Chaos Fractals and Dynamical Systems
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Lecture No. # 28
Analysis of Chaotic Time Series

Today we will mainly consider the situation where there is some kind of a system running and experiment running and out of that you are getting a time series information. What does it mean? For example there is an electronic circuit which is operating may be periodically, may be chaotically someway. Suppose there is a current that is coming and you are sensing the current. Whenever you sense the current, some natural questions crop up. For example suppose if that current passes through a resistor, how much power will be dissipated? Now for a DC current, this is a trivial question. For an AC current sinusoid, there also you know how to handle it. But what happens if it is a chaotic wave form that passes through a resistor, how much will be the power dissipated? You might also imagine the situation that the current is not just a sinusoidal quantity but a DC value on top of which there is some kind of a time varying quantity. One would naturally like to find out, what is the DC component of that. One would also naturally like to find out what is the AC component of that. AC means here I am not talking about the sinusoidal wave form but the component that remains after having taken out of the DC component, I am talking about that.

One would also be interested in finding out the variance, standard deviation. One would also be interested in finding out the spectral characteristics or how much is the state at a particular time dependent on the history. These are the issues that one tries to look for when one encounters a time series. By time series I will mean that there is a system running means it could be any dimensional system. It could be three dimensional or four dimensional but you are being able to access, observe, measure one of the state variables and thereby you are getting a waveform. Then you are facing the question what do I learn from this wave form? The same kind of problems are encountered in any signal processing application. Naturally the things that I will be covering, many of these are basically the same as that covered in any signal processing course. For those who have done that kind of courses, it might be repetition but since there is an inhomogeneity in the nature of the class, I have to cover all this.

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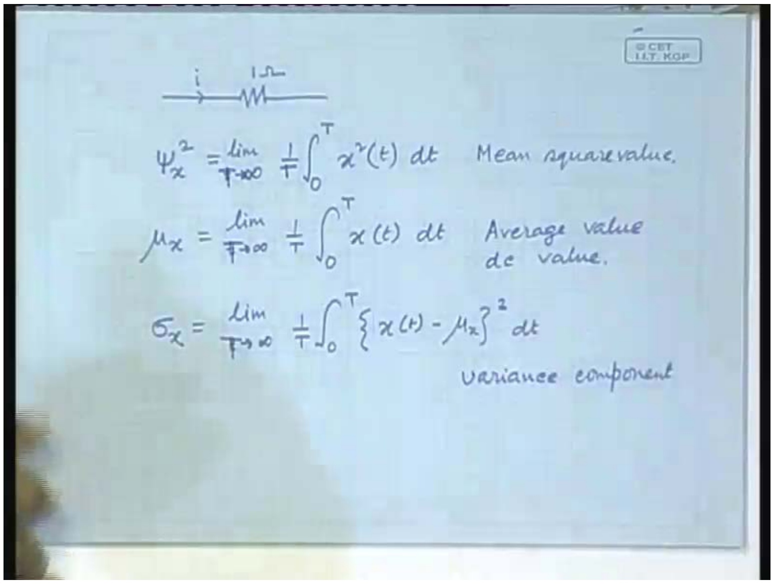

The first question, if say the current that I am talking about passes through a resistor. Suppose it is a one ohm resistor. How much power will be dissipated, how do you find out from that waveform. Now that is obviously given by the mean square value, square because i square R. It will be dependent on the square term. The mean square value will be given as that would be integral of 0 to T and square of the term x square t dt that has to be averaged out. Then you have to consider this t going to infinity. So limit T tends to infinity. xt is the quantity that you had, if you calculate this you essentially get an idea. This is the mean square value. It will give an indication of, if the current pass through a one ohm resistor how much power will be dissipated.
Obviously this could be just any waveform. You only need to have the time series availability. You would also be interested in the average value. What would the average value indicate? The DC value essentially. The average value would be, I will write it as mu_x is again limit t tending to infinity 1 by T but now I will be integrating xt dt.

Obviously here we have x square t to get the mean square value and this is the average value or DC value. If xt is simply sinusoid waveform then what do you get here? Zero. These are reasonably trivial stuff. The rest of the component which can be viewed as the AC component would be the variance from the average DC value. There is a DC value and on top of it there is some variation so that has to be then measured to give you the DC value. These are important for example in the context of say from electrical engineering point of view you have a DC to DC converter. You know that there is a DC value which you want at the output but there is inevitably a ripple component of it and the output actually utilizes the DC value. But there is a DC value plus an AC value that is contained in the final output waveform. The AC value is actually useless in this case. That is why it is necessary to measure and incase this waveform turns to be periodic, aperiodic whatever. All the time you will be interested in finding out that additional component.
The variance component would be again limit T tending to infinity 1 by T but what will you integrate in this case?

In this case you will integrate xt minus this quantity and then you square it because you want to find the mean square of that value. It's an AC component, you want to get a mean square of that value. (Refer Slide Time: 08:50). Essentially you are taking a very long, ideally infinite length of data. Obviously you will not get an infinite length of data but that is the total length T, I am talking about. Capital T tending to infinity means ideally you will consider an infinite length of data. This is the variance component. These are the three things that would be practically important for various situations and these would be the same for any wave form as good in case of the chaotic waveforms. But then we will come back to this and try to relate this with many other quantities that you will be interested in. By the way what is standard deviation? Square root of the variance.

In an earlier class, I had talked about the density function. Did I not? There our basic point was, let me recapitulate that a bit. The situation is that at that time we are talking in terms of maps and in the map or as the map was iterated, we try to find out the densities and the density function. But here we are talking about some experimental data. As we are getting the experimental data, there also a question of the probability density function will arise in the sense that suppose there is a data set that is coming. I ask what is the probability of finding the state, that particular value that is been measured within certain band of values. For example if I am measuring current, I say what is the probability of finding the current at any instant of time between 1 ampere and 1.1 ampere. This is a valid question to ask and in order to obtain that, all we will do is to measure this and to obtain this and find out, out of the whole range of time for how much time did the data set spent in that particular range, 1 to 1.1 amperes. Essentially what are you doing?

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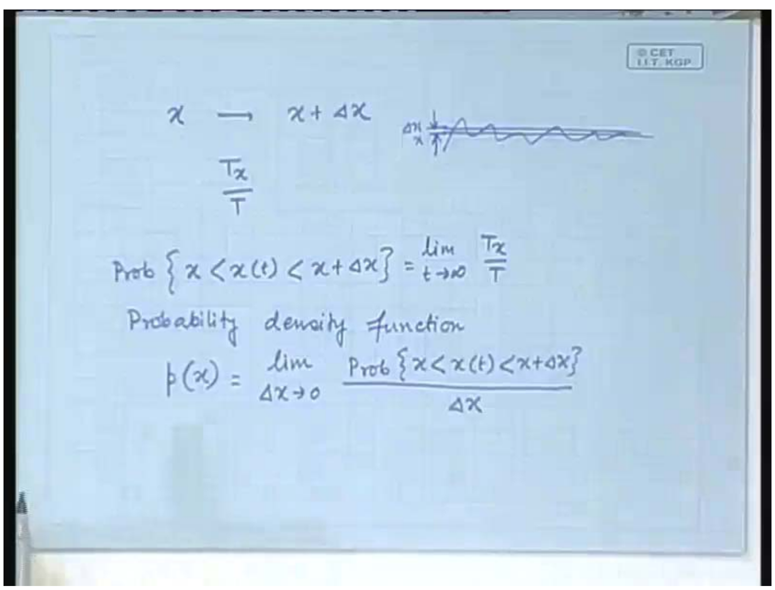

Suppose we are considering the range x to x plus delta x. Within this suppose it spends a time T_x and the total time is T. In that case obviously what is the probability of finding the state in that range which is T_x by T. There is no difficulty about it. We would say that the probability of finding x, also finding x (t) in the range x. The x (t) to fall in the range x to x plus delta x. The probability of that is nothing but T_x by T for t tending to infinity. We need to take large value. If you take a finite length of data then we basically get an approximation of this probability. Ideally you have to take an infinite length of data then only you can say this. We have a measure of the probability T_x by T and that gives you the probability of finding the state in that particular range. Now the probability density function will be then given where this delta x is squeezed to zero. If delta is squeezed to zero then what is a probability that will give the probability density. It is p of x is limit delta x tending to 0. Then we will talk about the probability of xt lying in the range… divided by delta x. This is the probability now I divide by delta x and allow delta x to zero.

If this is the waveform, I said that this is the range that I will be talking about. First we find the time that the waveform spent in this particular range. Where is it? At x. How broad is it? Delta x. Then we found the probability and then divided that by delta x and allow delta x to zero. That as a function of x that means where I am trying to find it, if I calculate that gives me the probability density function as referred to a data set, a signal. Here is the idea of the probability density function. Obviously the probability of finding the state between the range x_1 and x_2 would be...

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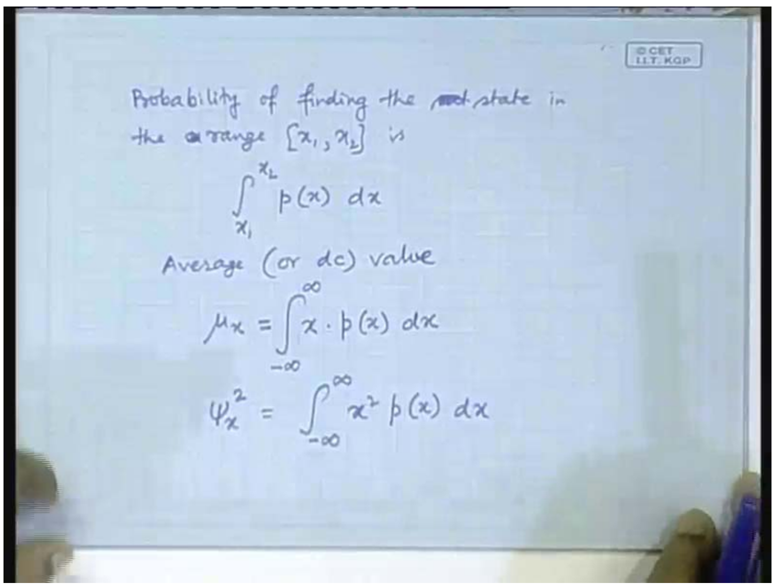

What is the range, what is the probability? It is nothing but x_1 to x_2 p (x) dx. Once we have calculated the density function using that I can find out what is the probability of finding the state between any two values. Not only that, there is another major advantage of finding the probability density function. What are you talking about? We are talking about the probability of finding the state at a particular value. Now from that obviously the average should be extractable. That means what is the average, what is the mean square? That information must be contained in the probability density function. Once we have obtained the probability density function, that contains the information about the average, the mean square and stuff like that.

For example what is the average? The average value or DC value would be, we will say x. What is a probability of finding the state at a value of x. That is times p (x). Then that will have to be integrated over the whole range. So integrated from minus infinity to infinity x p (x) dx, that will be the average. Notice what I am saying. Now I am allowing the x to vary over the whole range minus infinity to infinity and at each value what is the probability of finding? It is waited by the probability. Naturally if x has a larger probability of being very close to zero then that particular value of x will be multiplied with a higher probability. While larger value of x will be multiplied by a lower probability. As a result this integral will yield a value that is close to zero. That is intuitive because you had the average close to zero. This is nothing but what we had say mu_x.

Similarly can you say what will be the mean square value? x square. Mean square value we were denoting it by psi_x, x because it is function of x. Square is equal to integral minus infinity to infinity x square times p (x). This is the advantage of finding the probability density function. From there we can derive many information's. Once we have derived the probability density function, we can derive many information. That is exactly why we were so keen on working out those Frobenius Perron operators trying to work out the averages. Essentially trying to work out the probability density function so that these information which are of practical importance can be extended.

No, here I am not talking about only the AC component. Here we are talking about the total power that will be dissipated, if it went through a resistor of one ohm. If you take x minus mu_x essentially you are talking about the AC component. Mu_x is the DC component, x is the actual variable. If you subtract the DC component you would get only the AC component remaining. Its mean square that gives you the power that is dissipated by the AC component. But in this case you are interested in finding the total power that will be dissipated if this current went through. Probably this is common, this has been done in earlier classes. Only this is being used in this specific context. There is another thing that we are interested in which also probably has been covered in your math's courses. That is how the present state is correlated with the history. It is given by the auto correlation function. Have you come across that? Probably 80% of you have.

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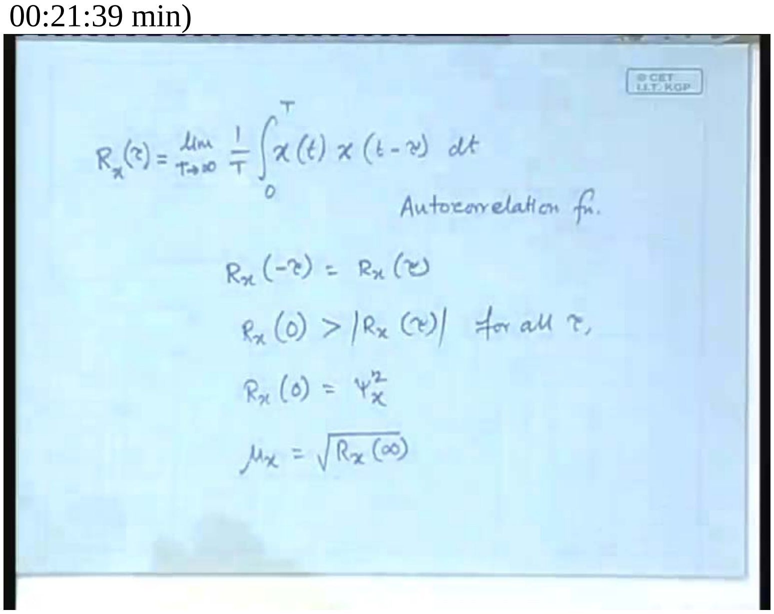

Just recall what it was? The auto correlation function is trying to correlate x at time t with x at time in the past or in the future, they would be the same. If I am standing here and I look into the past and try to find out, then I will write x (t) and x (t) minus tau. If I am talking about that state and how it is related to the state now, essentially we will take x (t) and x (t) plus tau, same thing. What we will get is actually even function. We will talk about x (t) and how is the correlation quantified simply by the product, so x (t) times x (t) minus tau. While tau is the amount of gap that I am trying to explore. That means if the state is here now at this point then how is it related to a state that is tau times in the past so that is a tau. We obtain the quantification by simply obtaining the product and then this is specific to a specific tau. But then here I have talked about a specific x (t). There was a waveform, at time t it had a particular value. How is it related? But basically what I am asking? I am asking how is the whole time series related to a history? I have to compute it over the whole time series which means that this will have to be integrated over 0 to T and it has to be averaged over.

Also we need to take a very large time series, ideally infinite size. Ultimately what we have arrived at is some kind of a quantification of the state dependence on history and that is called the auto correlation function represented by R of x and it will be function of tau. Function of the delay that I have taken. This is the autocorrelation function. We are generally interested in finding it for chaotic time series also. This is the autocorrelation function, we can easily imagine a few characteristics the way I say that it will be the same, if you take the tau positive or negative. Why? Say I am talking about tau is equal to one second, if I am asking the state now how is it related to the state one second earlier and if I talk about the situation how the state one second later is related to the state. It is the same answer because I am talking about one second.

Essentially it is an even function, we can write it as R_x minus tau is equal to R_x. It is an even function. Do you know what is an even function? That means if you imagine a mirror on the y the axis. It is a mirror image in the other side. That is the definition of even function. If tau is 0 then what do you have? It yields psi x square. If tau is 0 you essentially get the same thing as psi x square that means you get back what you have already done. One thing is sure that the value at tau is equal to 0 will be larger than all other values because it is very correlated with itself. The more you go away in time, it will be less correlated. It will be the maximally correlated with itself. The correlation function will have a peak at tau is equal to 0. We can write R_x at zero is greater than R_x at tau. We can say magnitude for all tau.

Now we have already arrived at the conclusion that R_x zero is psi square x. If the tau is allowed to go to infinity then what? Notice, try to argue it out. At tau equal to 0 that means I am asking what is the correlation between the state now and the state now. Then I get a large number and that is the psi x. If I ask what is the correlation between the state now and the state infinite time earlier. Will we really get zero? No, we will not get zero. Suppose there is a DC value that means mu_x is not zero. There is a DC value on top of it some waveform is there. If you go to infinity that correlation will go to 0 but the DC value will remain. Is that point clear? That is why if you allow the tau to go to infinity, what you extract is essentially the DC value. Since there is a square of it essentially you get the square of the DC value. You can write mu_x is nothing but the square of R_x at infinity.

We started with the premise of the DC value and the mean square value and we learnt how to obtain it from two different premises. One the probability density function and the auto correlation function. There is another thing that would be interested in most cases especially people working in electrical engineering are always interested in that, the frequency characteristics. That means how much power is available at different frequencies available in that waveform. If it is DC, the frequency is 0. If it is a sinusoid, the frequency has only one component. If it is a distorted but periodic wave form, we know that it has many component or many frequencies. If it is a chaotic waveform, it has all the frequencies but nevertheless what actually we are trying to obtain would essentially be the power contained between two frequencies f and f + delta f.

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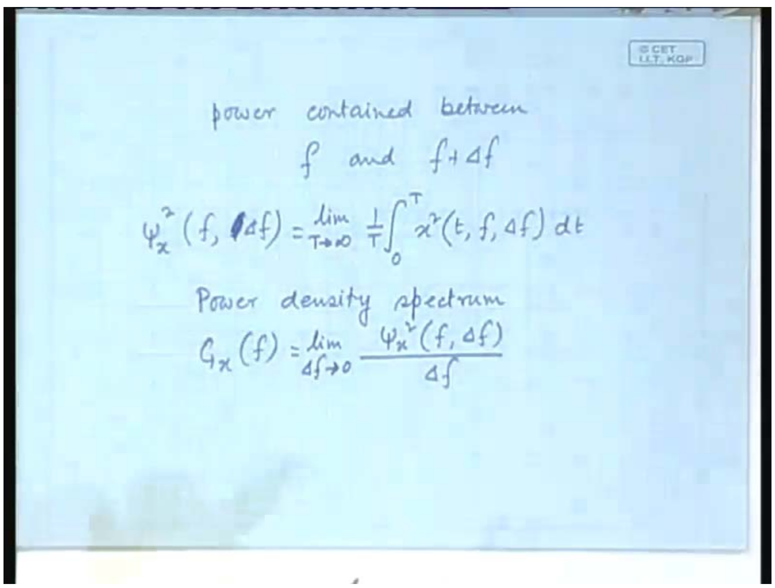

You would imagine that we have designed some kind of a band pass filter that allows only this range of frequency to pass. If in the input side we put that time series, in the output side we have only that component that falls within this range and we are trying to obtain the power so we obtain the mean square value. That is what we are up to. We will say the mean square value with the function of f and delta f that will be again integral of 0 to T. But what we will integrate? Now we will integrate x square that falls in the range x square of t and it will be also a function f and delta f. It will be function of time, it will be function of frequency and the frequency range. We have extracted that information and we integrate over time, averaged and we take the limit.

We have essentially the power that is contained in this frequency range that is what we have obtained. Now the power density spectrum is nothing but this situation where the delta f is slowly reduced to 0 and you divide this by delta f and let delta f vanish. That is how we get the power density spectrum. The power density spectrum, we will write it as G_x, it will be function of f. It will be this term psi x square f and delta f divided by delta f. This term divided by delta f while limit delta f tending to 0. Here we had the original signal x of t. We said that imagine we have placed a band pass filter that allows only that component to pass which falls in this frequency range.

We have essentially selected out of the whole signal, there are many frequency components allow only that which falls in this particular frequency range to pass. We get x of time but it is also function of f and delta f. We take the square of that and then integrate over time. That yields the power density spectrum PSD. You will find that these are now a day's automatically measured in some of the CRO's but obviously not through this route. It measures through a different route, I will come to that later.

This will always be a real valued function, the power density spectrum. For every frequency there would be a value of power, another frequency another value of power, another frequency another value of power. In case of a DC it will be picked at zero frequency, nowhere else anything. In case of an AC sinusoid wave form, it will be power concentrate at a single frequency. If it is distorted sinusoid, it will have many third harmony fifth harmony so on and so forth that means power concentrated on those. In case of a chaotic wave form, it will be a continuous power spectrum. But this is the definition of power spectrum you must understand that. Actually obtaining the power density spectrum is not done by this method because you really do not construct a band pass filter and allow it to pass. This is a good way of visualizing what it is but actually it is done through the Fourier transform. This comes to the root of the Fourier transform.

Probably you all have done Fourier transform. I know that some of the students are more comfortable with Fourier series rather than Fourier transform. Is that right? No, Fourier transform is easy easier. Good, I have seen many students who at the time of the viva voce test says that I understand what is a Fourier series but I don't understand what is a Fourier transform. You understand right? If you do let me give you a problem that I find very interesting. Let me write down the Fourier transform so that we can refer to that. I will come to that later. In the olden days, the view of the solar system that people had… do you know what it was the Ptolemaic picture? It was that the earth is at the center of the universe, sun rotates, the moon rotates and all the planets rotates around it. Do you know what is the problem of the rotation of the planets?

I mean if you go to any field in the evening and observe the motion of the planets. For example mars may be and how would you try to figure out its motion? You would simply take a graph paper like this and on one day, you will measure the positions of the stars and then say also the position of the mars. The next day you go out, take it like this and then the position of the stars will be the same. Relative positions of the stars will be the same but mars position will have moved a bit. That is why it is planet. It's a wonder. Again make a mark where it is. In this starry background next day again come and do that. Then you will be able to plot a trajectory as seen from earth. In fact that was done more than 2000 years back. Aristocrat did it first and he noted that it is a very peculiar wave form it is like this, it goes and then turns back and it goes forward and it comes back and goes forward so it is like this. Anybody who are trying to design a model of the solar system has to account for that.

What Ptolemy did was, he proposed that the planets do not really go around in circles but there are circles over circles. He said that inside there is the earth where we are sitting, that is the center of the universe and the planets are moving like this. But not only like this but they are moving like this.

For some time you will see them going backward, quite naturally. Then astronomers started pointing that they are trying to find out whether the actual motion satisfy that. They found that no, not really. Then the post Ptolemaic people said that, not only one such epicycle but there are more such epicycles. One big one, small one, even small one and that way they tried to fit into the picture. By the time it came to the time of Copernicus Galileo, that picture become horribly complicated because it turned out that they started to compound epicycles over epicycles over epicycles and it should get huge number. Then Copernicus said let's take a fresh look at it.

Imagine that the sun is at the center and they are rotating like that. But there is a theorem that says that if you go on compounding an infinite number of epicycles, you will see exactly the thing that we see from here. They stopped because you really cannot do that. Can you explain these observations on the basis of the Fourier transform? Post Fourier we know what really went wrong. Before that we did not. Think, it's a good problem. The Fourier transform is something like this. That means let me state the problem properly. The actual motion of the planets can be properly explained by considering an infinite number of epicycles. If you are really compounding any infinite number of epicycles you will be able to predict the exact motion of the planets. Still that picture is wrong, it's not that the earth is really at the center and it's just a picture. It's a representation but that representation would work.

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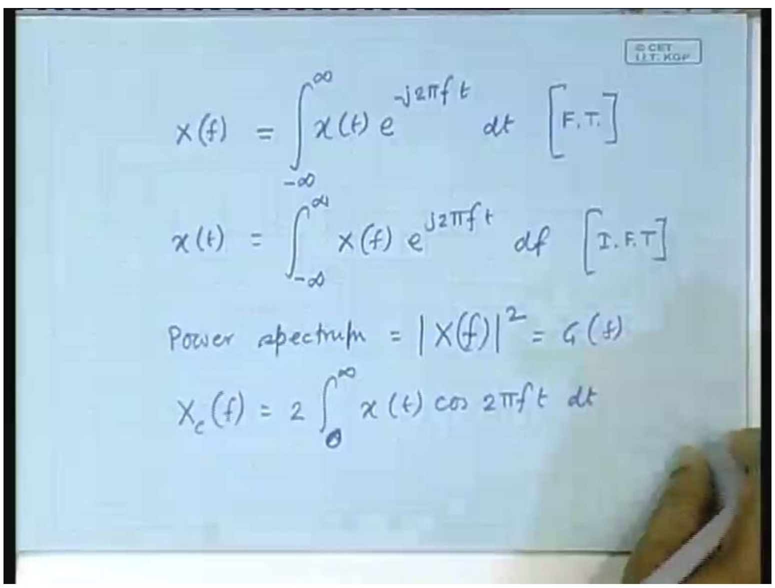

If you have a signal x (t), the Fourier transform is given by multiplying it with an exponentially decaying quantity because you wanted to be ultimately finite, the integral to be finite. It would be minus j 2 pi ft dt integrated over minus infinity to infinity. What you will ultimately obtain is X of f. This is capital X and this is small x remember. When it is function of time we represent it with small quantities and when it is the function of frequency, it is the capital. This is the Fourier transform. Of course the inverse Fourier transform can be obtained as x (t) from x (f) minus infinity to infinity, now added to integrate over f so x (f) e to the power j 2 pi f t df. So that is the inverse. This is the f Fourier transform and this is inverse Fourier transform. What is the relationship between this and the power spectrum?

Power spectrum is nothing but the square of the Fourier transform. That is the essential idea from which it is actually done by most of the modern storage oscilloscopes. Power spectrum is nothing but the square of this and what most first Fourier transformations do is to obtain a transform and then take the square of it. That is what is displayed. Probably you have seen those oscilloscopes which readily get the data and display the spectrum. It does this way. But remember we are interested mainly in the power spectrum. Engineers are interested mainly in the power spectrum not this one. For our purpose it makes more sense to talk about the power spectrum. Since probably you know that for even functions this can be simplified to a cosine term. You have heard of it. Now is a chaotic waveform an even function in a statistical sense.

In the statistical sense means there is a waveform, it is irregular waveform. You cannot really say whether it is even or odd but statistically if you see the waveform and place a mirror and see the other side, you will not be able to distinguish them statistically. That is why statistically the cosine transform also works on chaotic signals. In that case you will write. Now it is half of it zero to infinity x (t) then cosine, so this is a simplification that is applicable only in case of even functions. Since chaotic waveforms are statistically even that is why this works. We often obtain the Fourier transform by this method. Now the question is supposing we have obtained this, can we extract the quantities that we are talking about earlier? What quantities we are talking about earlier? About the DC value. What is the DC value, can we obtain it from the Fourier transform? Of course it is the quantity around zero so how will you obtain it?

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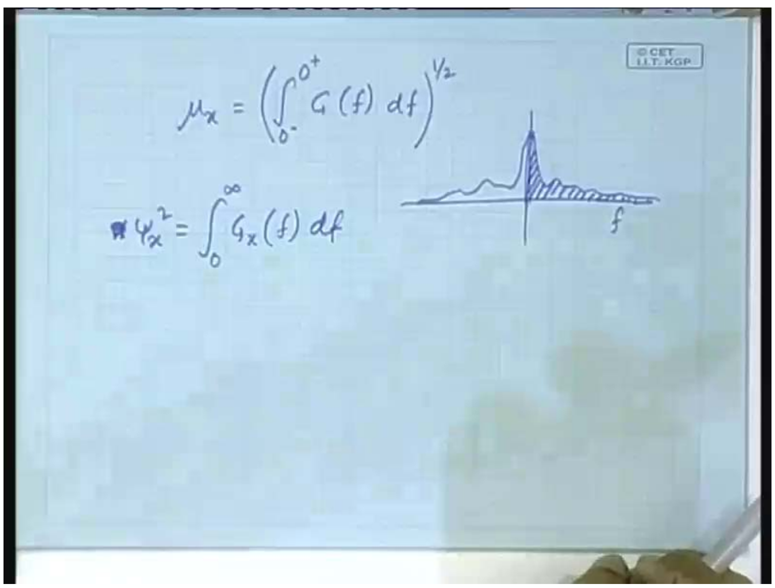

Mu$_x$ that is integral from 0 minus to 0 plus of what? Let me see what we have obtained. This we had represented as G (f). I will write G (f) here, G (f) power spectrum df. No, it is not right. The 0 minus to 0 plus, you have taken both the sides but now it has to be ultimately G (f) is square of what you wanted. You have to take a root of that.

Essentially if you have as the power spectrum, you take the very small quantity between 0 minus and 0 plus, this height and the power spectrum is you have already taken a square so you have to take a half. If you know the power spectrum from there, the average can be or DC value can be extracted. What about the mean square value? Can you extract the mean square value from here? Mean square is nothing but the total power contained in the signal. Now you have decomposed it into the frequency components and therefore you have to integrate over the whole frequency. What you get ultimately is intuitively speaking that is nothing but the mean square. You have to take the area under this curve that will yield the mean square. Here we talking about psi x square, again we will do it over half because the other half is not necessary, G_xf df. Here we are integrating from the left side of zero to the right side of zero. Here we are integrating over the all. We are through. Two things we have extracted, after that where did we go? Autocorrelation. The next question is how is the power spectrum related to the autocorrelation function? There is a very important implication and many of the modern gadgets depend on this implication. They actually obtain the autocorrelation function and then obtain the power spectrum from there because the autocorrelation is easy to obtain.

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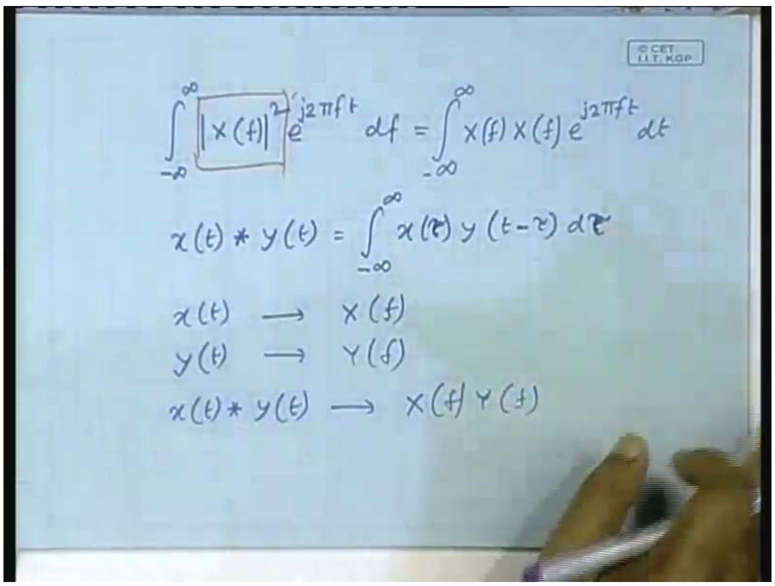

In order to do that let us proceed this way. Let us first take the inverse Fourier transform of this, minus infinity to infinity we will take the power spectrum which is x (f) square. We start from there, we take the inverse Fourier transform which means times e to the power twice pi ft df. We started from this quantity here and we are taking its inverse Fourier transform. Let us see what it yields. Now this can be written as minus infinity to infinity. Here I can write x (f) times x (f) e to the power j twice pi ft dt.

Now let us invoke the convolution law. Probably you all know the convolution law. One of the elementary things that one does in signal processing. What is the convolution law? Supposing there are two signals x (t) and y (t). Then x (t) convolution is denoted by a star. The x (t) convolution y (t) is integral minus infinity to infinity x (tau) y (t) minus tau d tau. You might also reverse the notation like x (t) y (t) minus tau dt, it's so simple. This is the definition of the convolution and we also know the convolution is the same as product of the Fourier transforms. We have learnt that I am not proving that we have learned. That means if x (t) has the Fourier transform x (f) and y (t) has a Fourier transform y (f) then x (t) convolution y (t) has the Fourier transform, x (f) times that is the convolution law.

Now imagine that I made x and y the same signals. Do we need to take the conjugate? (Refer Slide Time: 53:50) these are real signals I think. I will come back to this later because if it is real signal it is ok. If it is complex then you have to take a star. I will come to that later. If you take x and y the same signals then what does it yield?

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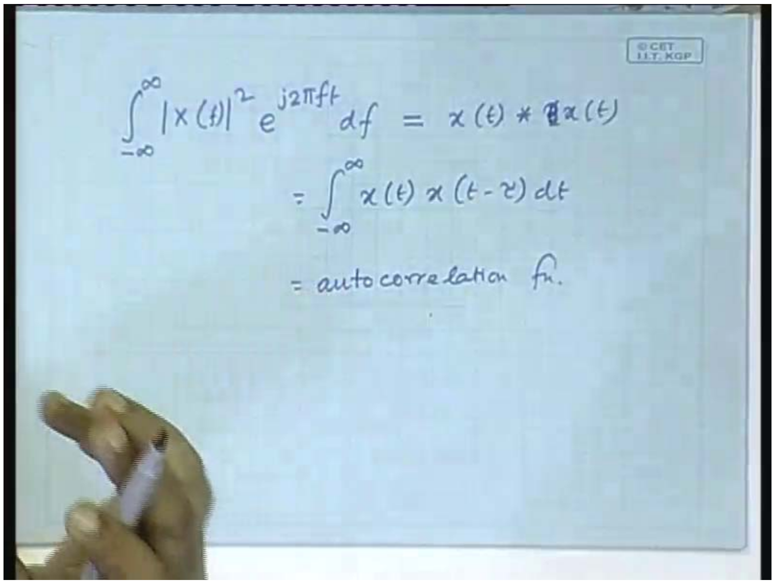

It means minus infinity to infinity, I was starting with x (f) square power spectrum times e to the power j twice pi ft df that is where we started. We said we are dividing these two. Now if these are the same signals then obviously this yields x (t) convolution x (t) which means integral minus infinity to infinity x (t) x (t) minus tau and this quantity is autocorrelation function. So which means if you take the power spectrum and take the inverse Fourier transform, you get the autocorrelation function and naturally if you take the autocorrelation and if you take the Fourier transform you get the power spectrum. The power spectrum and the autocorrelation function are nothing but the Fourier transform theory. That is the important result that is often used in practical instrumentation. We will stop here and continue in the next class with the same idea.