Chaos Fractals and Dynamical Systems
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Lecture No # 26
Dynamics on a Torus

In the last class we were talking about quasi periodicity, mode locking and things like that. In general I told you that such phenomena occur where there is a scope of having two or more frequencies in a system like for example suppose there is a pendulum and the pendulum's bob which is given an additional sinusoidal perturbation. There is one natural undamped frequency of the pendulum. I mean if the additional forcing were not there then if they have oscillated with a certain frequency and in addition to that you have added a frequency component. There should be an interaction between the two frequencies as a result of which the phenomenon that we are talking about might happen. There are also situations where you would normally not expect any sinusoidal oscillation. For example those who come from electrical engineering know that there are a class of power electronics circuits known as DC to DC converters in which you only have DC voltages. There is a DC input and a DC output so you don't expect anything more to happen except for the fact that these are switching converters.

There should be switch on switch off, switch on switch off so there should be a characteristic time period for which the switch on's and switch off's takes place. One frequency is easily identifiable but the other it has been found that when the parameters are varied across the certain parameter value, suddenly a very slow sinusoidal oscillation develops and that has been recognized as a very lingering problem of DC to DC converters. In such situations it will not be easy for you to visualize where did that additional frequency come from, other than looking at the mathematical analysis of it. You have a normal operating condition which is a limit cycle, you place a Poincare section, look at the fixed point, take a local linear neighborhood and if you find that a pair of complex conjugate eigenvalues are going out of the unit circle you know that is what should happen. That should result in the creation of another frequency component. These things are often very perplexing for main stream engineers but once you have this mathematical view point, things become relatively more straight forward to visualize. But in general when we are trying to understand the phenomenon of dynamics on a torus, it will be easier for us to picture situations where we can clearly identify two possible frequency components.

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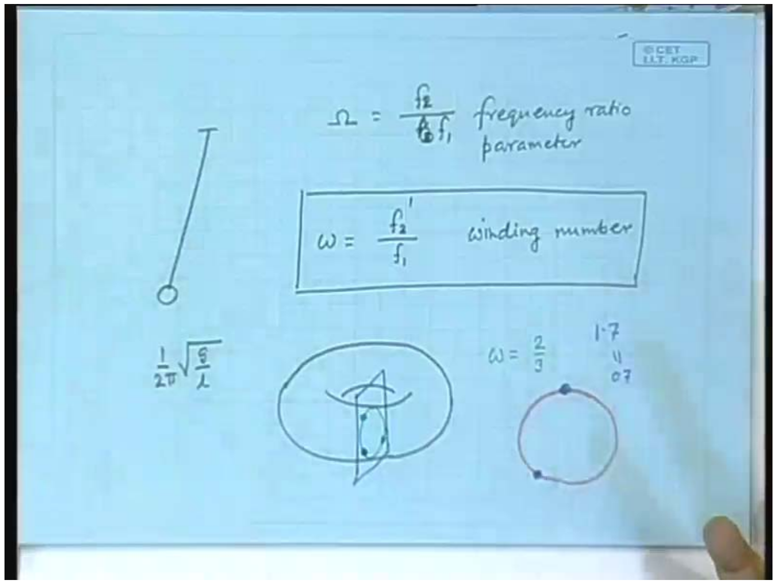

As I told you one possible thing is simply to visualize a pendulum with the bob which would move in a to and fro motion with a certain frequency, the natural undamped frequency and that would be one by twice pi root over G by L. That would be the frequency of oscillation, if there had not been any additional forcing. But supposing it is given an additional forcing a sin omega t, the frequency associated with that omega is additional frequency component. That immediately tells you that there are two ways to picture the frequencies. One frequency in this case is the additional frequency that has been imposed from outside but here what is the frequency of this fellow? This is what we have written that is the small amplitude frequency that means the frequency that you can identify under linearization. That is the natural frequency identified under the assumption that the oscillations are not very large. In that case it would be a linear system on which you have given a sinusoidal perturbation but of course in a linear system you are not allowing any scope for the two frequencies to interact.

In a linear system if there is no non linearity then they cannot interact. If you add some non-linearity in this case, if you simply do the modeling of the pendulum and allow the pendulum to oscillate over a large amplitude, it becomes a nonlinear system. In actuality the frequency will not be this, it would be something else. You can identify two types of frequency ratios, one is called the omega which is simply f_1 by f_2 where these two I mean if f_1 is the frequency imposed for outside and f_2 is the natural undamped frequency here then this omega is the ratio. Therefore it is not the ratio of the actual frequencies. The actual frequencies of the system, it's not the ratio of that. This omega is a sort of idealized the ratio, it is called the frequency ratio parameter. (Conversation between professor and student-Refer Slide Time: 07:15) Yes (but why you are forcing it is somewhat differential equation which is generally finalized this has been the forcing frequency by which it oscillates) that's true. Ultimately it will oscillate at the forcing frequency. But still we can identify a number like this because if you do not apply the forcing, it will oscillate at this frequency and you have given the forcing and therefore you can identify at least in theory. (Anyway that there should be these two frequencies) No, it's not the actual frequency oscillation.

It is just the frequency ratio parameter somewhat idealized but the actual thing is called the w parameter or winding number which is f_2 prime by f_1 where f_2 prime is an actual frequency of oscillation of the bob. In this case we are allowing the nonlinearities to play. In this case we are allowing the nonlinearities to come into the picture. (Refer Slide Time: 00:08:22) This one would be f_2 by f_1. The f_1 is the external imposed frequency, f_2 is the natural frequency and the f_2 prime is the actual frequency of this oscillation taking into account the nonlinear. This is called the winding number. Actually this is the more important thing because that is what actually you see in the system. This is sort of idealized value. It is not difficult to see that if the winding number is a rational number then you have a periodic orbit and if it is an irrational number, you have a quasi-periodic orbit. So which means that ultimately our point of concern is the winding number. That is what distinguishes between periodicity and quasi periodicity, the winding number.

Suppose the winding number 2 by 3, in that case what will you see happening on the torus? Imagine the torus and imagine that you have placed a Poincare section here. What will you see on this pointer section? Essentially you will see the section of the torus would be a circle but you will not see the circle because in this case w is equal to 2 by 3. In this case it is a rational ratio and therefore you will see a finite number of points. How many? Three number of points there. What will happen is that say the 3 points are here and say here. Starting from here it will go around it and will hit it somewhere here, will go around it and will hit it somewhere here and will go around it and finally come back here. That is what the behavior is. By the time it makes 3 rounds in the big circle, it makes two rounds on the small circle and that's why the ratio is 2 by 3. (Conversation between professor and student-Refer Slide Time: 11:50) (Sir how would you differentiate between irrational and rational, how can you say that it is irrational or rational in a real, we cannot always say that it is exactly 2 by 3 or) His question is how can you really say it is rational or irrational because in immediate neighborhood of every rational number there is an irrational number. In the neighborhood of every irrational number there is a rational number. How can you say that? Of course you say with finite precision of your calculation but soon we will see that we can do a lot of coarse grained calculations.

Just wait till the end of this class I will explain how. Because of the mode locking. Because as I explained in the last class, the mode locking happens for a large parameter range so the rational ratios will be happening for a large parameter range. That is what allows us to even by a rough calculation, even by a finite precision arithmetic to find out that it is really a rational ratio but we will come to that. The point is that let us now develop the circle here and see what is happening. Suppose here is a circle and suppose there is a point not exactly this situation, supposing another situation. Initial point is here and before coming to the next one it has gone around once and fallen here. It is possible.

Now will the Poincare section be able to distinguish between these two situations? No, that means started from here and when I just explained, I said that it goes around and comes here. Supposing it doesn't do so, rather it goes around and by the time it comes here, it has gone around this once and it has come here. Then looking at the Poincare section we will not be able to distinguish it. That is why when you talk about these mappings, there is no point talking about these 2 pi rotations. That is why you always take a modular 2 pi or if you say the whole rotation is one then we take a module one.

You understand what is the module one. That means you subtract all the whole numbers that means if you have something like, if the winding number comes out to be say 1.7, this is equal to 0.7. As far as a Poincare section is concerned, 1.7 is same as the 0.7. But we have to understand that something has gone on, that means it has gone around this small circle without intersecting here. It has gone around the small circle one more time. We have to keep in mind but on the Poincare section that might not be visible.

Now let us come to a very simple situation by which you can understand these things. Whenever we were talking about dynamics, we found that it is very convenient to understand things in terms of maps. We have understood the logistic map, we have understood the behavior, we have understood how to calculate the fixed point, we have understood how to find the stability of a fixed point and on that basis we can do a lot of things.

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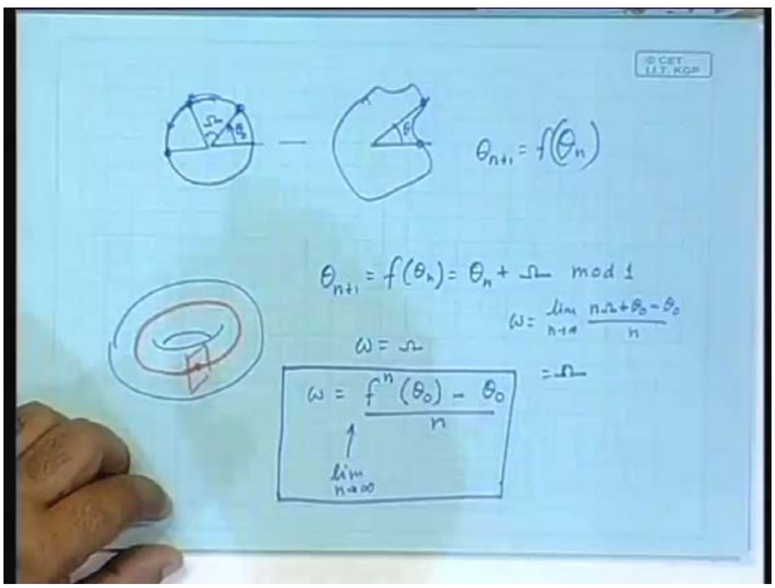

The question now is supposing you have the circular cross section. Is it necessary that the cross section of the Poincare section will be circular? No, it could be any way. For example it could be as well like this. It is still if you see this on a Poincare plane, it is still a quasi-periodic or a torus orbit. That is why when we define the map, it is always convenient to define the angle by which it rotates. Here also you can define the angle. For example if from this point or on the next turn, it comes here. Then we can define this theta and I don't care whether actually the orbit is so crooked or simply a circle. (Sir topologically it has to be a circle) Yes, it is topologically a circle. In that case it will be convenient for us to define the theta and if we can define something like theta n +1 is equal to a function of theta n. We are through and we can define the dynamics (is the cross section of the torus is circular) (00:16:58) Cross section of the torus is a topological circle means that if it is an ellipse, it is still topological the same as a circle. It is a closed loop that's all. (The control that is mapped, how can it be say anything other than the cross section) (00:17:15) Yes, it is the cross section. If you are drawing such a nice thing, it is fine but have you actually seen vada being fried in the actual fries. I mean you take a Poincare section, you will find that it is everything other than circle but these are topological circles.

(Sir how do you determine the center) you don't really need the center, though you do have the center but we don't need. Why do you have the center because how did this torus come into being? Initially it was an orbit like this which became unstable. When the orbit was like this, if you place a Poincare section it was a point that became unstable. When it becomes unstable and it develops into a torus, still this unstable periodic orbit is there. That can serve as the center. Though normally anything inside the closed loop can serve as a reference point from which you can measure the theta but it will be convenient for us if you can locate that unstable fixed point, it will convenient for us to locate it from there. But the point I am making is that whenever defining in terms of theta then you do not really need to consider whether it is a bend or crooked or whatever, as long as it is a topological circle it's fine. So long as you can eat the dough nut as a dough nut it's fine. We have to define it something like this.

Now if we can define it like so I will define a map and see what I mean n +1 is equal to the f of theta_n is equal to theta_n + omega. What is this omega then, what is it doing? Suppose initially the theta_0 was only this much that means initially it starts from this point. Then in the next iterate what happens? It turns by an angle omega, so it goes by an angle omega and comes here. In the next iterate, it again turns by an angle omega and so on and so forth. You might say (is this the same omega arise) yes, it is the same omega I will come to that. Yes, it is the same omega and it will turn out to be the same omega. (00:21:02) No, wait. Here things are normalized so that the full circle is one. We have normalized it so that going around the whole circle is one.

Now that is why his question is valid. This should actually be written as mod one because if this omega is greater than one then it really makes no sense. As far as the Poincare section is concerned, it will turn only by the remainder part. If omega is 1.1 it will turn by 0.1. Then what is it doing, what is this fellow doing then? It is turning the iterate by this much. What is happening on this system description? See starting from here it is going around it and then landing here. In that time it has turned by an angle omega. Again it is gone around and by that time it again lands here it turns by an angle omega and so on and so forth. (Sir omega or omega naught) no, it is omega because it is a linear system. There is no interaction between the two frequency that's why I am saying that. It is omega I will explain how. What is it doing? It is simply turning by this much. Now if I ask you, what is the winding number. Can you figure out the frequency ratio?

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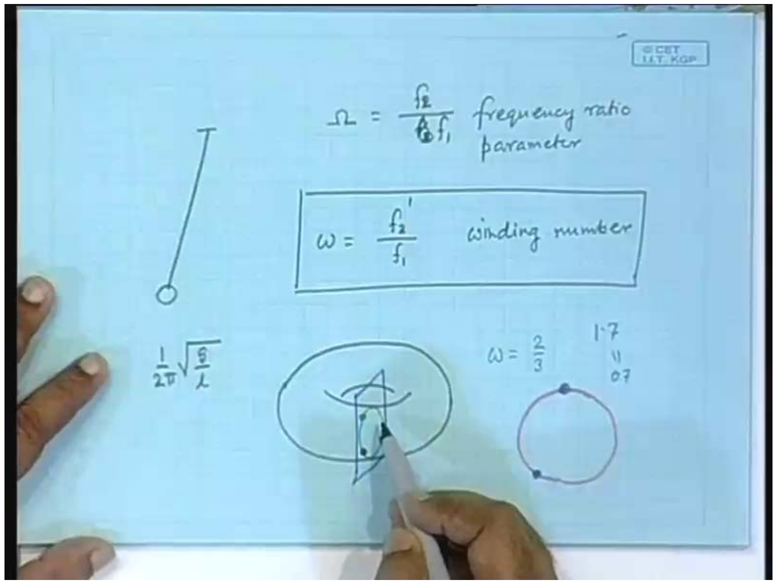

By the time it comes through one of the turns, it has turned by an angle omega. What is the w in this case? Omega by one. It has turned by omega around the small circle and by that time it has gone around in the big circle and therefore it is omega one, it is omega. In this particular case your winding number happens to be equal to omega because it is a linear system. Normally how would you define or calculate the w, the winding number looking at the map? See here we had defined it depending on the frequency ratios, we assume that we do a fifty and identify the frequencies but suppose we do not have that. We are only available to see what is happening on the Poincare section. Then you only have this map available to you. Then the definition of w is, in each mapping it turns by an angle omega. If n number of mapping it turns by angle n omega.

Generally speaking if such a map is given, if you take f to the power of n some theta$_0$ and subtract theta$_0$ then you find out how much it has turned in n iterations. Remember do not take mod one at this stage? Because you are really wanting to calculate how many times it turns. Divided by n will give you the average turning per iterate and that is omega. In general only in this particular system that comes out to be your omega. In this case in this system only the w is equal to omega. By the way this has to be, I have to put in here limit n tends to infinity because we have to consider the transient dyeing down and that should converge on to the actual value of the winding number. In this particular system what is happening? In this particular system you will say w is equal to limit n tending to infinity. Here it is n omega plus theta$_0$ minus theta$_0$ divided by n, simply. That is why it is omega for this particular system. This is our definition of the winding number, given any specific map. Is the grounding clear? Now on that basis we will take a greatly forward.

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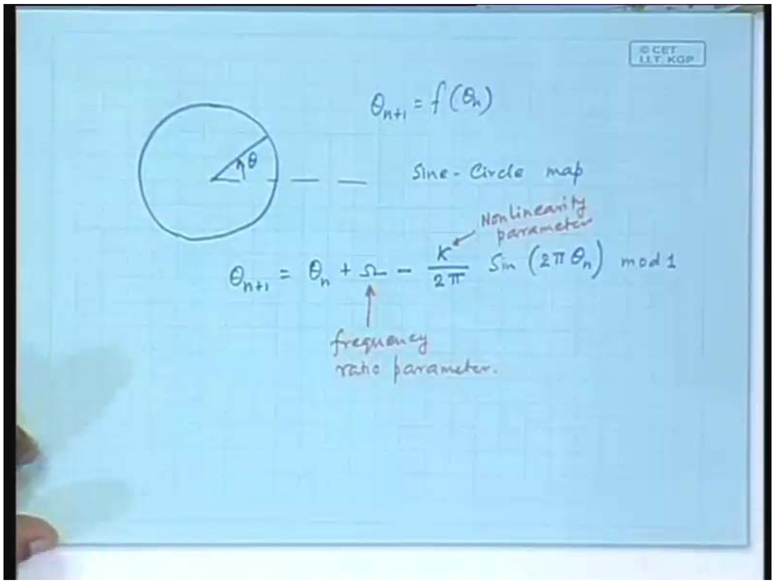

What are we doing? We are defining a topological circle. We are defining some kind of a datum level, we are defining theta from here and we are defining theta n +1 is equal to some function of theta n. We are trying to understand all those things I have been talking about in terms of such a map. The map is in theta as it turns. Now once this grounding is clear, now we can introduce a very special map that has been very well studied because almost all the phenomena that happen on the torus can be understood in terms of this map. It is called the sine circle map. In some cases, some books you will find simply it is said to be the circle map because it maps on a circle. It is theta n + 1 is equal to theta n plus omega is same as the earlier one, minus k by twice pi. This is only to normalize it, sin twice pi theta n. we have to do a mod one so this is the expression for the map.

Now you would notice that if k is equal to zero, it is same as the… this is equal to circle map that we have got. This is a linear map and this is a nonlinear map and the non-linearity has therefore quantified by this term k. This is the non-linearity parameter and this system actually has two parameters. What is this? It is actually the frequency ratio parameter. We had started from the frequency ratio parameter, their frequency ratio. What we have here is same as their frequency ratio, when the normality is said to be zero so this is the frequency ratio parameter. There are two parameters in this system. Now let us try to understand how the behavior will be? First if k is equal to zero, can you draw the graph over the map? k is equal to zero means the old system. It's a map so you can draw a graph, theta n +1 versus theta$_n$ can you draw?

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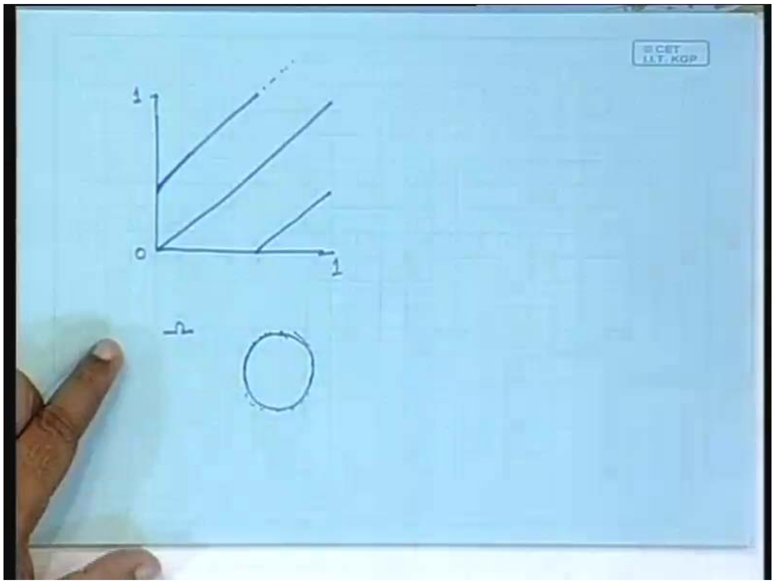

Suppose this is the 45 degree line. Then here $theta_n$ times one so it is something with the slope one. It goes till the value of one then mod one. It again jumps here and goes. That is the character of mod one. It comes down here and then it goes. It is actually not a discontinuity in the real sense because it is actually continuing like this but since we are taking mod one because there is no point in looking at how many times it turns around the small circle. That is why we have studied this function. This system can only keep on turning since the omega gives the amount by which it turns. The omega is same as w and therefore if this omega is rational we have got a finite number of points. If the omega is irrational we have got an infinite number, drift we get. That means it will have a ring with all the points filled up, if it is irrational. That is the behavior of the original system without k that means k is equal to zero.

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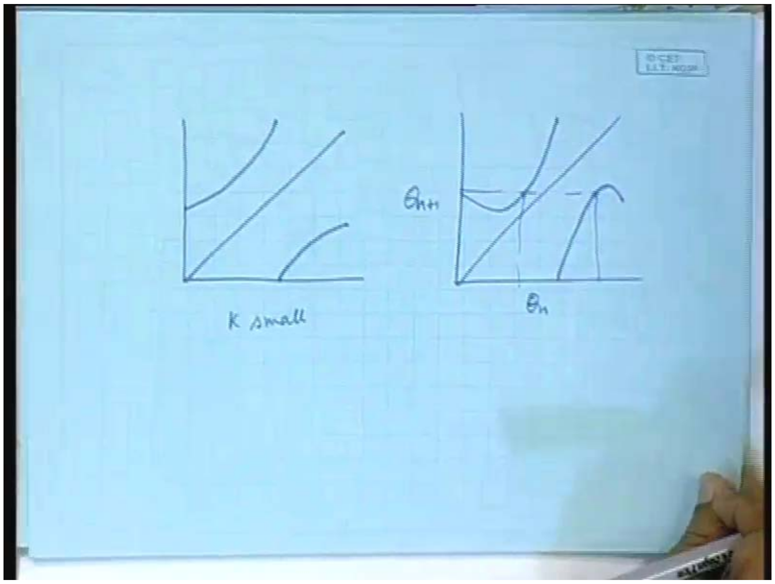

Now if you have k values that means take a small value of k then it was going, it will result in a bit of non-linearity something like this and again it will start from here. It will have a some bit of non-linearity. Instead of this being a line, straight line as for k it will become like this, for k small and for k large this bends will be large bends and then it will short here. This bend will become large bends as a result you see at this stage it is invertible. But here it loses invertibility because from here for every theta$_n$ there is a unique value of theta$_n$ plus one but for every value of theta$_n$ plus one there are two values of theta$_n$. You can come down in two places and we have seen in the case of the logistic map that leads to folding so there would be a lot of complicated dynamics that is possible in such systems. But let us work things out somewhat through logic now.

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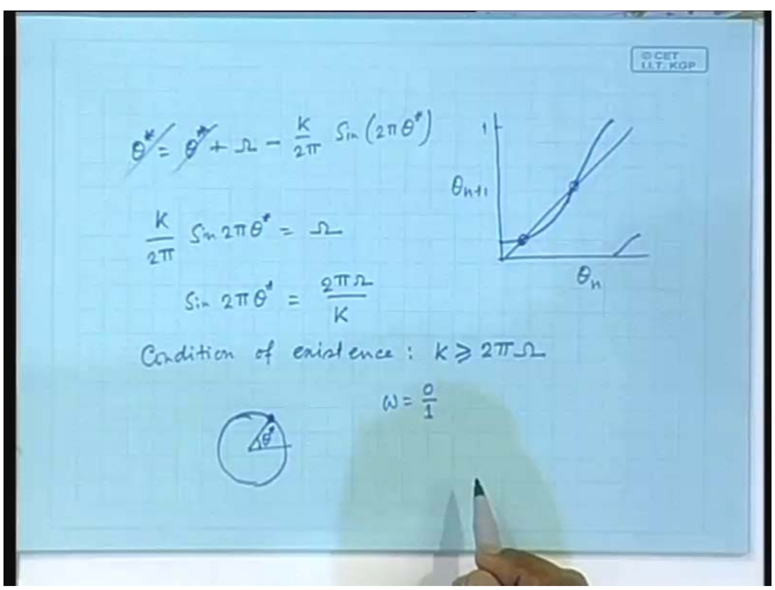

Now let us write down the map again. How to find out fixed point of this one and when will fixed point exists? In order to calculate the fixed point, we will say theta star is equal to theta star plus omega minus k by twice pi sin twice pi theta star. Theta star is the fixed point, it cancels off. That leads to k by twice pi sin twice pi theta star is equal to omega or sin twice pi theta star is equal to twice pi omega by k. As you know, sine something has to be between zero and one. So long as k is above, 2 pi omega we are through. We will have this fixed points. The condition for existence of the fixed point is that k is greater than equal to 2 pi omega. We have two parameters k and omega and if this relation holds, you conclude that there will be at least one fixed point.

Suppose we located a fixed points, can you figure out what will be the behavior B on the torus? Suppose we have located a fixed point. We have satisfied this condition and depending on this condition, we have located a fixed point. How will the fixed point be? Here in the graph at some value it will go like this. You can easily see that here is a stable fixed point, here is a unstable fixed point. Beyond this value of one, it will again start from here so it will be something like this. But nevertheless we have identified that there is a fixed point here, stable fixed point here. Starting from any initial condition it will ultimately converge on to that and suppose it has converged. Can you figure out what the behavior will be on the whole torus? Yes, what is your question? (00:36:37) No, I have said a fixed point is existing. It doesn't mean that it is stable.

I have only worked out the existence condition. I dint not work out the stability condition. We will have to work that out separately but supposing we have satisfied the condition and found a stable fixed point because you can see that I was saying, the more you increase the value of k the more bend will be there. It is not difficult to see that beyond some value, it will have a behavior like this. You are expecting a saddle node bifurcation as a result of which one fixed point will start to exits and when it starts to exists it will be stable. Later that may lose stability it's a different issue.

Now when it is stable, what is the behavior? See here is the circle on which I am talking about the thetas. Suppose this is your theta star that means once if it is here, the next iterates also it is here, next iterates also it is here and so on and so forth. What does it mean? It means that it does not rotate in the circle. It does not rotate in the small circle, it only rotates in the big circle and comes back here. What is a frequency ratio? Zero to one, so here the winding number is clear or zero. By the time it comes back through the big circle, it does not rotate in this small circle and therefore the winding number is zero. It is possible but under some condition where it goes around this circle once and then lands here. That will be one by one. Under what condition will that happen? If you forget about this mod one and calculate this, you can easily calculate the condition under which that will happen. You can calculate the condition under which the omega will be one.

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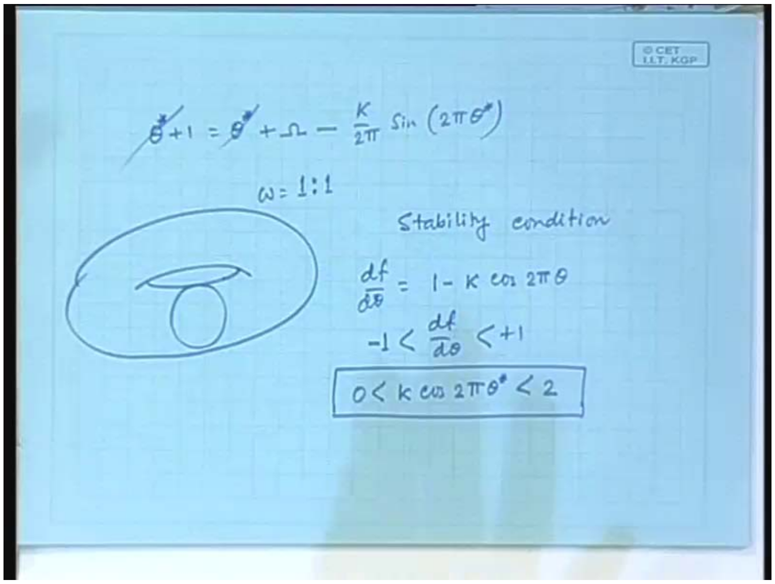

Can you calculate the parameter condition like this for which the winding number will be one? Since you have raised the question let’s do it. What will you say? You will say theta star plus one is equal to theta star plus omega minus k by twice pi sin two pi. Do it and you will get. You solve this equation, you will get the condition. There will be another parameter range for which, so this theta star solution will lead to omega is equal to one to one ratio or omega will be one. I am writing one to one because to let you understand what is happening. Here this is the circle and this is actually the cross section of this. By the time it goes around, it comes back again here so that it turns in the small circle once.

You can easily find out the condition. Do that but now you raised the question that supposing this fixed point exists but will it be stable? How will you find out the stability condition? Slope. All you need to do is to calculate the slope so calculate the slope. So stability condition, we will be able to say this fixed point is stable when you know that there is a range. A point at which it comes into existence and a point at which it becomes unstable. In order to calculate that we will have to differentiate it, so df d theta is one minus k cos twice pi theta. Check it out. One goes off, twice pi comes forward it gets canceled, so k cos so this is what (Refer Slide Time: 42:10). It becomes unstable when this fellow is one, so you have to put this as one and solve for it.

It will be stable for a parameter value between minus one that immediately gives you a range of theta. That will give you the parameter range zero less than k cos twice pi theta star less than two. That is the condition under which the fixed point will be stable. We have found the existence condition as this and the stability condition as this. When both are satisfied, your zero to one ratio is satisfied. Again I showed you how to calculate the range for the one to one ratio. You have to do the same thing to find out the range in which it is stable. You can easily see that there is a parameter range in which the zero one ratio is stable. There is another parameter range in which one to one ratio is stable.

Natural question is how can you calculate the two to one ratios, one to one ratios? That means we are talking about mod locking now. We are essentially probing the phenomenon of mod locking I say moon is mod locked. Moon is mod locked in what ratio? One to one ratio. Mercury is mod locked in one to one ratio but there are many other satellites that are mod locked in other ratios, they are still mod locked. They are some 23 moons of the Saturn and many of them are mod locked in different ratios. Obviously we are trying to understand these ratios and in practical systems, in electrical engineering experiments, in mechanical engineering experiments you will get enormous number of situations where you have these sequences of mod locking windows. This is a very common in fact. We are trying to understand that in terms of a toy model and the toy model is simple enough. Let us try to understand when and on what condition will there be a quantitative ratio.

Intuitively speaking when a period two orbit exists in the system, let's try to understand what actually happens on the torus. We have got the map. I am talking about the period two orbit of this map. Can you find out when the period two orbit will exist? Simple, we have already calculated the stability condition and we know the period doubling happens when the ratio becomes minus one. We take that condition of minus one, that would be when the two which is on this side. This is equal to two, if you push the parameter k beyond that range then you get a period two orbit. That's what we have learnt from earlier classes. We need to find out the ratio. we can we can also do this, we can obtain the theta n plus two in terms of $theta_n$. Same thing as you did in case of map and find out its existence as well as stability conditions. Both are possible, both roots are possible.

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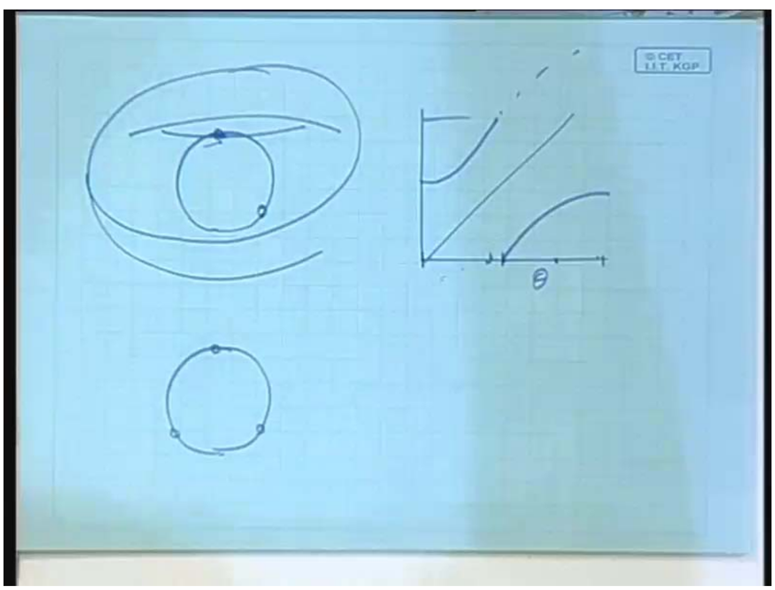

Now once we have found that what we will get? On the circle we will get two points say here and here. What does it mean? It means that starting from here, it went around the big circle and landed here, went around the big circle it landed here. That is the behavior here, it may be also true. Look at the graph of the map, when that happens what is the ratio? No, there is no three here. Started from here, went around it and landed here.

Again I try to visualize sorry so started from here went around it landed here, went around it and landed here. By the time it went through two circles here, how many times did it go around this? Once so it is 1:2 ratio. It rotates like this once. While it rotates like this twice, 1:2 ratio. Likewise there would be some parameter range at which it will became a period three behavior. Say it becomes period three means there would be 3 points, starting from here it will go around it, land here, go around it land here. How many times did it go around the big circle? Starting from here once twice thrice, three times. How many time did it go around it? Once, so you have a one by three ratio but not always. See the graph of the map has this kind of a property.

This is the range at which the next iterate is between zero and one and if it is in this part, theta is in this part then it has actually gone beyond. So it has been brought down which means that if any iterate lands here, it has actually gone once around it. Here the mod one has come into picture but you have to understand that no, if any iterate lands here it means that in between it has gone one additional time. Now suppose you have got a period three orbit out of which two points lie here and one point lie here. What is its corresponding behavior on the torus? Try to visualize for example I have a situation I can show you, probably from the book this is Heilbronn's book.

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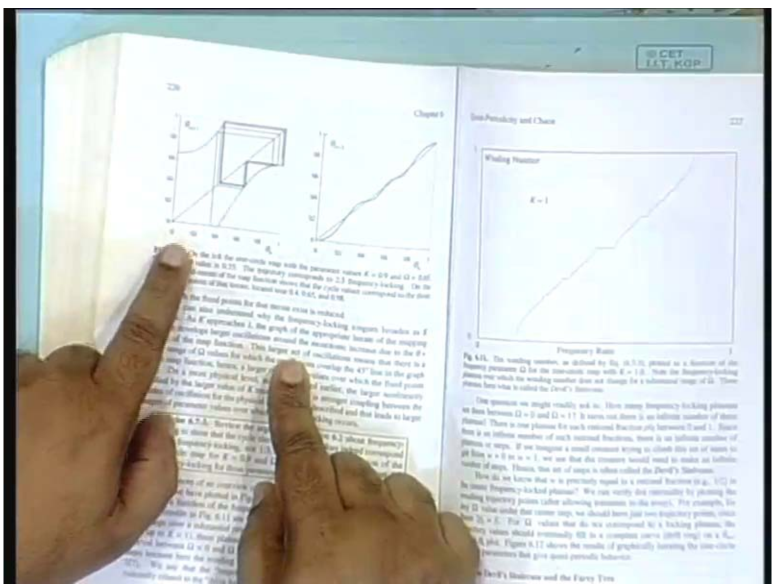

Can you see this picture? You see here there is a period three orbit, out of that one was in the smaller range and there are two in the bigger range. That means where iterates falling in this part. Iterates falling in this parts means it had gone twice. What does it mean? What is this ratio simplifies? Two by three because in between it has gone twice, there are two iterates here so 2 by 3. This particular period three orbit means 2 by 3. You see we are able to understand the mod locking ratios in terms of this simple map and in fact all the higher periodicities are possible and all the higher periodicities means different types of mod locking windows. Now you might ask how big are these windows? For example if I keep say your k constant and vary omega then what do we expect? Do you understand my question?

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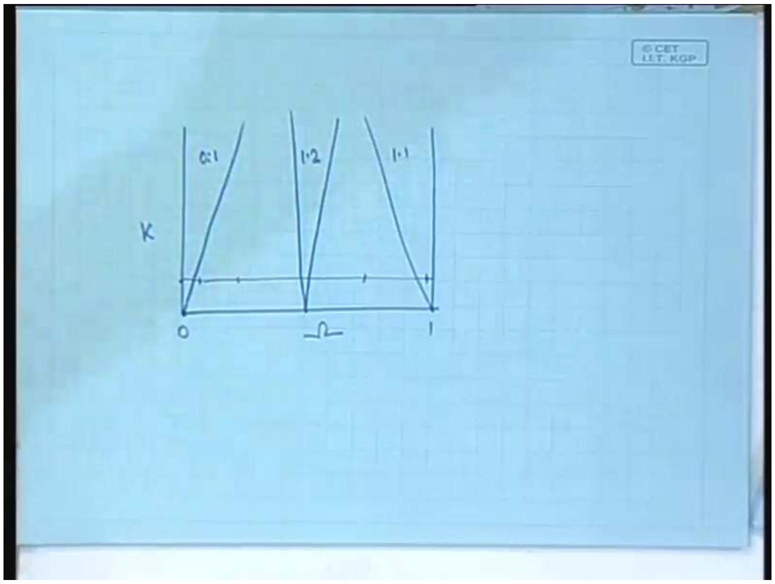

This mod locking windows, in that case we can paint a picture in the parameter space with omega here and k here which means that if you take a slice here, this slice would be that for this value of k as you change the omega, there would be some range for which zero one mod locking will occur. There will be some other range for which one mod locking will occur and there will be all those intermediate ranges in which the intermediate mod locking's will occur. It so happens that if you draw graphs, it will be something like this. This is the zero to one ratio, here is the zero to one, here is the one to one ratio and like so is the one to two ratio. You might ask why and how. These widths are relatively difficult to calculate from the map itself because the map is a nonlinear map but you can always do that simply numerically.

Simply numerically find out for which range of omega do we have a zero one mod locking, which range of omega do we have a one to two mod locking? Can you visualize how the bifurcation diagram will be? If you keep k constant and vary omega. You already have the program written for obtaining bifurcation diagrams. You have got the system, you have got the map, just do this exercise keeping k as constant and vary omega as the bifurcation parameter and draw the bifurcation diagram. You will see a wonderful thing. Do this exercise because you already have the program written. Now that was an exercise given some time back so you already have. So do it and come back with that knowledge in the next class. Then I know that you will come back with a lot of questions why is this happening and then it will be my job to explain it. We will continue in the next class.