Chaos, Fractals and Dynamical Systems
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Lecture No. # 03
Limit Cycles

At the end of the last class, I gave a problem but before I go there, I think I should mention a few things that come from the problem that we tackled before that.

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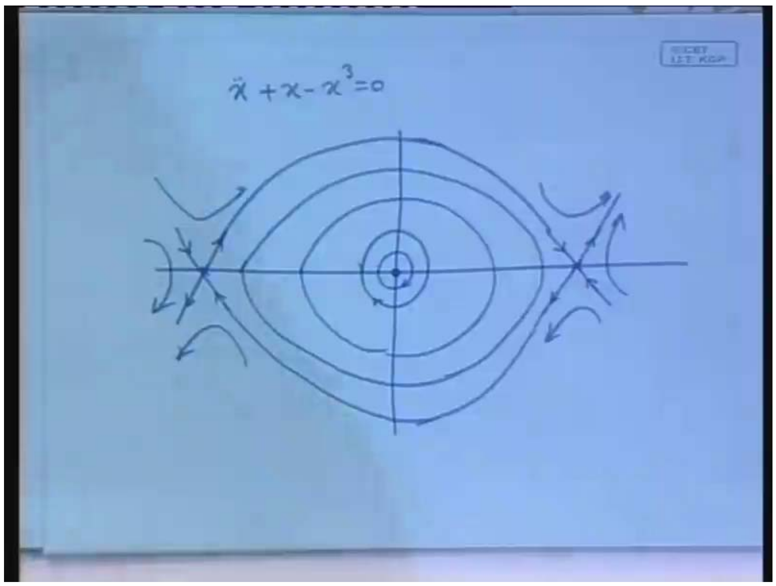

x double dot + x - x ^{3} = 0. You have done that. That teaches us a few things. Let's talk about that and then go to the next one. So its vector field as we have obtained, there was one equilibrium point here, another at +1, another at -1 and we had noted that for the equilibrium point that was at the origin, its character was like a center. The Eigenvalues were purely imaginary and so far, we had inferred that they would be circular orbits. Here what was it? It was a saddle. So it should have two outgoing Eigenvectors and two incoming Eigenvectors. Likewise here you had the same thing. Then we said that these guys will bend around and will meet these. So we said that here there will be orbits like this and so on and so forth. It will be more or more distorted but nevertheless closed loops and here there will be orbits like this. That is what we say. It tells us a few more things. First how did we infer that the orbit here would be close loop? We had locally linearized it and obtained the Eigen values and had found that Eigen values are perfectly imaginary.

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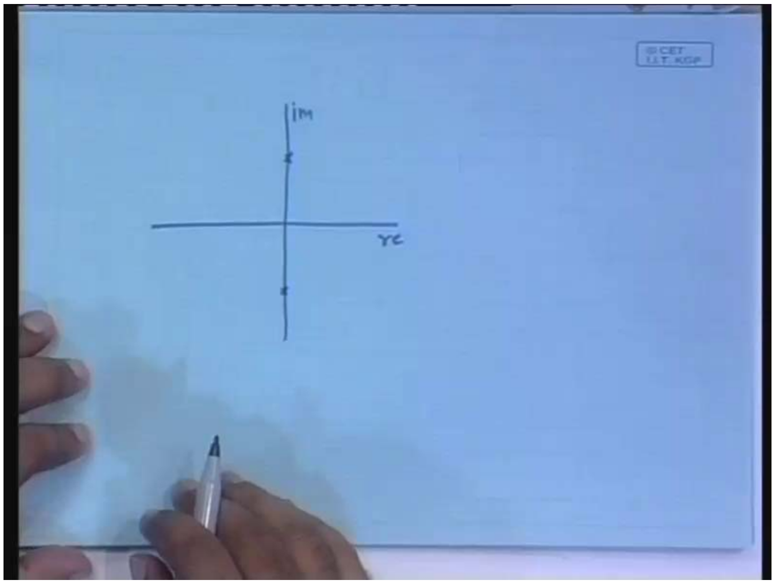

Now the perfect imaginary Eigenvalues perfect imaginary Eigen values would be in the complex plane somewhere like this in the complex plane. Now you very easily notice that if you give a slight perturbation to these Eigenvalues, you will either go this way or that way. So the purely imaginary Eigen value is a marginal case between two possibilities. Either it goes this way or it goes this way. If it goes this way, it would be incoming spiral orbit. If it goes that way it will be outgoing spiraling orbit. So notice in addition that the basis of our argument that whatever we talking about is valid only in a small neighborhood of the equilibrium point. Naturally that question will be to how small a neighborhood and if you extend the logic you will realize that the smallness is arbitrary smallness. So only as you look at it at an arbitrarily small neighborhood, its character will be perfect circles. else it might not be right because after all you had a non linear system you are locally linearizing if you go for it will be it will assume the character of sort of perturbing the eigen values. So it will either go this way or go that way which makes you conclude that the conclusion that here the orbits will be circular or periodic orbits is questionable. Why was it questionable? It was questionable because our Eigenvalues were at very special positions. If it is to the left side just perturb it and not much change happens. If it is in the right hand side, it unstable and perturb it, the behavior remains more or less the same. But if it is on the imaginary axis, that is not true. Part of it either belongs to one type or it becomes another type. There is a very large difference that happens. that is why the systems were the eigen values are on those very very special locations are given a separate name because we need to be careful about them. They are called non hyperbolic orbits. Hyperbolic systems are where the Eigen values are not on the imaginary axis. The ones where you have Eigen values exactly on the imaginary axis the idea would say that “Be careful!” what you are arriving at by the local linearization? It may not really be true. So how do you check whether it is really true?

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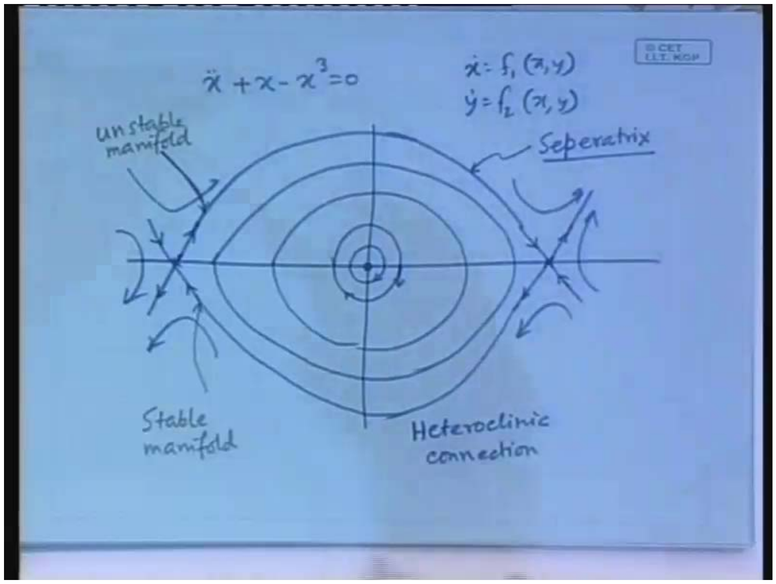

You simply start from somewhere here. Then use the original equation and you can evolve it. Using the fourth order Runge-Kutta method or whatever you have learned in the numerical analyses classes. If it really comes back to the same position, you know that "okay, till here the local linearization is valid. Else it is not." for example this you can go back and run this particular system on Matlab. You will find that it becomes an incoming spiral orbit. So locally that you would infer that the local linearization is valid for that system in an infinitesimally small region. Our conclusion obtained from the linear systems theory is then questionable.

The question is that, in order to extend the ranges, initially we obtained in these three ranges and then we extended it with the equivalence with magnetic lines of force. How do you know really the magnetic lines of force analogy is valid? What is the logic behind it? Well if you look at it you have the equations of this form x dot is equal to f 1(xy) and y dot is equal to f 2(xy). the moment these are given obviously at every point there can be a unique vector. in that sense, at every point there is a unique direction of the magnetic lines of force. So in that sense you have got a similarity. And why did I bring it? I could have introduced at initial from here. But then we have some kind of a concept brought from our school days about the character of the magnetic lines of force. If we see the similarity it becomes easier for us to understand. You cannot infer the clockwise or anticlockwise direction from there because the result of an imaginary Eigen value or a complex conjugate Eigen value could result from both clockwise rotation as well as the anticlockwise rotation. You cannot simply look at the Eigen value and say it's clockwise or anticlockwise. You really have to look at the vector field and the way to look at the vector field is place your pen here at this point and calculate the actual vector direction. If it is this way then you know that it is clockwise rotation definitely.

His question is that supposing somewhere it is a circle and then somewhere it is a spiral so at some point they have to intersect. They cannot because the moment they intersect transversally, at a point it will result in two vectors.

That cannot happen because the vector at every point is given. That immediately gives you the concept that the scenario that you are talking about cannot happen. If there a spiraling orbit, the spiraling orbit will asymptotically hold on to the circular orbit. Nothing else can happen really. The moment you understand these concept many thing fall in place. There is another idea. That is you see the the line here what does what character does it have? Let’s start from the idea. What is the definition of an eigenvector? The definition of the eigenvector is that if you have a point on the eigenvector, then it will always remain on the eigenvector.

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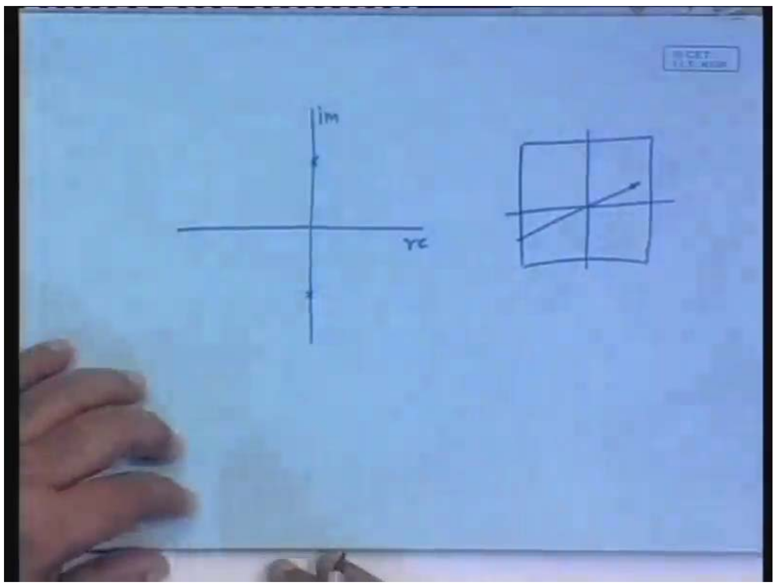

In a linear state space given by a linear set of equations, then you have the axis here and you have got an eigenvector here. What does it mean? It means that if you take an initial condition here throughout its evolution will always remain there. That is the concept of the eigenvector really. In mathematics books, you may find more abstract concept but this is what I find visualisable and easily understandable. Now in a nonlinear system, the same idea can be extended. In the sense that notice that this is no longer a line. This line actually bends. Why does it bend because the system is nonlinear. Had it be linear that means these lines would be extended at infinitum. However since the system is nonlinear, these lines bend. But nevertheless they retain the character. If an initial condition is on this line, it will always remain on this line. So they have in that sense a similarity with the question of the eigenvector. Now the eigenvectors locally can be stable eigenvector and unstable eigenvector. This one for example is unstable eigenvector. This one for example is a stable eigenvector. So this line has the character of an eigenvector but not really an eigenvector. Such lines are called unstable manifold. This line is called the unstable manifold and this line would be called a stable manifold. So let us now talk about some more mathematically correct definitions. The manifolds are subspaces. It is a two dimensional space. A subspace would be one dimensional in which a specific property can be assigned. Now in this case, what is the specific property that on that subspace, the orbit on this subspace, for example the orbits are converging. That is a stable manifold. On this subspace, the orbits are diverging. That is an unstable manifold.

But then in a nonlinear system it may so happen that the unstable manifold as it goes around, it may become a stable manifold. It has become the stable manifold here. So start from an initial condition here what will happen? Exactly on this line it will go along that it will converge on to this. So an unstable manifold has become a stable manifold. So there is some kind of a connection between this point and that point and the connection is established by an unstable manifold going around and becoming the stable manifold. So that establishes the connection. It is called ‘a heteroclinic connection’. Why /hetero’? Hetero means more than one. One is homo. So it is easy to see that if there is a heteroclinic connection there is also the possibility of homoclinic connection.

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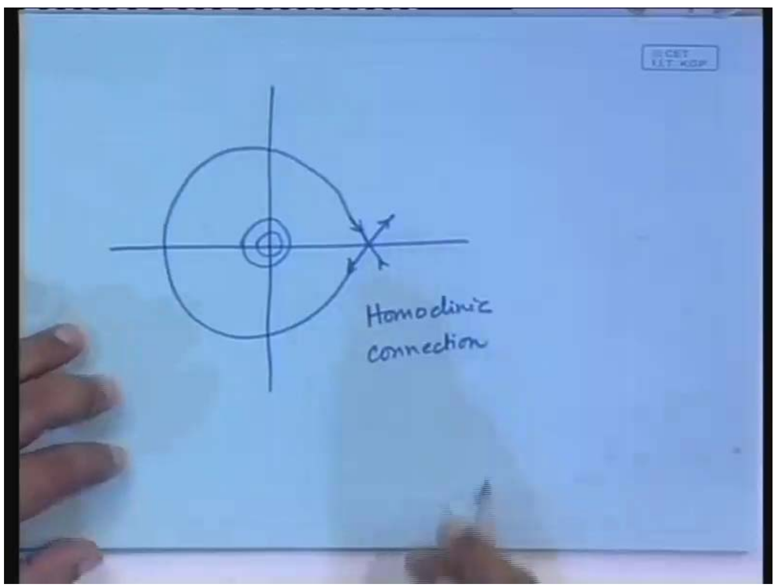

It is not difficult to imagine that there can be vector fields of this type. Suppose this is the equilibrium point. These are circle equilibrium point for example and here there will be a saddle equilibrium point, then it is possible to have it this way. If that happens, then it is a homoclinic connection. The same equilibrium point is connected through both its unstable manifold and the stable manifold. It goes around and connects it. These are also possible and happens. So this is a homoclinic connection. In some books, you will find another nomenclature. For example you would notice that these two homoclinic connections established a sort of island. Within this island the behavior is stable. Outside the island the behavior is the unstable. So it sort of separates out two different types of behavior. That is why in some books you will find that these lines have also been called seperatrix. But I prefer not to call it seperatrix because if you are calling them unstable manifold and the stable manifold, just call them. I don’t prefer having one name for the one thing and the same thing you call by some other name. That confuses to you. but for your convenience because you will be studying from many books in some books the same thing has been called seperatrix. But for our purpose we will call them stable manifold and the unstable manifold. Essentially that is extension on the idea of eigenvectors. Remember if you can identify the unstable manifold and stable manifold, then at the saddle fix point, the eigenvectors are a tangent.

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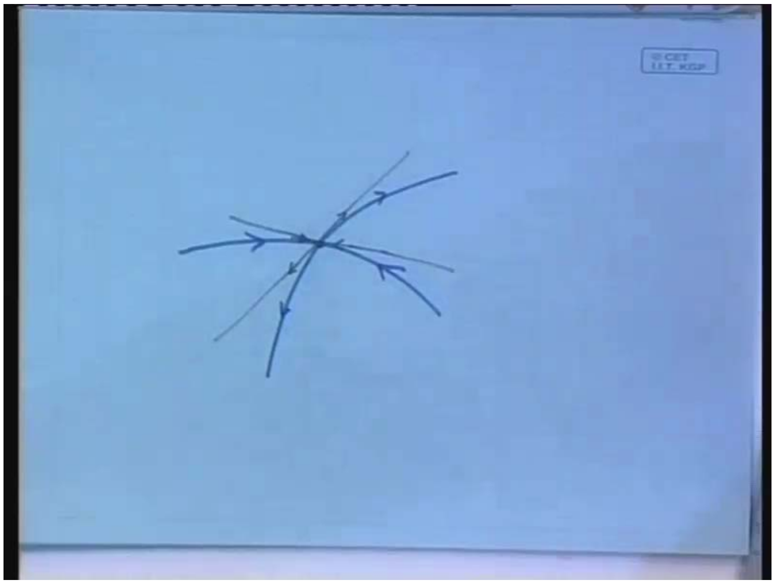

Suppose here is a saddle point and you can identify that this is an unstable manifold and say this is a stable manifold. Now see I am drawing unstable and stable manifolds as curved lines because normally that will happen in a non linear system. But then you can also conclude that if you locally linearize them you can always obtain the unstable eigenvector and the stable eigenvector and they would be tangents. So the unstable eigenvector is tangent to the unstable manifold at the equilibrium point. The stable eigenvector is tangent to the stable manifold at the equilibrium point. Unstable manifolds are normally curved lines in a nonlinear system. So we have just extended the idea of Eigen vectors. How we extend the idea of Eigen values? I will come to it little later because that will be a little more but at this stage, you can understand what these are. Now let us come to the problem that I gave in the last class.

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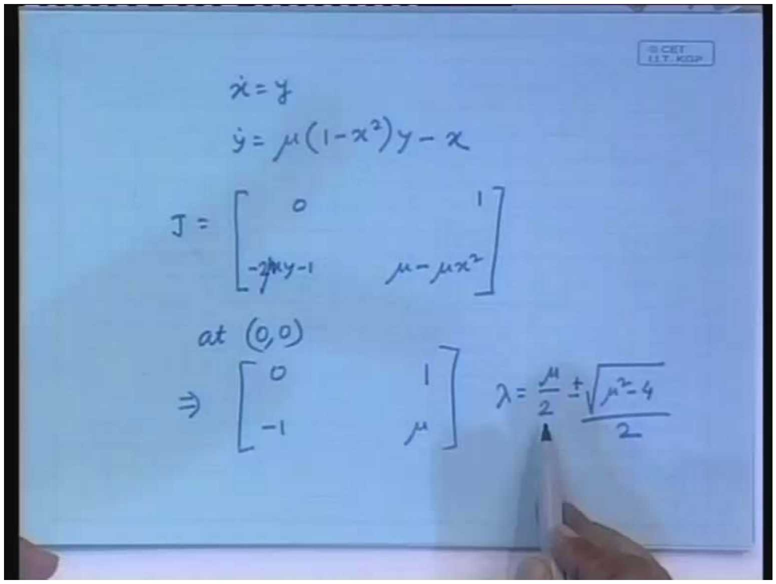

What is the problem I gave? It was x dot is equal to y and y dot is equal to mu (1 – x 2) y – x. so what its behavior. Of course, the equilibrium point is only one and that is at (0, 0). If you locally linearize the equilibrium point you get the Jacobian matrix as zero one minus twice mu xy - 1, mu - mu x^2. At (0,0), this will take the form zero one minus one, mu. Now I say that now you obtain the Eigen values and see what happens as mu changes. It changes to zero. First you consider a negative value then you consider the zero value then you consider a positive value.

What will lambda be? Lambda = (mu/2) +/- (root mu^2 – 4)/2. The transition happens at =/- 2. So at that value of mu, there will there is a there is a transition happening. Suppose I take the value of mu as plus two. +2 means this fellow is positive. So you have got lambda = 1, a positive value. Now slowly start reducing mu. At that point the two Eigen values are exactly the same. If you have it greater, then the Eigen values are different, so greater than +2, they are different. At plus two, they are the same and below plus two they become complex. Complex with positive Eigen values. As you go on reducing in further, at mu = 0, it becomes exactly imaginary Eigen values and when it goes to to negative it becomes negative real part. Now you imagine that you have a system in which mu represents some kind of a parameter which can be continuously varied. The parameter in this physical system will have the mass. In case of the the pendulum the length of the string, air friction and things like that these are all parameter there can be parameters that has variable also. In an electrical circuit, you may have an, input voltage that can be varied you may have a rheostat which is a very good resistance and that can be parameter so things are variable. So imagine this mu represent such a variable parameter. Suppose you are varying mu from a negative value to a positive value. For a negative value what is the behavior? Complex conjugated with a negative real part.

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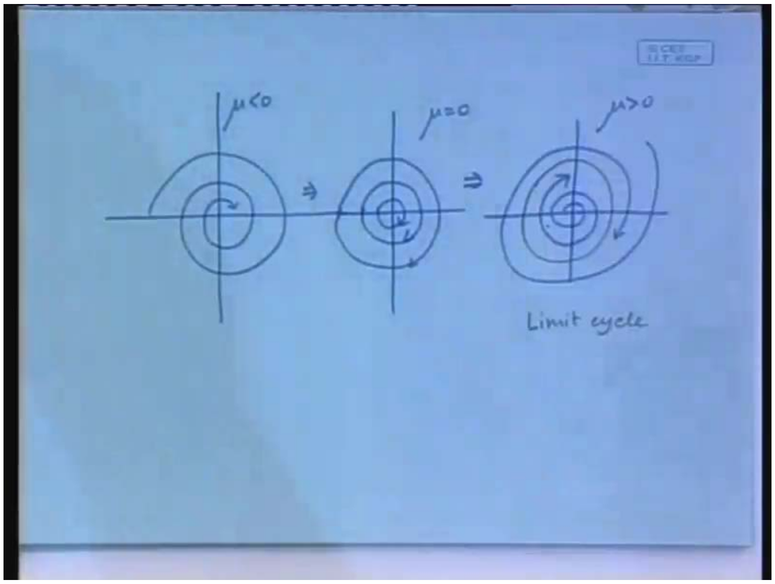

So it is a spiral thing. You expect a behavior like so (Refer Slide Time: 23:51). You don't know whether it is clockwise or anticlockwise. I am just drawing one. Depending on this specific system you will draw it correctly. At mu is zero what happens it becomes a perfect circular behavior and at mu positive it becomes outgoing spiral. This is mu less than zero, mu equal to zero and mu greater than zero. Now will this orbit go to infinity? There is no guarantee because we derived our conclusion based on the local linearization. So locally yes, it will diverge but nobody can say that it will go on increasing because elsewhere as you go away from the equilibrium point there is no guarantee that the local linearization is still valid. So there the orbit can still remain incoming. There is no guarantee that it will go on diverging indefinitely. After sometime you may encounters still an incoming spiral inversion.

So imagine that is happening. Outside you have an incoming spiral orbit. What will happen then? In between there must be some orbit that lies in between which is stable from both the sides. There must be such an orbit. This is how stable oscillations are created anywhere in nature. These are called limit cycles. At the outset, let me show you the difference between this orbit this is a periodical orbit it goes on like the pendulum but there is a very important device with the pendulum. If you have a perfect pendulum without any air friction, then also you have got a periodical orbit like this. But what is the difference between this orbit and that orbit? If you perturb the initial condition, it will settle down into another orbit but not the same one. While if you perturb it, it will come back to this one. In that sense this is a stable orbit. That is why I said that whenever there is a stable oscillatory behavior anywhere in nature, you know that it is created by a limit cycle. Some phenomenon like this and you can easily see that that is a nonlinear phenomenon. You cannot have a stable oscillator behavior other than having in nonlinear system. A linear system can never have a stable oscillator behavior.

If you perturb it more than it will spiral out and finally converge on to that if you spiral out if we if we start it will spiral in and convergent on to that but that again did not guarantee that if you start from here it is still as not a spiral orbit I didn't guarantee that I can only say that there exist a neighborhood from which this will happen clear in an non linear system you cannot say sitting here studying here you cannot say that if I am there what will happen you cannot say that but at least we can say with confidence that if this fellow has become unstable and if I find that there is some kind of an incoming orbit somewhere then in between they are most exist a stable periodic orbit. That's the limit cycle and when I say that wherever you see any oscillatory behavior stable oscillatory behavior that must be created by this kind of a phenomenon. That's a very strong statement. Now an oscillator has to be a limit cycle. We cannot help it. So, all oscillators that you have heard of are a limit cycle.

Any other oscillators that you have encountered? Regular periodic oscillation. There are many that you see alive. Heart beat example. Heart beat is a regular periodic oscillation and has to be a limit cycle can't help it. Try to imagine what would happen if the heart beat were of this type with imaginary Eigen values. What would happen? Let me tell you that people measure Eigen values of peoples' heart. Suppose the heart were like this. A linear system with perfectly imaginary Eigen values, a known hyperbolic system. What would happen? Our heart is anyway going on oscillating. You cannot say our heart will go on oscillating. It is anyway oscillating. That's why we are alive. So what would happen if we if we if we burst a burst a cracker here then it will be perturbed and it will be staying there. All your life you will be locked with that orbit. But that cannot happen and that is why it has to come back to the normal rhythm and that happens because it is a limit cycle. There is an outgoing spiral behavior and an incoming spiral behavior. Are you convinced that all stable oscillatory behaviors in nature in engineering are limit cycles? The heart stopping is a different issue. I will tell you why the heart stops because that is very widely studied but I will come a little later to it. Presently I am talking about healthy heart and healthy heart exist because there is a limit cycles.

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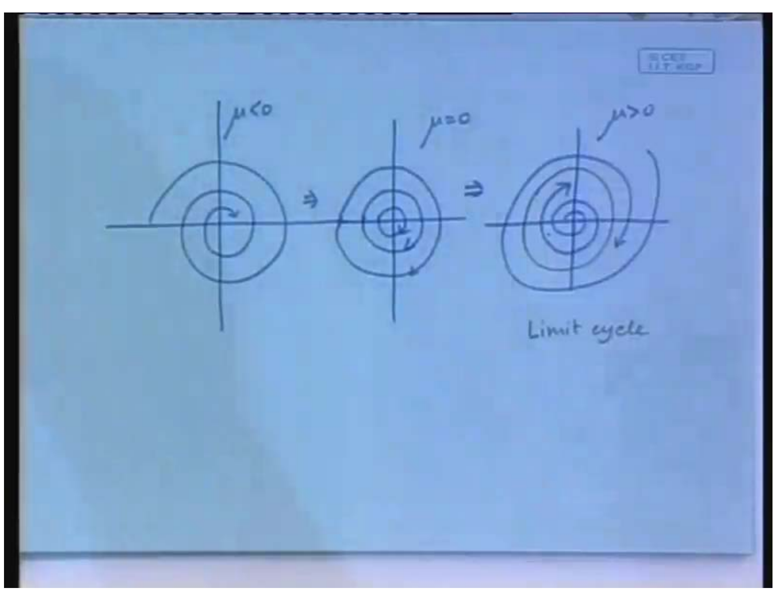

Convince yourself then you will get the real message why nonlinearity is so important. Why you the thing that you study in linear system theory cannot get you far because most of the things that you see in nature or in engineering are oscillatory. There is some kind of oscillation and in order to understand that, if you want to have oscillation then it has to be a limit cycle. In power electronics switching circuit, sometimes it is on and sometimes it is off. So it is going on oscillating. That behavior must be a limit cycle. that's why limit cycles are so very important and stability of limit cycle are also very important because in engineering whatever you are concerned with, if I have to work with a limit cycle I have to design a oscillator I have to design some kind of a oscillatory system, then I must ensure that this fellow is stable. So there has to be some way to ensure the stability of the system but of course the systems may lose stability and things may happen and we will learn how to understand the stability of limit cycles of those things. Those are important part of our discussions but now if you have a stable orbit like this, what will its time domain response be like?

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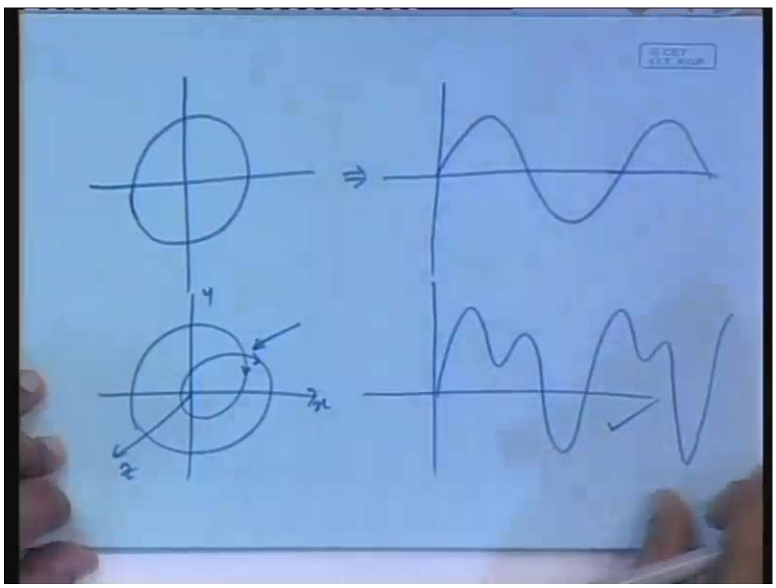

An orbit in the state space looks like this (Refer Slide Time: 34:00). Its time domain response will be a periodic orbit. So your signal generators must have something like that inside it. Your cell phones generate microwaves. Microwaves are oscillatory generated by an oscillator inside that oscillates at the microwave frequency. So there must be a limit cycle sitting inside and it is your job to ensure its stability and stuff like that. But then can it be an orbit like this (Refer Slide Time: 35:01)? Now go to another level of abstraction. If you have this kind of orbit what would its picture look like in the state space? It will be something like this (Refer Slide Time: 35:45). What is happening here (Refer Slide Time: 35:51)? Is this possible? it is not possible in a two d system because if this system is two dimensional, then this must be an intersection and the orbit cannot intersect itself the the moment it intersects that becomes the initial condition and then it will be the same periodic orbit. So you cannot have this fellow going this way and this fellow going that way from the same point. Then obviously where does the vector field point to? It must be a unit vector field. Obviously this is not possible in a 2D system.

So a simple geometry tells you that such an orbit on the oscilloscope cannot be seen if the system that you are studying is two dimensional. You don't have to solve any equations to prove that. Simple geometric equations tell you that. In order to have an orbit like this, you of course need to have a third dimension so that this orbit as it's seen on this particular diagram is a projection. It is not the actual diagram. It is a projection. There has to be a third direction.

His question is: how do I get this from here (Refer Slide Time: 37:39)? Think it in the opposite way. If this state space behavior is like this, what would the time domain behavior be like? Try to work it out.

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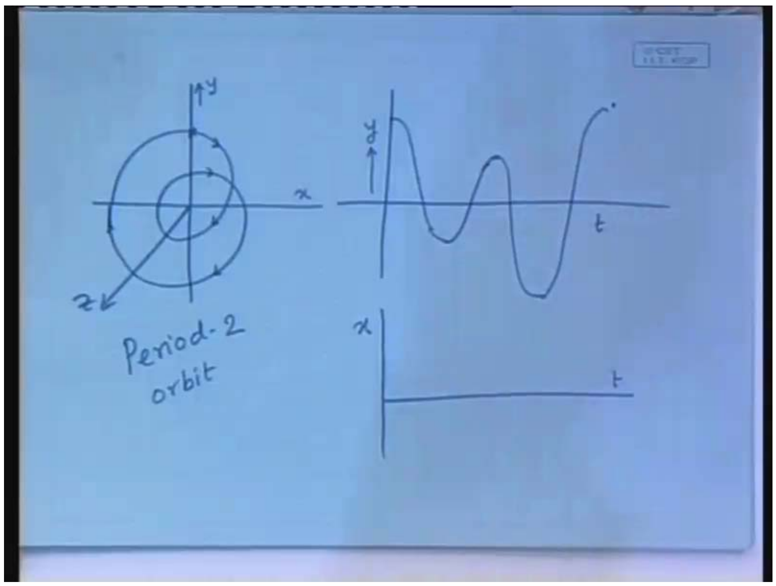

So I am starting with an orbit something like this and I am asking you what would be the time domain behavior like. You start from a point and you will trace its progress. Of course when you are plotting it against time, you are either plotting x or y. suppose you are plotting y, you start from a point which is the positive y. I can see that afterwards the y is reducing. y will be reducing. It goes to the negative value while time goes up. Then again it takes a positive value but it doesn't go as high as this stops here again comes down go to negative value and again comes back to this value and that repeats. Now plot x verses t in the same logic. Do you notice that the character of the waveform in the y verses t is that it comes back to the same value after two cycles. It comes back to the same value after two cycles. That is seen in the y axis and in the x axis also that is true. It cannot be so in the y axis. It is coming back to the same value after two cycles and x axis just one. It is not possible there.

So whichever direction you may look at, it comes back to the same position after two cycles. That is why this is called a ‘period 2 orbit’. It may so happen that you had a period 1 orbit. As I said that you may have the birth of a periodic orbit as you change the parameter. In this example, as you change the parameter through mu = 0, the periodic orbit was born. Earlier it was not there. As you change the parameter further a periodic orbit may become a period two orbit. So long as the system is 3D, there is nothing to stop it. You cannot really prove the theorem to tell that this cannot happen. You can prove on the theorem to say that this cannot happen if the system is 2D. If this system has just one capacitor another inductor, this cannot happen. Probably you have seen this kind of a waveform in the saturation of transformer. The immediate conclusion is that it must be a 3D system.

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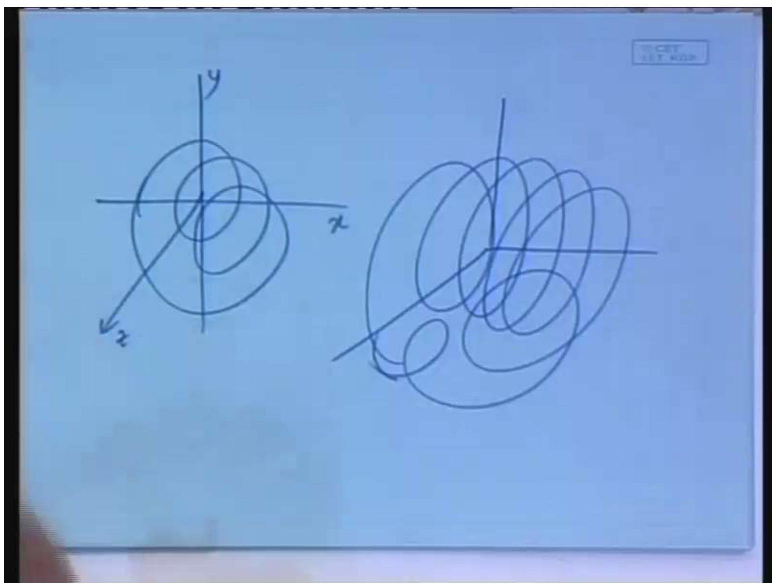

Can you have an orbit something like this (Refer Slide Time: 42:20)? If that can happen this can also happen. it has to be 3D. Otherwise it will stop in the first step itself. But if it is three d then this can happen. can this happen (Refer Slide Time: 42:48)? There is nothing to stop it really. you cannot say that this will not happen. All this can happen then. So you see these are all limit cycles. so you have you can have a period 1 limit cycle you can have a period two limit cycle you can have a period three limit cycle you can have period four limit cycle period twenty-seven limit cycle period {aha} (00:43:13) one hundred and thirty-eight limit cycle nothing will stop it. in fact all of them do occur. Period infinity orbit? If one is possible two is possible three is possible, then why not infinity? What is the meaning of period infinity orbit? That means it never comes back to itself. A periodic orbit is essentially a period infinity orbit. That’s also possible. So you see the moment you start the nonlinearity you know there’s a whole lot of possibility. Earlier you had only a few types of behavior in linear systems. You can either have a sink kind of behavior, a source kind of behavior, a saddle kind of behavior or spiral kind of behavior. Nothing more. So, linear system theory is so simple because of this. The moment you take the nonlinearity into account, all these are possible and in fact bounded. Periodic orbits go by a special name that is called a chaotic orbit. So chaos is nothing but a periodic orbit- period infinity but bounded. An unstable orbit is also a periodic orbit. Unstable orbit means something that collapses. It didn't come back to itself so it is also a periodic orbit I am not talking about that a system that is stable stable in the sense that it doesn't collapse yet its behavior is aperiodic. I will give examples to work out. I asked you in the last class to get a custom to some of the computation softwares. Which one are you most used to? You solve one set of equations and see its result.

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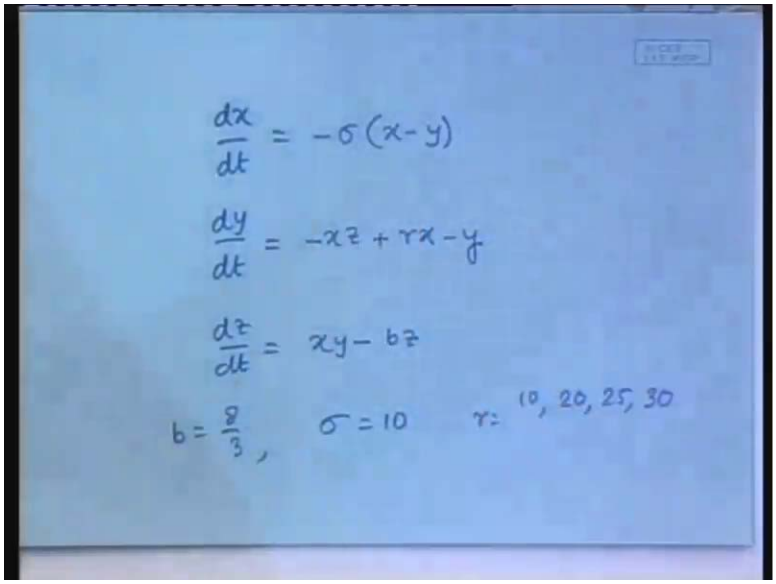

Take down this set of equations and then I will go ahead. As you can see, x, y and z are the state variables. It's 3D system. So everything can happen. I can easily see three parameters sigma r and b normally we take two parameters are constant and study the effect of variation of one parameter. So take these parameters. Take r as a variable parameter and calculate it for say a few values of r. r = 10, 20, 25, 30 and so on and so forth. See the result. That will be your assignment to be done before you come to the next class. By the way what is the equilibrium point for this system? (0, 0) is the equilibrium point. What is the behavior at that equilibrium point? You can obtain the Jacobean; you can find out the behavior find it. So it will be 3 x 3 matrix. (0, 0) is an equilibrium point but that is not the only equilibrium point. In this case, you notice that there must be more number of equilibrium points. Let me tell you the process that you have to follow because we will study this system for sometime in the next class and later. You will proceed in the same way. You will take this, put the left-hand side is equal to zero. Solve the right hand side, you will get the relationship between x and y here. There is an xz term here. There is an xy term there. All put together will give three equilibrium points. All the three equilibrium points you will then need to locally linearize. That means you will have to obtain the Jacobean matrix and put the positions. After that, you have to obtain the behavior at the individual equilibrium points.

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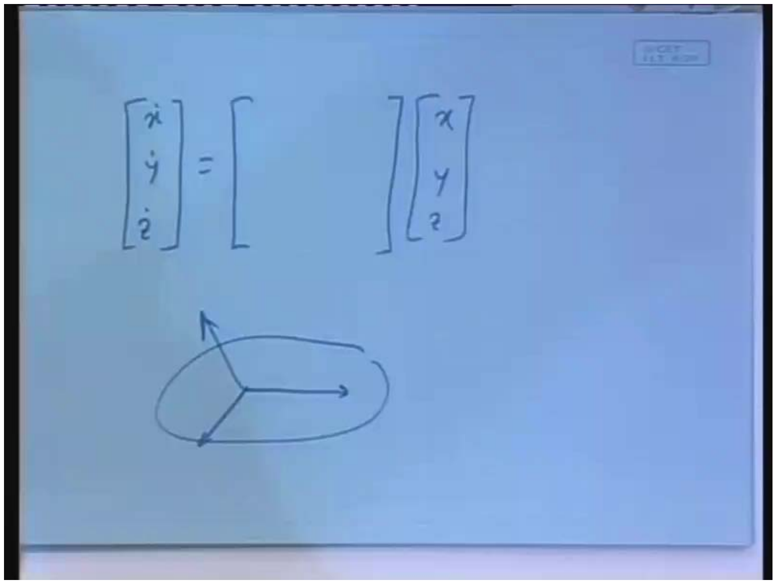

Suppose you have got an equation, something like x dot y dot z dot is equal to 3 x 3 square matrix. How do you solve it again the same way you obtain Eigen values or Eigenvectors. What are the different possibilities? You can either have all the Eigen values real, three Eigen values real, all can be negative in which case it will still be a sink. All can be positive in which case it will be a source. One can be positive to negative in which case it will be a saddle. but remember in that case, suppose this is one eigen direction, this is another eigen direction and this another eigen direction and in these two eigen directions, it is negative and in this eigen direction it is positive, then I can easily say that there will be plane compressing these two eigen directions. A plane in which it is stable and in that direction it is unstable. So that becomes the stable Eigen direction, that becomes unstable Eigen direction and these become the unstable manifold. I say sub space in which it is stable or unstable. So this plane becomes this stable manifold and that becomes the unstable manifold. What if the Eigen values are the complex conjugate? Two can be complex conjugate and the other fellow cannot be complex. It must be stable and real. If it is real then there are immediately a few possibilities. Can you draw for example associated with the complex conjugate Eigen value? Then I can associate a plane.

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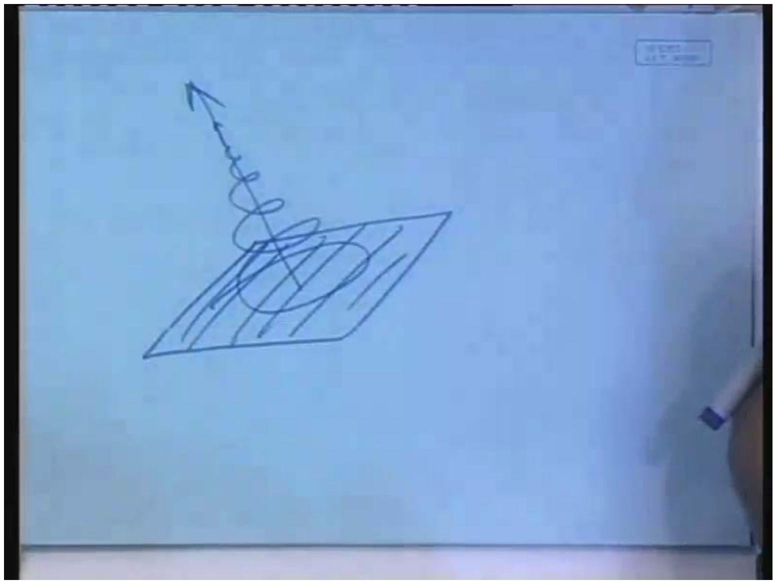

Suppose it is this plane and this is the other other one associated with the real, in this plane this behavior could be either incoming spiral or outgoing spiral depending on the real part. Supposing it is incoming spiral and this fellow is positive, what will be the behavior like? It will start from here and then it will go like this. It will be incoming spiral here along this direction, outgoing in that direction. It will be a behavior like this. Similarly you can imagine it opposite if the arrow is that way which means the real Eigen value is negative and this fellow has a positive real Eigen part. Then what will happen? It will go on increasing and expand it along this plane. It will converge on to this plane. Before coming to the next class you try to figure out what are the different possibilities of such quality of behavior in 3D and on the basis, we will continue. Without understanding that it will be difficult. That’s all about it today.