the way before going to targeting let me give you the idea of another algorithm that has been very widely used. That is supposing you have got a chaotic system whose data is coming and here is the plant which is chaotic. You make a feedback loop, here is the input and here there is a feedback loop. Here is the plant that is now behaving chaotically and you want to control it into one of the unstable periodic orbits. What is this fellow? This is nothing but a delay. What are you doing? You are taking the output, giving it a delay and adding to the system behavior. What will be the result? This is the error that is going into the plant. This error will be, if the error is non-zero then it will lead to some kind of a control action. It will be zero only when the delay is such that it is exactly the same delay as or same period as the periodic orbit that you want to stabilize.

Suppose you have got a delay and you have got a means by which you change the delay. Then what will happen? So long as its period or delay is not the same as the period of the unstable periodic orbit nothing will happen. But the moment it becomes the same as the unstable periodic orbit, immediately this fellow will go to zero and you get a locking on to the… This algorithm was invented by a person called Pyragas and that is why it is called as the Pyragas state and this has also been applied to many chaotic systems. Now let us come back to the issue of targeting. In targeting essentially what are you trying to do? We want to reach that state, some desirable state in the least possible time. Again we will use an algorithm which has no equivalent in non-chaotic systems. These are very dependent on the sensitivity dependence on initial condition.

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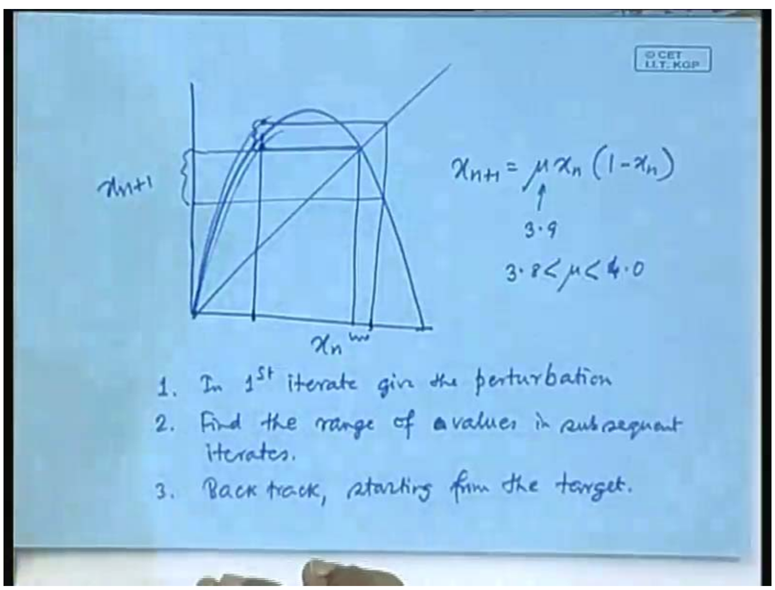

Let us illustrate that with the logistic map. You have got the logistic map here. This is x_n and x_{n+1}. Suppose you are here and you want to reach here. One logic would say start from here and keep allow it to oscillate for long time. Soon or later it will come here, yes that's true. But then we said we don't want to wait till then. The equation here is as you know x_{n+1} is equal to mu x_n 1 minus x_n. Here is the parameter. Assume that you can vary the parameter as your control action by tiny amount in any iterate.

Now if you have the variation. Suppose it is now for a chaotic behavior, you would need something like 3.9. The 4 is fully chaotic but suppose it is 3.9 which will give you a chaotic behavior. Normally it is 3.9 but suppose you have scope to vary it in the range 3.8 less than mu and 4.0. That means this way 0.1 and that way 0.1. That is the extreme amount that you can vary. Then what will happen? Supposing these were here and this would map to this point. If you had used 3.9 then it would map to some other point. If you use 4.0, it would map to some other point. Because for 3.9, the graph would be slightly different like this. For 4.0, the graph would be slightly different like this.

In one case it maps here, another case it maps here. There will be a range over which it maps. It has a capability of mapping for this range of mu. If you have the option of varying mu over this range, you have the capability, you have the option of reaching this range in the next iterate. If you now withdraw this perturbation then this range will map to some range. How will you obtain it? This range you bring to the 45 degree line and you bring here so you have got this range. In the next iterate this range will map to this range. Do you notice that because of the stretching behavior, this range is slowly increasing. After some time this range will increase and within couple of iterates it will include the point that you want to reach. What was our logic? I will apply a small nudge now, directed in such a way that 3 iterates later I will be where I want to be. Do you see that is possible now? What you need to do? You need to find out that if I give the parameter perturbation now, what is the range of values I will reach in the next iterate? Which is the range of value that I will reach in the next iterate? Start from that range of values and find out the range of values in the iterate after that.

Very soon, you will find that the target is included within the range. The moment you have found that the target is included, you have found how many iterates did I need in order to reach that. Now notice the argument. One, in first iterate give the perturbation. Two, find the range of values in the subsequent iterates. Suppose you have found that after the third iterate, the target is contained in that set. Then it is only pressing the calculator. Then what will you do? You start from that target point and back track. Starting from there you will back track that means x_{n+1} you know, you calculate x_n, back track and finally at the first iterate then you will land up in an equation like this where you need to know mu. You will be able to calculate that. It only requires pressing a calculator not even a computer. So 3 is back track, starting from the target which tells you that in the first iterate if I give only that much of perturbation say from 3.9 it needs to be made 3.83 may be.

In the first iterate you change the parameter to 3.83 and then bring it back to 3.9 and let it run. Automatically it will come there, you understood the point. How could this be possible? This was possible only because of the sensitive dependence on initial condition. No other reason, it was possible because of sensitive dependence on initial condition. Here we are using the sensitive dependence on initial condition, it’s not difficult to see how to apply this. This I illustrated with a map. How will you apply it to say 2 D map?

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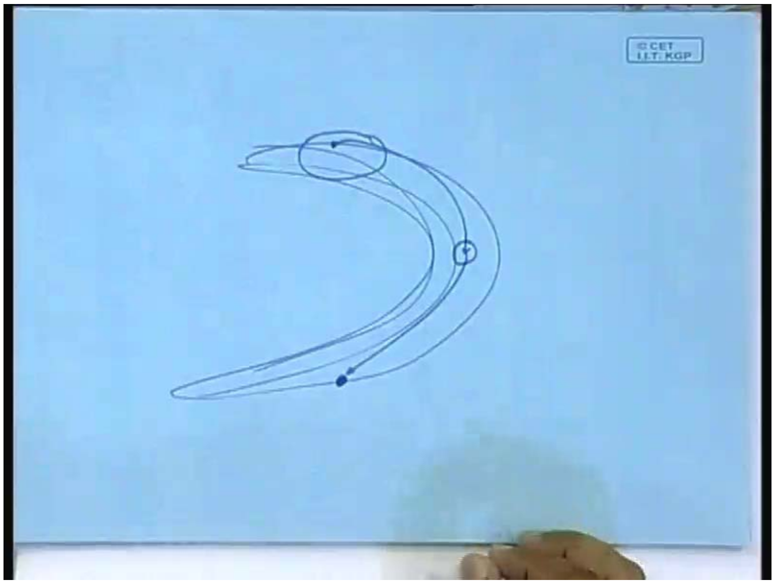

Suppose in the Henon system, the orbit is something like this. Suppose you are here and you want to reach this point. How will you do it? Give the parameter perturbation in the first iterate that means there is a range of parameter. If you give that range of parameter this point maps to say this point, this range. Now withdraw that parameter perturbation. This range will map to say this range and then you find that this fellow is included which means that it is possible to reach from here to there with only two iterates, two jumps. Then you calculate how much was the exact change necessary in order to get from this to this point to this point, easily be done.

In case of continuous time system essentially the logic is the same. You start from a particular state, you are trying to target that state. Now if this state is my target then how would I go? From here I will apply the tiny perturbations for some span of time. In this case one iterate, in that case some span of time say 5 seconds. I say that I will apply a tiny perturbations for 5 seconds. Apply the tiny perturbations for 5 seconds, you can easily find out by solving the equations. You can easily find out the ball in the state space that will be reachable. Then let that ball evolve, after some time you will find that the target is included in that ball. Find out the time that was necessary in order to reach from here to there, then back track. You can find out the exact amount part of perturbation that would be necessary in order to reach from here to there.

This logic has been applied in a very unlikely scenario I suppose it was 1988. There was a satellite that was almost nearing the end of its time. There was a comet coming, the NASA scientist decided that this particular spacecraft could be used to observe that comet, you wanted to have a cometary encounter. But the distance was something like that 50 million miles and the fuel was almost exhausted. There was only a tiny amount of hydrazine fuel left and you could give only tiny nudges and you wanted to reach 50 million miles. How is that possible? They calculated because the three body system earth, moon and the satellite is a chaotic system. Therefore there is a sensitivity dependence on initial condition and therefore it should be possible to hold the spacecraft over such a large distance just by using a tiny perturbations.

They calculated that, they did that and they reached there and that was the first planetary encounter. It essentially involved 5 rounds around the moon. The actual orbit was rather complicated but they had to give only tiny perturbations so that the gravity would steer the system to that point. These are typical applications of this logic, happens only in chaotic system you cannot have this kind of things happening in non-chaotic systems. We are very close to the end of the course essentially through this course we have learnt some very typical features of nonlinear system. We have learnt that not all nonlinear systems are chaotic but all chaotic systems must be nonlinear systems.

Normally in regular control theory course or whatever course you have learnt in engineering, they will look at only the linear system behavior. All these possibilities are essentially left out of ambit of what you learn. That's sufficed more or less for a 19th century engineer or 20th century engineer. For a for a 21 century engineer that often does not suffice because firstly most of the things that one has to deal with are nonlinear. Secondly, earlier we used only the linear theory in order to design control systems, now no longer. Probably you know the fighter aircraft are all open loop unstable. They make it open loop unstable because otherwise there is no other maneuverability and then you need to stabilize with your hand. Here is a control system, control that you exercise on a system that is chronically unstable. So likewise many other systems are designed in the same way to have maneuverability.

You can easily see that chaotic systems offer additional advantage in maneuverability because in a chaotic system there are enormous number of unstable periodic orbit involved and you can switch from one to the other. You do not need to design say 100 different systems for 100 different works. Just one system stabilize a particular periodic orbit, you have the behavior that you want. The versatility, the width of the different types of behavior that are possible, these are offering many advantages to chaotic systems. So that now it is even thinkable to design systems chaotically to be used in engineering applications. That was say 5 years back nobody was thinking about that but now they are. By the time you guys become full-fledged engineers, you will find more applications and then the things learned in this course might prove to be useful. I suppose that will be enough for this course.
Thank you.