Chaos Fractals and Dynamical Systems
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Lecture No # 27
Dynamics on a Torus (Continued)

Let us briefly recall where we were in the last class. We said that when there is a periodic orbit like so in continuous time and suppose you have placed a Poincare plane here, you would normally see a point but supposing this orbit becomes unstable which means the fixed point here becomes unstable and not just any type of instability. It is that instability in the map where the Eigen values are complex conjugate and their magnitude is just equal to one. When that happens on this, if you blow it up you will find that in the neighborhood of that you will get an outward spiraling orbit while outside there will still be an incoming spiraling orbit as a result of which there would be a closed loop develops. A closed loop on the Poincare plane would mean that there is a torus in the whole continuous time.

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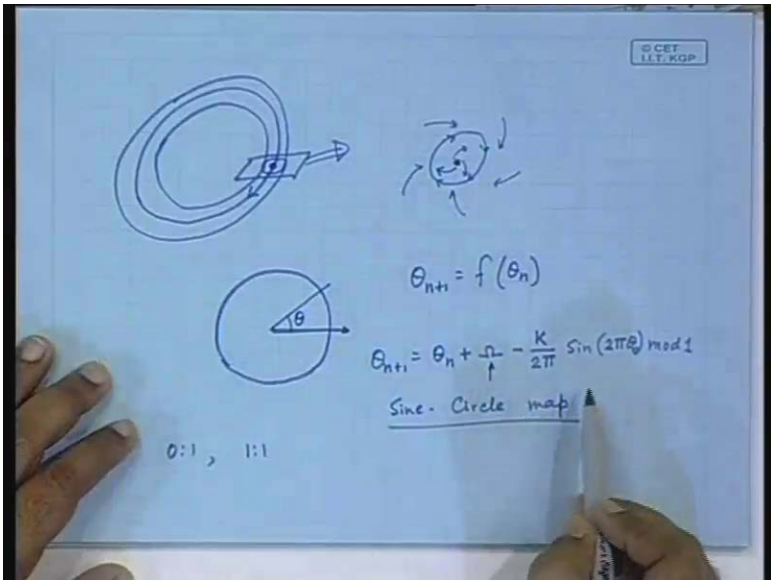

That is the picture that we have seen. We forget about what is happening elsewhere, we keep it in the back of our mind but let us concentrate on the Poincare section. Since on the Poincare section we will see these are rotating orbit, since the fixed point that is sitting inside has become unstable while the Eigen values are complex conjugate, there should be a rotational motion and the rotational motion will give rise to a turning this way. We said that suppose we consider the rotation on a topological circle which is no different topologically from any closed loop and then we defined the angle and said that let us define theta$_{n+1}$ is equal to a function of theta$_n$. Let that be the equation. We define a 1 D map that represents the angular rotation or angle in any particular iterates measured from any data.

That is where we were and we had used for that purpose a specific map given by theta$_{n+1}$ is equal to theta$_{n}$ plus omega minus k by twice pi sin twice pi theta$_{n}$ and then you have to take mod 1. This is the sine circle map that we have taken. We had taken a sine circle map and then we had gone ahead with this argument but if this is the representation of the dynamics, dynamics as it turns then we can identify very easily. The situation when there would be a 0 to 1 mode locking, we can easily identify the 1 to 1 mode locking. What would represent the 0 to 1 mode locking? The condition where it does not rotate at all in the small circle. It goes around the big circle and comes back to the same point which means nothing but the period one fixed point.

To find out the condition we had set this equal to theta star, this equal to theta star, this equal to theta star. Solved it and we obtained the condition. Likewise what would be the condition for 1 to 1? 1 to 1 mode locking would mean physically that while it goes around the big circle once, it also goes on the small circle exactly once. Going around the small circle exactly once means addition of one in this case because it is normalized to one. In order to obtain that condition we had simply put theta plus one. We have solved it and we got the condition. But that may be convinced ourselves that there would be two parameter ranges. There are two parameters, the omega is parameter and k is another parameter. There would be two parameter ranges in which you would have a 0 to 1 mode locking and one to one mode locking. We had more or less come up to this stage in the last class. Do you exactly recall what the ranges were? The 0 to 1 mode locking condition was 0 less than or equal to omega less than or equal to k by 2 pi. The 1 to 1 mode locking condition was (Refer Slide Time: 7:05).

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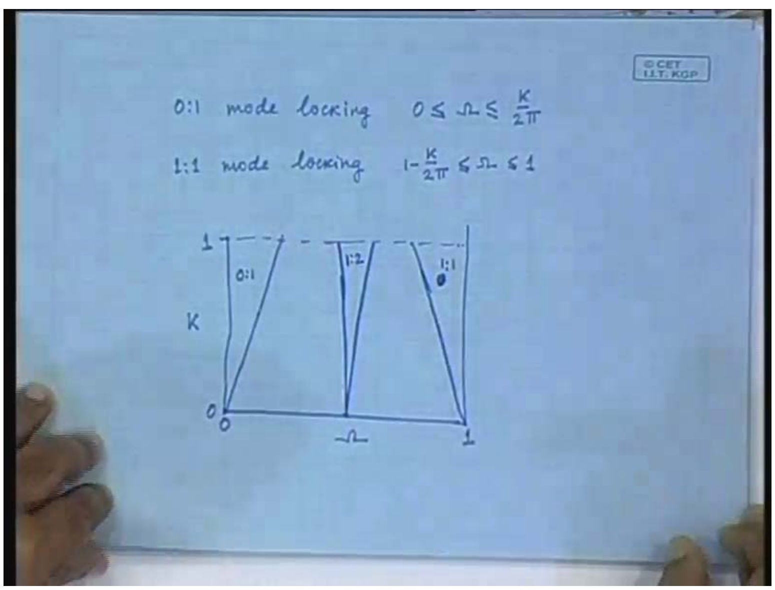

Notice that as k goes to 0 because k is a parameter and we have already understood that k is a nonlinearity parameter. If k is set to zero it is a linear map and then omega becomes the frequency ratio parameter and k is a nonlinearity. It brings in the nonlinearity. If k is 0 then you would notice that there is no range. But as k is increasing you get some range and if k is equal to 0, you get a lot of range.

Now k is equal to 1 you get some range for this one and some range for this one and it is customary to visualize this as a graph in the omega verses k plane in which how would this condition look. This is 0, this is 1 and here it is 0 to say here it is 1. For 1 I will draw a dash line. This would be nothing but a straight line condition that goes up. For k is equal to 0 the range of omega for which this happens is 0. For k is equal to 1 it is 1 by 2 pi, so you would get a range like this. No, we are not limiting. We are only now studying what happens between 0 to 1. It is not really limited to one. We will see what happens if it goes beyond one. Then here also this is a range that is limited to one but its starts from 1 minus k by 2 pi and so here is also a linear range that goes like this. This is the range in which you will see 0 to 1 locking and this is the range where you see 1 to 1 locking.

Now the question is that in between these two, will there be other mode locked conditions? Now other mode locks conditions means what? It means as you can see that in case of this kind of a map, the mode locked condition is represented by periodic orbits. If it is periodic then it comes back to the same state after some time which is nothing but the mode locked condition. By the same procedure that we have already learnt, we can find out the range of parameters for which it becomes period two for example. If you try to do that yourself this will give some trouble because you will land up in a transcendental equation. That needs to be solved numerically. If you do so then you will find that here also there is a range that goes like this. It is 1 to 2. Let's try to figure out how the 1 to 2 situation looks in the graph.

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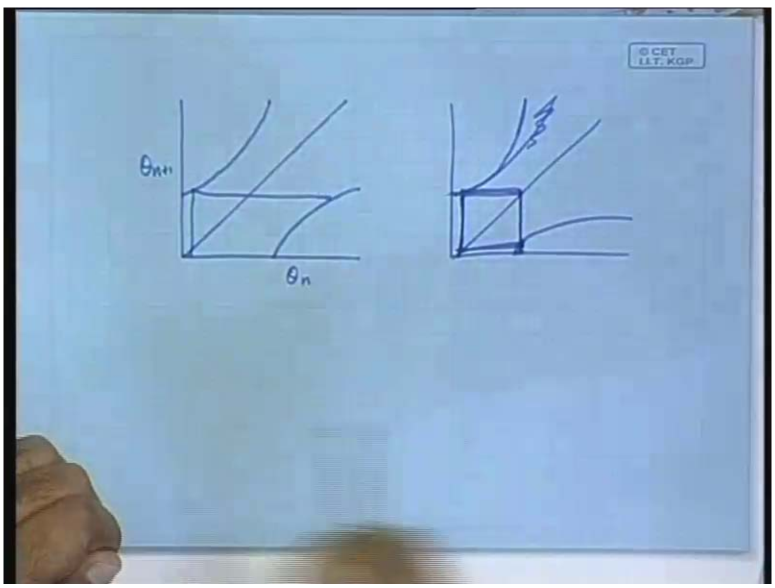

Here is your θ_n here is your θ_{n+1} graph of the map. Here is a 45 degree line so it would be something like this and then from here it will come back here and it will start like this. You can visualize an orbit something like this. The drawing on the left side of the above slide is wrong. I will the draw again. Supposing it goes here, it comes here so it has to come back somewhere here. An orbit can be something like this. If it is like this that means one iterate falls in this chunk and another iterate falls in this chunk; falling in this chunk means it is actually gone up like this, not here.

Here it is continuation of that line and it has come down here because it is mod 1. Actually the graph is going up like this but since you are taking mod 1, it starts all over again here. As a result if there is a iterate falling here, it means that it goes around in one circle. This orbit would mean that it goes around the small circle once and period two means it goes around the big circle twice, so it is a 1 to 2 mode locked orbit. Why it is 1 to 2? Similarly you will be able to visualize a 1 2 3 mode lock orbit something like this.

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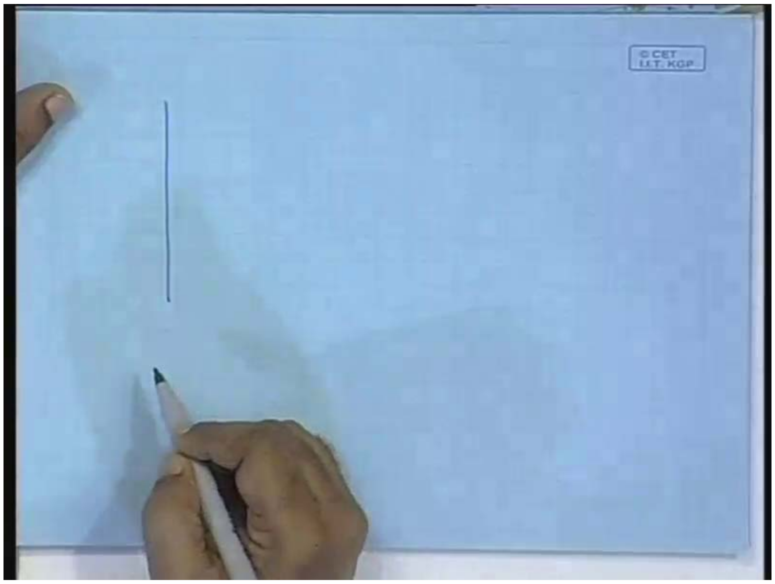

Drawing this is difficult, so I will just show you the graph from the book.

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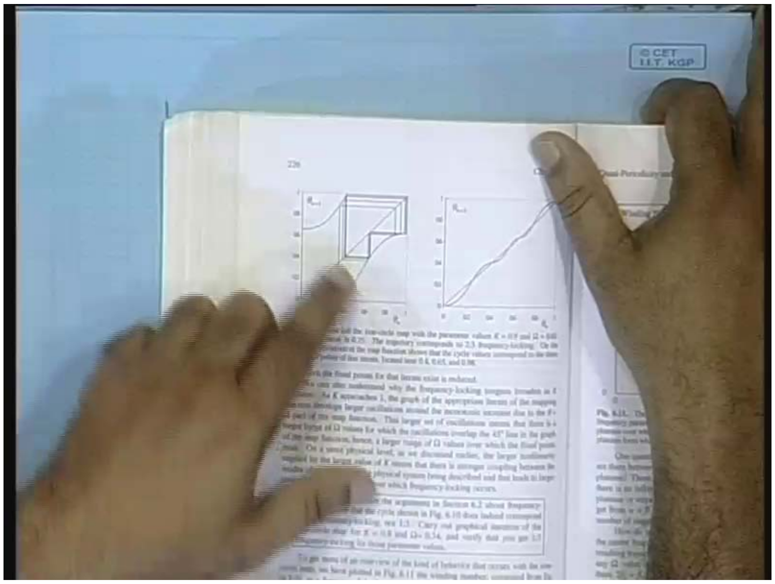

Can you see this graph? This is the period three orbit fine and in the period three orbit I mean they have shown it starting from some initial condition with some iteration that means the Kobe wave diagram. Finally it converges on to the period three orbit. What would be the winding number for this one? Looking at this, can you figure out? Not difficult that all, because there are two iterates in this chunk which means that it goes around the small circle twice. While in total there are three points therefore it goes around the big circle thrice. It is 2 by 3 mode locking window.

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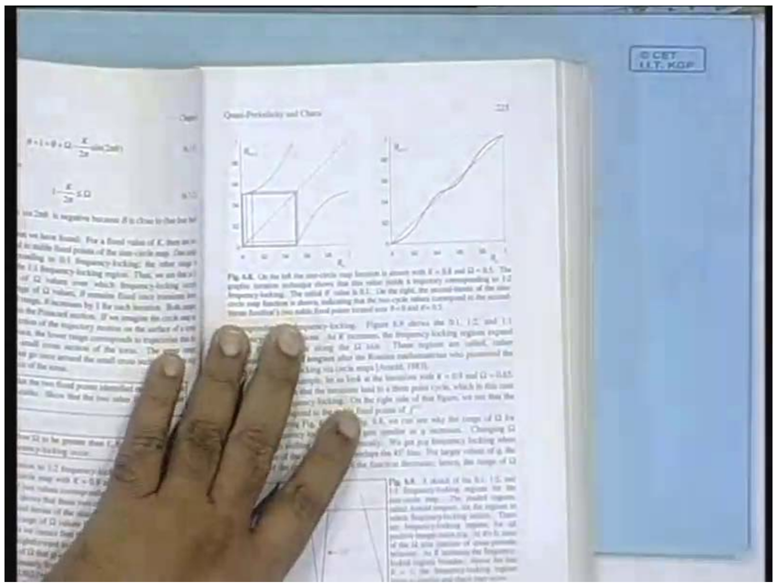

Now for these two situations, one was this situation and the other was that situation. For these both you can plot $theta_{n+2}$ verses $theta_n$. Can you see or is it too faint. I will draw again on the paper.

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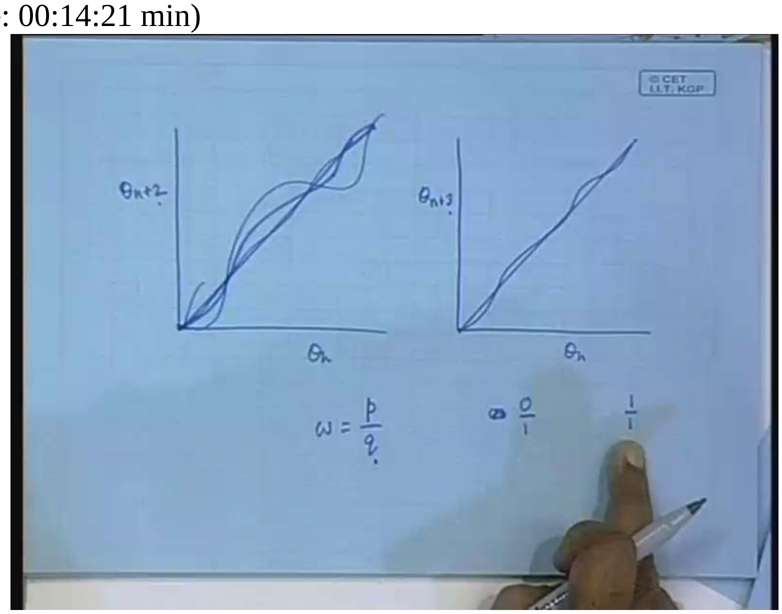

In one case I will have to draw $theta_{n+2}$ verses $theta_n$. The way we did for the logistic map in order to understand the character of the period two orbit and the other case we will need to do three plus. In this case there would be this kind of behavior so as a result of which there would be few fixed points. In this case there would be more number of this things and it is not difficult to see that the more you increase this number that means which iterate I am talking about, the more would be these oscillations. Is it difficult to visualize that? No, because we know in case of the logistic map we have seen that in case of period two there would be two of these, in case of period three there would be many more of these. As you go on increasing there would be more number of oscillations, same thing will happen here. The point that I am trying to make is that therefore the periodicity I am investigating will have to take n plus that many. The more there would be those numbers of oscillations.

Now look at the character of this. What does omega do? Imagine I am drawing the graph of this and what is the role of omega? It is just moving in vertically. As a result if you move this vertically, the intersections will exist only for a certain range. I am trying to give you a sort of intuitive argument that you can appreciate. Mathematically everything can be worked out algebraically but very long calculations will be necessary. I am trying to give you an intuitive argument. There would be oscillations in this graph. Now the more this periodicity, the more will be the oscillations. As a result, as the omega is increased, this will be moved and naturally the higher the oscillations, the smaller will be the range of omega for which this graph will intersect the 45 degree line.

Let's put it this way. First let us understand the argument for k. We find that this broaden as you go for higher values of k. What actually happens for higher values of k? If k is zero there is no oscillation at all, they are straight lines. If you introduce k, it becomes slightly like this. The more it increases, it becomes larger oscillations. As a result the intersection will exist for a larger range of omega as this goes up and down. That is exactly why for larger values of k, we have a larger range of omega for which these are stable. Is that argument convincing? Now we will get into the interesting part of it, if these arguments are understood. The larger the value of k, the larger will be the range of omega for which we will get the periodic mode locking windows.

Second point is that higher the periodicity, the narrower will be the range in which they will occur. Now you would notice that if the periodicity is p by q, if the winding number is p by q then periodicity is this q, the denominator. As a result you immediately conclude that larger the denominator, the smaller will be the range of the parameter for which it will occur. At this stage number theory comes in because here we are talking about rational ratios. Rational ratios starting from 0 to 1 or I will write it this way, 0 to 1 basically 0 and ending it 1 to 1. In between these two, how many rational ratios are there? Infinite. As a result it is not difficult to see that within these two ranges, here is a range of occurrence of 0 to 1 mode locked window. Here is a range of occurrence for 1 to 1, in between there should be all infinite period mode locking windows. We also conclude that higher the periodicity that means higher the denominator, the narrower will be the range.

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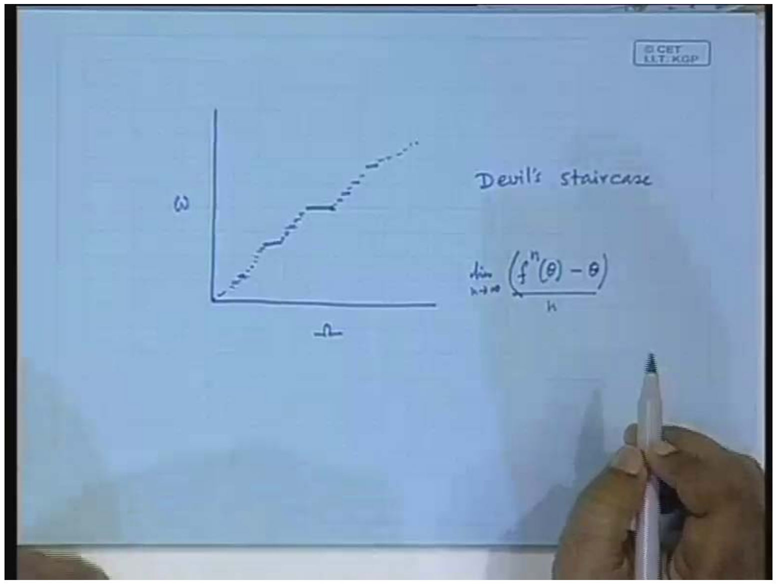

A nice way of visualizing that is if we plot omega versus the winding, it must start with 0. It must end with 1. It goes from 0 to 1. Now you are using omega as a parameter say you are setting some value and you are slicing through this. You will see all these ranges in which there would be mode lock windows. As a result this graph looks like this. It is extremely difficult to draw. There will be a range here 1 to 2 mode lock, there will be a range here, there will be a range here, so I will not draw it to continuous line. It will be a graph like this, so you will get a graph like this (Refer Slide Time: 21:36). What is the specialty of this graph? Each of this horizontal lines represent the parameter range for which a periodic orbit occurs, a mode locked window.

Since if I ask you how many of these mode locked windows are there? Your answer would be infinite because there are infinite number of rational ratios between 0 and 1. If you imagine that this is a staircase and then n starts from here and tries to climb up to 1. How many steps does he has to climb? Infinite. No I don't want to get up to that level. I only want to get up to that level. How many steps? Still infinite. No I don't want to get up to that level, I only want to get here. How many steps? Still infinite. You see in order to go from any place to any place, he has to climb infinite number of steps. He actually cannot climb. That is why this typical structure is called Devil's staircase structure. This is called Devil's Staircase structure because you never can climb such a staircase but it is a staircase. That follows from number theory.

How will you actually calculate this? Omega is a parameter, for every value of omega you will actually have to calculate the value of omega. Every value of omega you will have to calculate w. How will you calculate w? The average number of turning so you will have to take fn of theta minus theta, you have to take limit n tends to infinity. This way you have to calculate w, the winding number and then you have to plot this graph. It's possible to plot, I mean the ant cannot climb it doesn't mean that you cannot plot. It is possible to plot this graph. Of course you will be able to calculate these to a finite precision of the computer but nevertheless it is possible to plot these graphs. Another very interesting thing follows from number theory. See there is a range with zero winding number, there is a range for one winding number.

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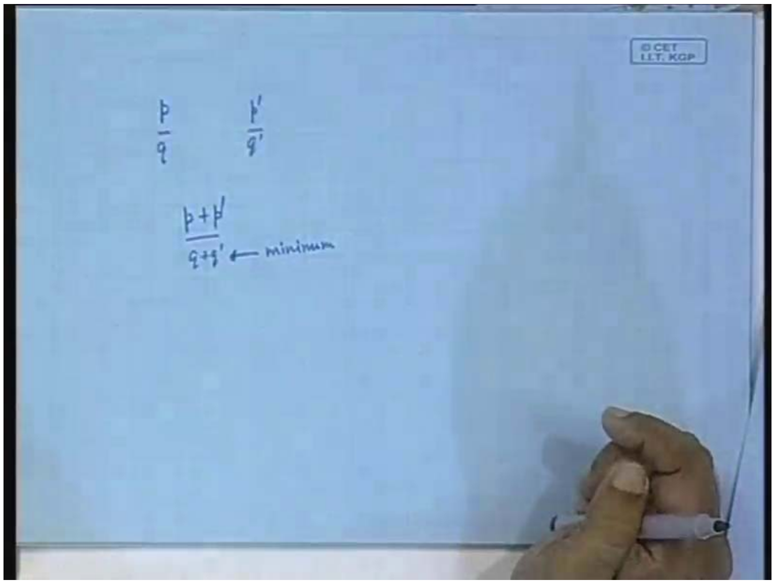

Suppose there are two rational ratios I am talking about. Now forget about dynamics just consider the number, two rational ratios p by q and p dash by q dash. Suppose this is 2 by 3, this is 4 by 7 and suppose these are brought down to the minimum level that means we are not talking about 4 by 6. If it is 4 by 6, we will talk about 2 by 3. Suppose there are two rational ratios then number theory says, there is a theory in that which says that the rational ratio that occurs between them with the smallest denominator is p+p dash by q+q dash. This doesn't follow from dynamics, this follows from number theory was proved pretty long back.

It is a number theory proof but that initially becomes applicable here. Why, because here we are talking about rational ratios and we are interested in mostly how big is the parameter range over which a specific ratio will occur. For example suppose I am trying to find out which is the mode locked window that will occur between this and that? Obviously for practical purpose we will be talking about the window that is largest. Essentially we are talking about this. Then number theory says that can be simply found by p + p dash by q + q dash. Why, because the number theory says that this denominator is the minimum. So between this ratio and that ratio, if i ask you that which is the rational ratio that has the minimum denominator? Then this is the rational ratio. We have already said that once the ratios with a small denominator will occur for a larger parameter range and therefore you can conclude that this particular ratio will occur for the largest possible parameter range between these two. Now we can start from say 0 as a ratio and 1 as a ratio and try to build. What will happen in between?

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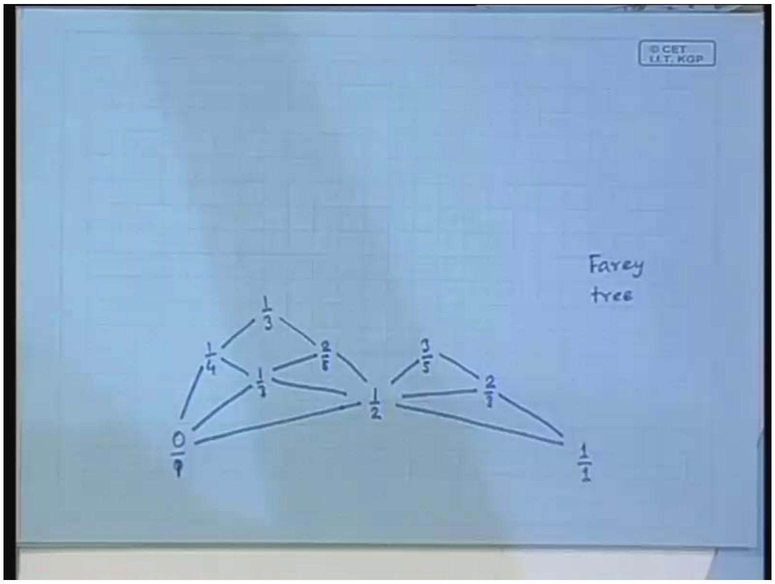

Obviously between 0 and 1, 0 means 0 by 1 and this is 1 by 1. This is p by q, this is p dash by q dash in between there should be largest ratio of 1 by 2, these two added one, these two added 2. In between there should be 1 by 2. In between these two what ratio do we expect? 1 by 3 yes, in between these two what ratio do we expect? (Refer Slide Time: 28:15) 2 by 3. In between these two what ratio do you expect? 2 by 5, yes. In between these two what ratio do you expect? 3 by 5. In between these two, you can easily construct. Can you see that it goes up and as it goes up you have the ratios with a larger denominator and smaller denominator ranges occur to the base of this tree. What will be here? One fourth. What will be in between these two? 3 by 9, one third. 2 plus 1, 3; 4 plus 5, 9. 3 by 9 is nothing but one third and so on and so forth. You would be able to build up the tree yourself and actually that is what happens in dynamics. If you see a bifurcation diagram of this system then you will be able to identify exactly these ratios as the mode lock windows occurring between them. This tree is called Farey tree. Now we can do some filling up of these.

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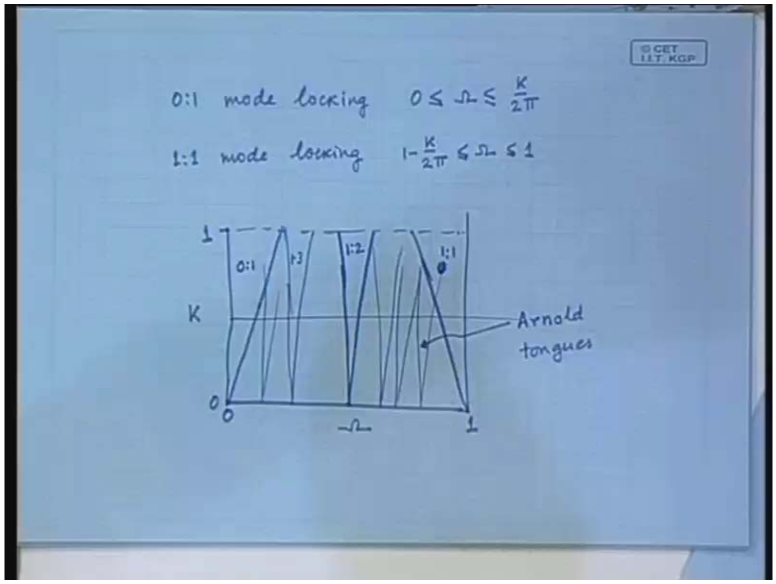

In between these two what do expect? In between zero and one you expect a 1 to 2 ratio, it happened. In between these two, you expect which ratio? 1 by 3 so that should be something like this. This should widen up like this. In between these two what will you see? This is 1 2 3, 1:3. 1 by 4, yes. There will be a range like this. Here will be a range like this, here will be a range like this. Of various widths depending on the denominator. Now these ranges, this geometrical structures after the person who invented them are called the Arnold tongues. In nonlinear dynamics people are very imaginative, they give all sorts of funny names. It is his named after his tongue. These things are the Arnold tongues. He was a mathematician in the 1960’s who noticed this structure. You have these nice unknown structures. What are there in between? Is there any gap in between? Yes. There are gap in between because these ranges broaden.

Suppose they almost fill the top part then obviously that also filled with bottom part because they are broadening. If you take a slice here, in between there would be ranges where there is no mode locking. Then what is there? Irrational ratios. The irrational ratios are quasi periodic behavior. In between you have the quasi periodic windows in these gaps and it is easy to see that the ranges in which quasi periodicity occurs will be large for K close to 0 and the ranges in which the mode lock windows occur, they will be large as K goes to 1. I will get back to some more concepts of number theory little later. By the way have you heard of Fibonacci series? Where? C programming. Good programmable things. Yeah it says that between the number 0 and 1 there would be an infinite number of rational ratios. But then the question naturally comes is that is there any ordering or they occur in any way?

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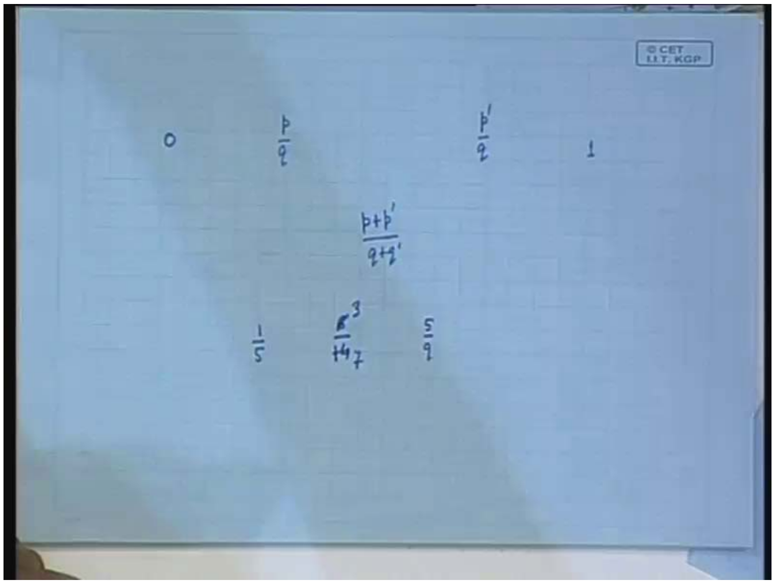

Mathematician showed that no. They occur in some specific ordering sequence and they showed that in between somewhere if you have a ratio p by q and somewhere here, a ratio p dash by q dash then in between, there must be the ratio p plus p dash by q plus q dash. Not only that, this particular ratio has the lowest denominator in between these two. In between these two there would be infinite number of rational ratios but all of them will have a larger denominator than this one. Take 1 by 5 and 5 by 9. In between you have 6 by 14. In between the number half also occurs. Is it? No, not just anywhere. Actually this has to build on the Farey tree concept then only you will get it.

That means here you have this ratios. In between the maximum is this one. In between maximum is this one, in between maximum this one and so on and so forth. That way you have to build on. No, ultimately all the fractions will occur somewhere on the Farey tree. Somewhere but building up is this way which means that in between these two the maximum ratio is half. In between these two, the maximum ratio is this; in between these two maximum ratio is this so on and so forth. No it has to build this way. This will not be allowed (Refer Slide Time: 00:36:23). 1 by 4, 1 by 3 you can take between the consecutive points. No, between consecutive ones you can take like this. Between consecutive ones you can build up like this. p by q by p dash by q dash is…
No, all the fractions will ultimately occur. Because if anything actually contain infinite number of fractions. Your point is correct, you cannot just join anything. Before I go into the Fibonacci sequence and all that, I will have to talk about something. All this is fine but what actually happened on the Poincare section?

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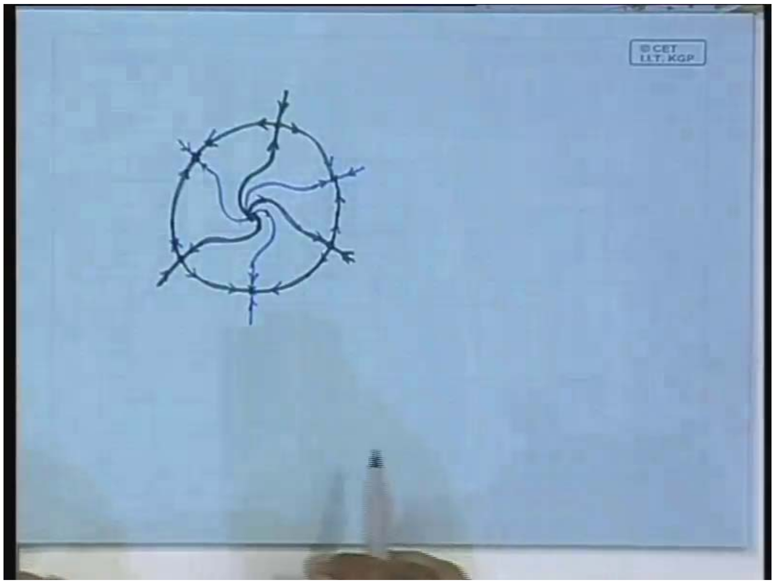

I said that there was a fixed point here which had become unstable. Even though that became unstable and now there is a closed loop around it. That fellow is still there as an unstable periodic orbit. That fellow is still there. But as we say that supposing a periodic orbit has developed which has one point here, another point here and the third point here. The question then is what makes the loop? I can find three points, on what basis do I say that there is still a loop? On what basis do I say that it is a movement on a torus? Obviously valid point, I have to have some construction that shows how these parts are constructed. Now whenever there is a birth of one of these periodic windows, all periodic windows are born through saddle node bifurcation and there is no exception here. Here also there is a saddle node bifurcation that happens through which these are born. That means when this periodic window, this period three is born there is also a saddle that is born.

Suppose the saddles are here, that means when they are born this point and that point was very close. This point and that point was very close and this point and that points are close. As you change the parameter then they move apart. If these are saddles and if these are nodes then we can draw the stable and unstable manifolds. The saddles will have the stable and unstable manifold like this. These would be the stable manifolds and these would be the unstable manifolds (Refer Slide Time: 39:50). Anything that is inside because of the action of this stable manifold goes outward to this loop. Anything at the outside because of the action of this stable manifolds go into this loop but then these are the stable points. From here this is wrong; from here it would converge onto this, from here also it would converge onto this. from here it will converge onto this, from here also it will converge on to this like so and on this there would also be stable manifold because these are stable equilibrium points. If you individually take this point and obtain the Eigen vector, there would be four, two directions. Here also there would be this, here also there would be this. These are all incoming because these are all stable.

What happens to these guys if you extrapolate them further? They must all converge on to this point or diverge from this point but this point is a spiraling orbit. If you take a close neighborhood and obtain the Jacobian this will lead to spiraling orbit which means that it would be a spiraling thing and so on and so forth. It is actually like this. Here it is a spiraling orbit but as it diverges they become the stable manifolds that are shown here. I would better draw some of these with different colors, I am drawing with black pen. These are the stable manifolds of the saddles and these are the saddles. This is a saddle so this is another saddle. Now do you see who is creating the close loop? The close loop is actually formed by the unstable manifolds of the saddle fixed points and the stable manifolds of the nodes.

The unstable manifolds go outwards and converge onto the stable manifold of the nodes so on and so forth and that is what creates the closed loop. That is the geometric process of creation of this closed loop. This closed loop is actually there, it is a geometrical construct not out of our minds, it is there. Even if I can identify only three points still we can say with confidence that it is a mode locked window not just any type of period three window. We have already seen that there are period three window in say the logistic map. One way of looking at is that if you see that means in the logistic map there is period three window. That is also a Poincare section of some kind of continuous time system. I take the data, do some kind of data acquisition and do a … (Refer Slide Time: 00:43:55). What will you see? We will not see two sharp lines but here you will see two sharp lines. Meaning that it is still an orbit on a torus. Geometrically what is happening is that on the Poincare section, you will have this structure.

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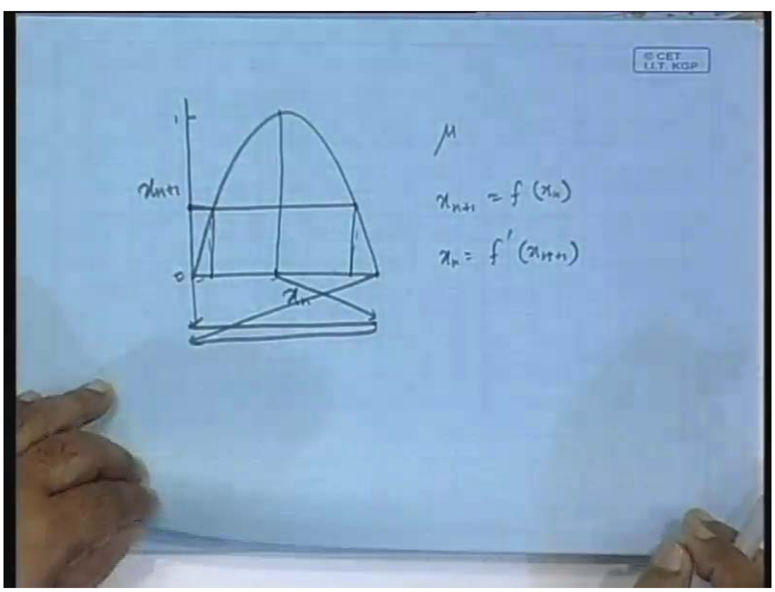

Now let us see what creates chaos? In order to understand that properly, what created chaos in case of the logistic map? See in order to have chaos you have to have sensitive dependence on initial condition. What causes sensitive dependence on initial condition in the logistic map? See logistic map let us draw it was like so, x_n verses x_{n+1}. What created the sensitive dependence? Notice if you take an initial condition somewhere between 0 and half, this range is stretched to the range 0 to 1.

So 0 to half range is stretched to 0 to 1 and half to 1 range is folded. Actually this range goes to this and this range is here and this range is there, so it actually folds. Hence you have the stretching as well as the folding. Stretching and folding are therefore an essential consideration for creation of the chaos but that is only if you have a stretching and folding, it doesn't necessarily guarantee chaos. Because you know that if mu is say something like 3.1, it is not chaotic. It is some high periodic orbit say period four, periodic eight something like that but in that situation also there is stretching and folding. But what I am meaning is that the stretching and folding is a necessary condition for creation of chaos.

Now let us look at it from another angle. What causes the stretching and folding? If you see this for every value of x_{n+1}, there are two values of x_n. It is not invertible, you cannot invert it. If you have x_{n+1} is equal to function of x_n that does not allow you to say x_n is equal to another function of x_{n+1}. Geometrically what is there? If you have the x_{n+1} given that could have come from two values of x_n either here or here. As a result if you want to take the inverse map you do not know which point to take that is the noninvertiblity condition and that is also at the base of this stretching and folding.

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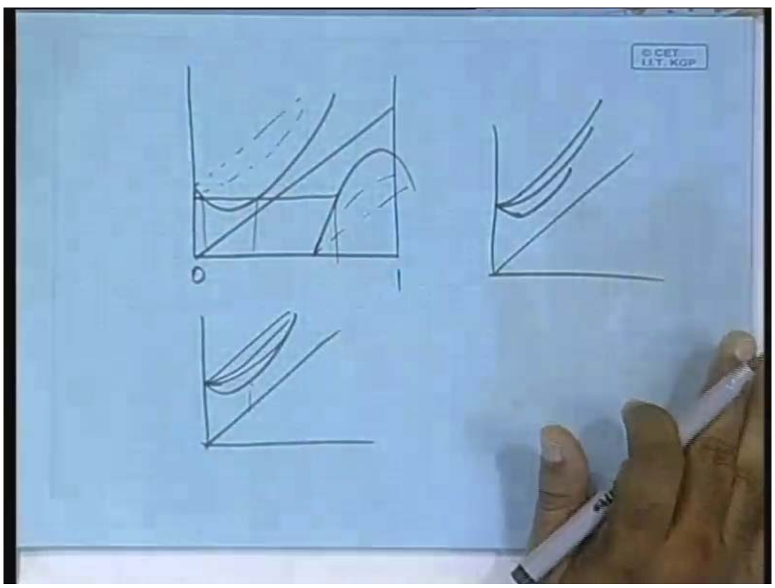

Now look at the sin circle map, its behavior. We have already shown how it looks. It looks like this if k is equal to 0. It looks like this if you have some value of k and it will look like for k larger than one. Somebody were asking why limit k to 1? If you increase k to a value larger than one so you have this situation so that for a value of x_{n+1}, there are many values of x_n that are possible. The map becomes noninvertible. It is also easy to see under that condition there is a stretching and folding. If we extend that what was the behavior? See if you extend without taking down, if you extend that what will happen? It was actually like this and then it becomes like this.
No, it is between 0 and 1, I am not taking a part of the map. Map is defined between 0 and 1 because of the mod 1 condition. I am taking the mod one thereby I am limiting to 0 1 1.

I am not taking a part of the map, this is the whole map and the whole map so long as these lines was like this and finally it becomes like this. When it becomes like this, there is a noninvertible condition that has coming and that happens when k is one. That is why so long as we were considering the situation where k is less than one, you have essentially invertible maps and that is why for k less than one, you don't see chaos. Because the condition is not satisfied, the stretching and folding condition is not satisfied. But the moment it becomes like this, the stretching and folding condition is satisfied and therefore the condition is now right for generation of chaos. No, I am not saying that. I am saying that stretching and folding is necessary for chaos and noninvertibility is a condition that causes stretching and folding. It doesn't really matter. If something is noninvertible then you have stretching and folding, the reverse is not true.

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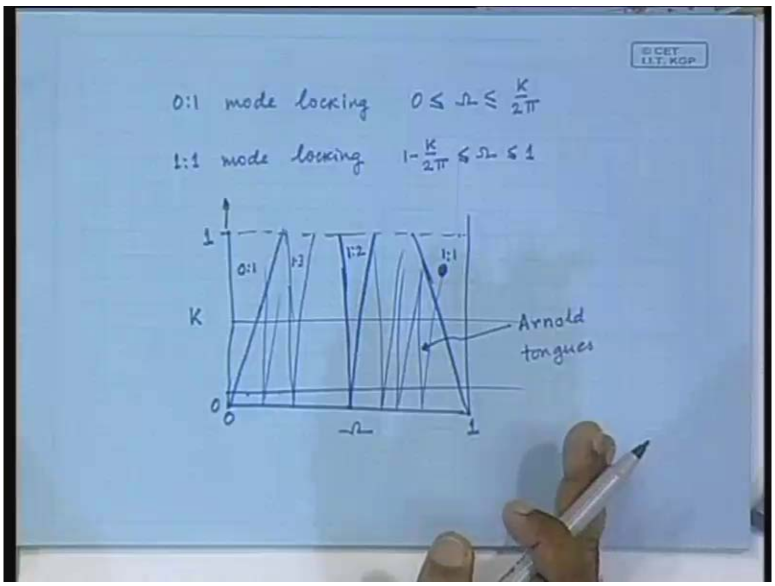

We conclude that if you stretch k further up there we expect chaotic behavior but we have also seen that in this part there would be more of quasi periodic behavior. In this part there would be more of periodic behavior, mode lock periodic behavior. Therefore as it goes up, see even though here you can expect of more of periodic behavior, the quasi periodic behavior is still there. It is not that it was nonexistent, only the ranges become very narrow. It is possible to imagine that I am keeping omega constant and I am varying k, through a line which goes all the way through quasi periodicity zone. It is possible to imagine that.

It is also possible to imagine that I am varying it through a line that goes through a periodicity.
In this two cases, we need to consider the situation. In one case it is the transition from periodicity chaos, in the other situation it is a transition from quasi periodicity chaos. We need to consider both these situations. We will take this issue up mainly in the next class. Different ways of transition we will take up in the next class.

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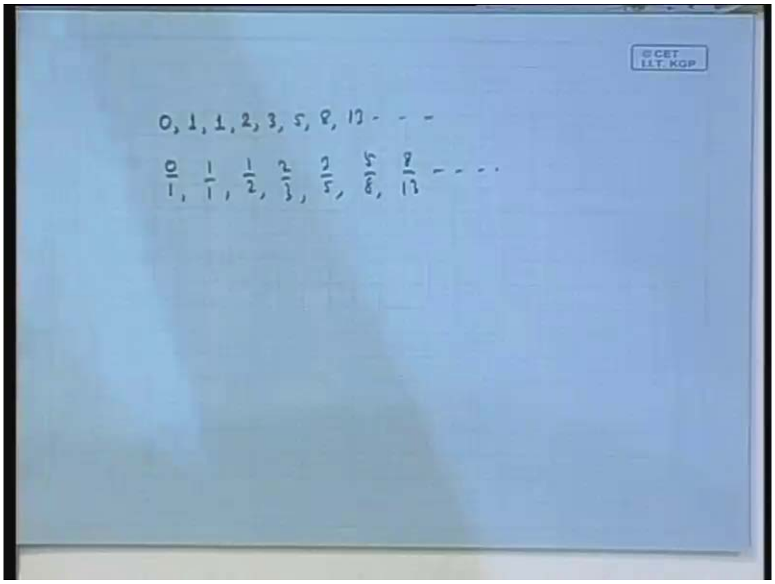

But before that since you have said that you have some background on Fibonacci numbers, so you know that the Fibonacci series is essentially 0 1 add them, add them so on and so forth. What is the Fibonacci ratio? Fibonacci ratio is or golden ratio its sometime called is if you go on making ratios of the subsequent numbers that means the first ratio is o by 1, then 1 by 1 etc. This series actually converges onto a ratio. That ratio has been found to be very important in nature, in various ways. When saplings grow, the distance in which the leaves sprout, if you measure the distances, you will find that they follow this Fibonacci ratio. There are those shells in which there is a winding. Those windings follow a Fibonacci ratio. Even our aesthetics follow the Fibonacci ratio, post cards are made in Fibonacci ratios.

There are various ways in which the Fibonacci ratio becomes very important. For example this I don’t have to measure. If it looks good to our eyes this ratio, this rectangle; if its look good to our eyes the ratio is harmonious. Immediately you can say with confident that the ratio must be Fibonacci ratio. Then go ahead and measure it, you will find that it is a Fibonacci ratio. So that has various ways in which Fibonacci ratio appears in natural sciences. In the next class we will see how it appears in dynamic also, very interestingly it appears in dynamic also. That’s all for today.