**Chaos Fractals and Dynamical Systems**
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**Lecture No. # 25**

## Frequency Spectra of Orbits

When we were talking about dynamical systems and their dynamical behavior, we are essentially talking about wave forms. Depending on whether or not the wave forms are repeated, you said it is a periodic or a periodic. But essentially we are working with wave forms. Those of you who are coming from electrical, mechanical or physics or like backgrounds you know that whenever we encounter in a wave form, one of the things we always like to look at would be its spectrum. Probably you all have come across Fourier series, Fourier transform and stuff like that. I am not going into that specifically. I am proceeding with the assumption that you know. Normally if you have a system with a period one behaviour then like what would the Fourier spectrum be? An impulse.

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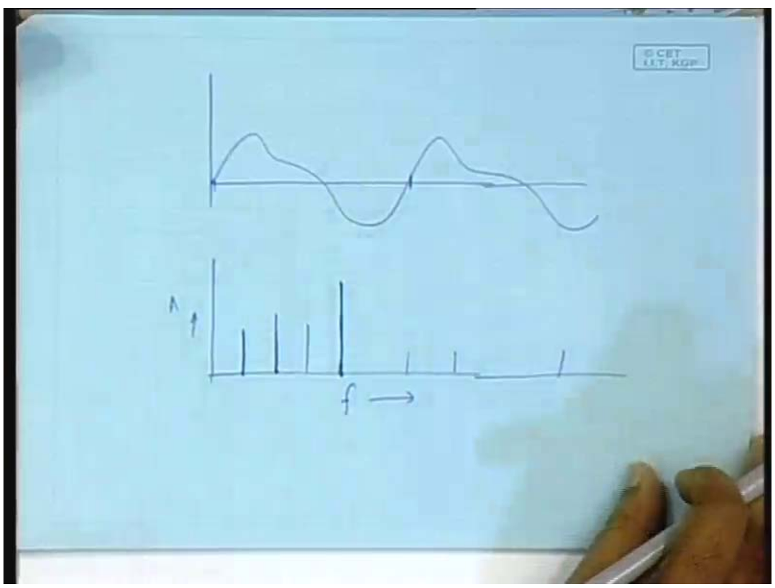


My question is that if you have a period one behaviour which could as well be like this then it would not lead to just one frequency. Obviously this is a wave form with a periodicity something like this and if you decompose that there will be a fundamental and there will be harmonics. But the point that I am making is that this would be line spectrum that means very discrete spectral components, discretely spaced at the fundamental and its components. This is the frequency axis and this is the amplitude for every frequency say A then there would be large magnitude at the fundamental frequency and there would be those smaller ones at various frequencies. One would normally expect this kind of things. For the sake of our understanding or going further, let us confine our attention to the fundamental frequency and things that happens below that.

These higher frequency components would understand that they are there but presently let us put that out of our attention, let's concentrate on this. Now suppose as you change a parameter, the behaviour becomes period two. What change do you anticipate in the Fourier spectrum? Assuming that you recall those ideas that you learned regarding the Fourier spectrum. Can you logically tell that this should happen? No, twice the fundamental frequency will that happen? Yes, the point is that, earlier the fundamental frequency was related to this time period but now the periodicity of the wave form will be double which mean that a new fundamental component will appear whose frequency is half. Another component will appear whose frequency is half.

Now the way when we had this fundamental, we had those harmonic components. For this one when this is fundamental then there would be those harmonic components also and it is not difficult to see that one will coincide with this fellow, coincide with the already existing component that was there and there would be the further components also. If it becomes period four then obviously a new component will appear that has a frequency that is the fundamental frequency at one fourth to the original one. If this is the fundamental component because of the systems non-linearity, its harmonics will also appear and naturally there would be components at these points. The overall effect of the period doubling scenario is not difficult to see that new frequency components appearing whose periods are less than the original fundamental frequency. That is why these wave forms are also called sub harmonic wave forms. The people who have been in electrical engineering they must have heard this term sub harmonic before hearing about period two, period three, period four. Which one are you calling $f_0$, is this one? Yes, first fundamental if in period one. If that is called $f_0$ in period two another will appear which will have a frequency half the earlier frequency. That is why this is called a sub harmonic oscillation. Let us understand them properly.

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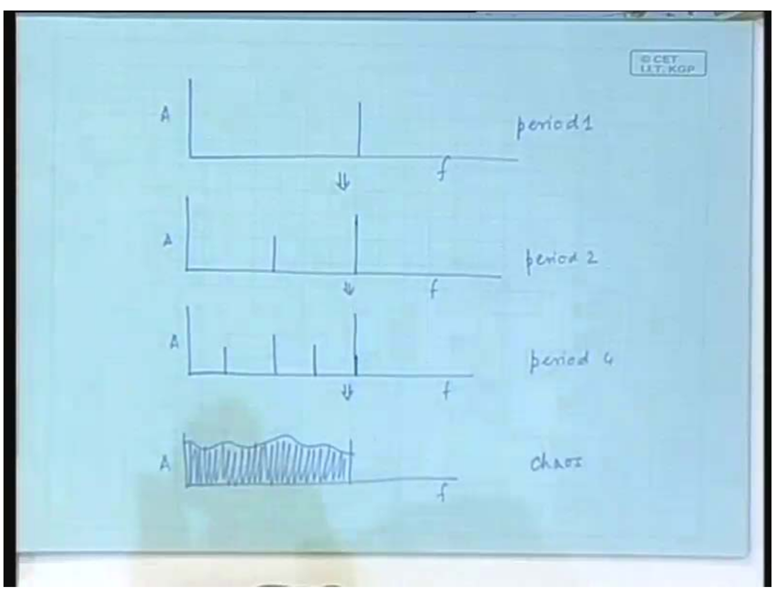


You have one situation where you have one fundamental component so period one. When it becomes period two then it will have this as well as this and its harmonic will be coinciding with this one. This might be little longer so but nevertheless there will be this. In period four, this will remain and there would be a new intuition of frequency at one fourth which will have components at half and three fourth also because they are the harmonics of this frequency and there will be one component on this frequency also and so on and so forth. You see as the period doubling cascade proceeds, more and more lines appear within the range between zero and the first fundamental frequency. Now can you anticipate what will happen, as the systems goes to chaos.

All the intermediate frequency levels will be filled means you will instead, at the stage let me draw the chaotic spectrum. This is the period two, this is period four and when it goes to chaos you will have the whole range filled that means you will have all the place filled that is why chaos has the effect of spreading the spectrum. Earlier these were line spectra. Now this is a spread spectrum and those who are coming from electronics backgrounds they have heard of specific applications of spread spectra and in all these, chaos can be used. For example the CDMA cell phones, they use spread spectra technology and now there are specific propositions of using chaos generators for this purpose. Chaos is as good as a generator of a spread spectrum wave form. In chaos you have a continuous frequency spectrum, this is f axis, amplitude. Yes it is possible but that would be a little more tricky thing to discuss for this class. We will leave this out. It is not impossible to control the spreading but that requires more rigorous mathematical treatment.

His question is, is it possible for two different systems to have the same form of chaotic motion. If by form of chaotic motion, you mean the same frequency spectrum. Yes, it is possible but actually one system may be electrical system, another system may be a mechanical system. That doesn't really matter because when you bring them to the model, if the model has same kind of parameters you will have the same kind of behaviours and therefore they have the same kind of spectrum. Now if that is the spectrum, we have one more situation you had already considered that is the orbit on a torus.

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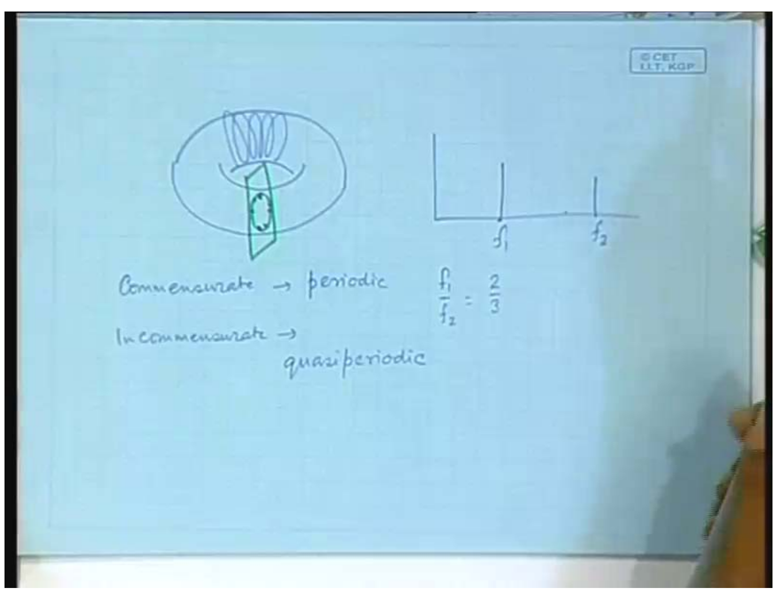

We considered the orbit on a torus, an orbit that goes like this. What would your anticipation be? If you observe that kind of a wave form and take a Fourier spectrum what would you get? Two frequencies, exactly two. If you observe it, there will be one frequency here, another frequency somewhere here. That is one very clear way of distinguishing a quasi-periodic waveform from a high periodic wave form or a chaotic waveform, a very clear way of distinguishing. If you simply pass it through a spectrum analyzer, you get very distinct two frequencies. (Conversation between Professor and a student: Refer Slide Time: 12:43). No. His question is if you get two frequencies, is it always quasi periodic? No. It's not difficult to see that if these two frequencies are commensurate. Commensurate means you have learned in school, there is some numbers so that these two multiplied with this number you get that number.

If these are commensurate that means their ratio can be expressed as a rational number. Then do you understand what will happen. One frequency is related to motion around a big circle, another frequency is related to the motion in the small circle. Say this is $f_1$ and this is $f_2$, $f_1$ is related to the motion around the big circle and the smaller one that is $f_2$ is related to motion around the smaller circle. Then if I ask you, will the orbit come back to the same initial condition after going around? It's not difficult to see that if they are commensurate, it will. For example say $f_1$ by $f_2$ is 2 by 3, what does it physically mean? It physically means that by the time it goes around the big circle twice, it goes around the smaller circle thrice. If you start from a point, after three rounds around the big circle, it would come back to the same position. In that case the orbit will become periodic. That's also a periodic orbit even though there are two distinct frequencies. Is that point clear now?

Even if it is an orbit on a torus, even if there are two distinct frequencies that does not mean the orbit is quasi periodic, it could be a periodic also. A periodic orbit so far we have been dealing with periodic orbits not on the surface of the torus but now we also have to consider periodic orbit that lie on the surface of the torus. If the frequency ratio commensurate, you have periodic orbit. Now what will happen if you have incommensurate? Actually it will not come back to the same state, the whole torus will be filled. The orbit will progressively fill the surface of the torus, it will never come back to same state, if the ratio is incommensurate and that is called quasi periodicity. Now you have learnt one way to understand these behaviours would be to place a Poincare section and see the behaviour on the Poincare section. What would it be like, if you place a Poincare section and see the behaviour in case of commensurate frequencies? It might not be single but a finite number of points, because every time it starts from here and goes around it, it comes to somewhere else.

Again after sometime it will fall on the same point, so there will be a finite number of points on the Poincare section. If you have quasi periodic orbit what will you see? A ring. You will see a ring because every time it starts from here, it falls in another place again it goes, it falls in another place, again it goes it falls in another place but ultimately none of these points will fall on each other. As a result the whole ring will be filled, you see a ring. It's also called a drift ring, you will see the point drifting along the ring and it will go on doing that way. The signature of quasi periodic behaviour is a closed loop on the Poincare section. How does all that happen? Let us tackle that question at this stage. Let us come back to our concept of a continuous time dynamical system and let's develop from there.

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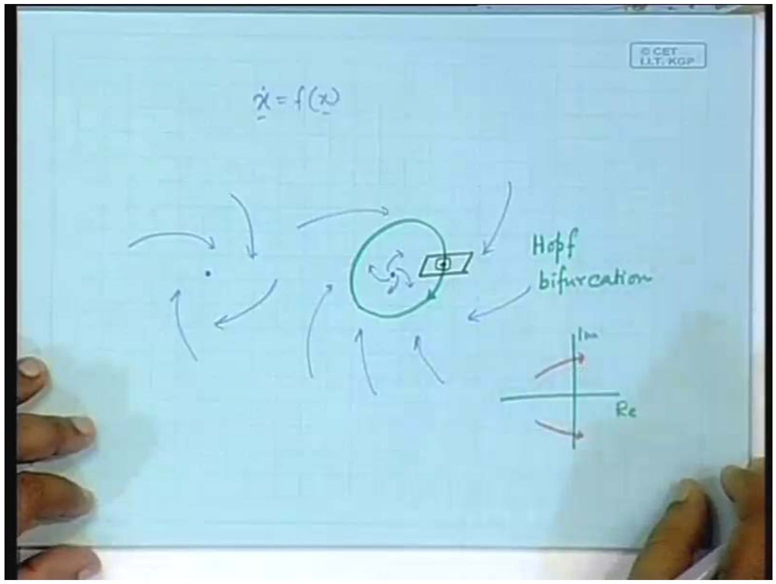


In a continuous time dynamical system your description was x dot is equal to f (x) where x is a vector and we are considering three or more than three dimensions so that the period two, period three chaotic motions are all possible. Suppose initially you have the parameter such that you have got one point that is stable equilibrium point and you have the orbit vector field like this. What would your immediate conclusion about the Eigen value would be? The Eigen value must be complex conjugate with negative real part and this part I have already done just let me repeat. If suppose you change the parameter, vary the parameter and as a result of that the Eigen value will move across the imaginary axis to the positive side. What will happen? In the immediate neighborhood of it, it will become repelling but that does not mean that away from it, it will still be repelling. So away from it, it could be attracting and as a result there would be a limit cycle. This is the (birth) (Refer Slide Time: 00:19:44) of the limit cycle that we discussed. This phenomenon is normally called hopf bifurcation.

This is obviously a bifurcation because the asymptotic character of the orbit change, there is a fundamental change in the character of the orbit so it's a bifurcation. It's called hopf bifurcation. Hopf bifurcation is actually related in the complex plane. The Eigen values actually move like this. This is the real axis, this is the imaginary axis and the Eigen values is like this. At this point we have the occurrence of the hopf bifurcation. We have the birth of the periodic orbit and thereafter when we try to study the stability of it what did we do? We placed a Poincare section and that is how we did it. Thereafter we found it more convenient to study it in discrete time, so we say that let us place a Poincare section and now let us study the stability of this. That's how it proceeded. Then we said that now we can locally linearise around that fixed point of the map, if the Eigen values are less than unity in magnitude then you have a periodic orbit, this fellow is stable and so on and so forth. We were considering the situation where we have the Eigen values inside the unit circle and we have already done that, I am not repeating that. The point is if the Eigen value of this fixed point of the map, not the continuous time dynamical system of the map, at the fixed point are complex conjugate and say they are stationed somewhere like this.

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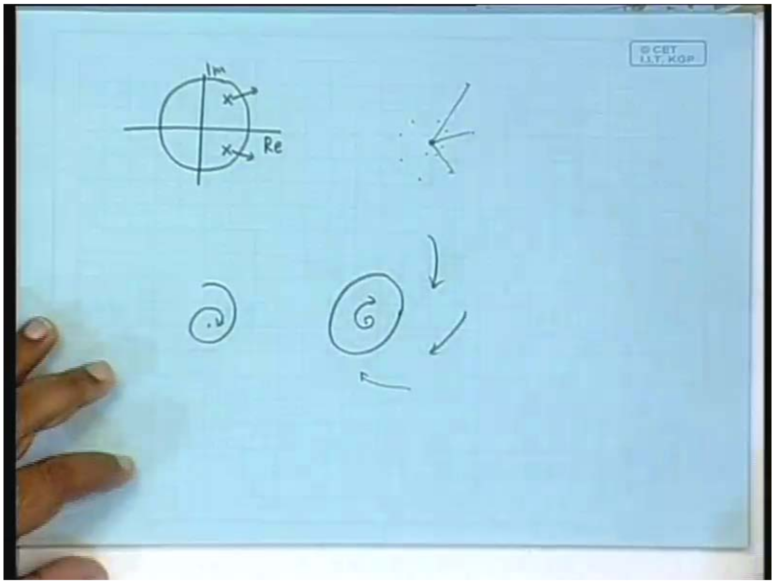


This is the real axis and this is the imaginary axis. What would the behaviour be in discrete time? In discrete time I have done that. Suppose this is the fixed point then the behaviour would be a spiral. It should go through discrete jumps but then progressively the distance between the points will reduce and the vector will rotate, the rotation will be given by the tan inverse b by a term and the shrinking will be given by the magnitude of the Eigen value. Essentially we get an incoming spiral. If say they move like this, what will happen? It will become outgoing spiral so again a similar situation is unfolding but in discrete time.

Initially it was an incoming spiral behaviour, I am drawing continuous line but it is not continuous really. It is discrete jumps, it was like this and then it became like this. Again we can raise the same issue that outside away from this fixed point, you cannot guarantee that it is still outgoing. There can still be incoming directions as a result of this there will be a… In this case we will not call it a limit cycle because now it is happening in discrete time. We will not call it a limit cycle, rather we will say that now what has happened is that we have seen the birth of a closed loop in discrete time. What is it in continuous time? A torus in continuous time.

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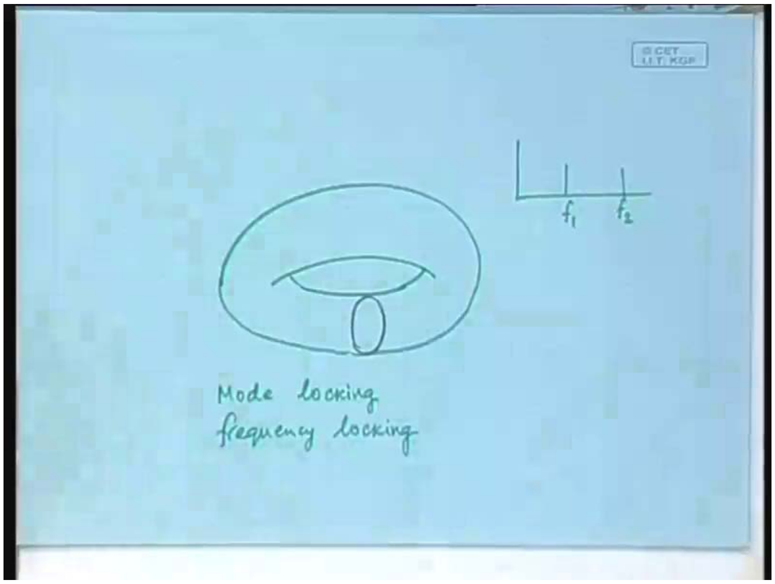


Whenever there is a birth of a closed loop in discrete time, you immediately know that there is a torus in continuous time. This is how a torus is born in a continuous time dynamical system. When we try to understand that we try to understand that in terms of the discrete time behaviour, in terms of the birth of a closed loop. But now these closed loop has been born. Quite a natural question is that this closed loop has obviously two frequencies and I said depending on the frequencies and their ratio, it is either a periodic orbit or a quasi-periodic orbit but physically how can two frequencies be there in a system? To get a physical idea you might imagine it this way.

One, suppose the system has some kind of a LCR circuit means there would be some characteristic frequency and supposing it is excited by a sinusoid which is the different frequency. Obviously there are two characteristic frequencies in the system and they will interact with each other. It might also be so that there is no external periodic input but there are two parts of the system which have their own LCR circuits or may be an electrical component which has its own characteristic time constant and there is a mechanical part which has its own characteristic time constant because there are springs, masses, frictional elements which are similar to LCR. There are two characteristic frequencies and as the dynamic unfolds, there will be interaction between these two characteristic frequencies. That is what gives rise to this kind of behaviour.

Normally you would expect quasi periodicity to occur in systems which if you look at the system description, you should be able to identify what creates the frequencies normally. But there are also situations where it may require a lot of insight to find out what exactly is creating the frequency but nevertheless there should be that kind of a situation. You have already done in mathematics course or in other courses, supposing there is a simple LCR circuit, a linear system excited by an external harmonic sinusoidal excitation. What will be the behaviour like? Sinusoid. It will be a sinusoidal behaviour. Yes, it will be excitation frequency only.

In case of a linear system, the effect of the original systems characteristic time is not all that visible but if the system is nonlinear that is not quite so then there is interaction. Then we can have more complicated behaviour that all I mean at the moment. In such systems you would anticipate the possibility that when you have only the excitation frequency visible then it is a periodic orbit. In the state space it is a closed loop, take the Poincare section it is a point but then because of the nonlinearity as you change the parameters there might be situation where this closed loop might become unstable and as a result an additional loop may develop on the Poincare section. It is possible. But then there is something more to it. What is more? Supposing there is a frequency $f_1$ and there is another frequency $f_2$ and as I said if this ratio is commensurate then you have the periodic orbit. If that is incommensurate you have the aperiodic orbit.

Now supposing I keep $f_2$ fixed and I vary $f_1$, I can do that. What would the anticipation be? In order to get a periodic orbit, you have to actually fix $f_1$ with infinite precision because if you slightly vary, you will get an irrational number or if you want to obtain a quasi-periodic orbit, you have to really fix it at an irrational number very accurately because around an irrational number there is always some rational number. That's a fundamental number theory. The natural anticipation would be that in order to get any desired behaviour, you need to fix the frequencies very accurately. No, because of the non-linearity. What happens is that if you keep $f_2$ constant and vary $f_1$ or keep $f_1$ constant and vary $f_2$, you will find that mysteriously the two frequencies get locked for some range of the parameter. Get locked means they get locked into a particular ratio, that's it for long ratio.

As you change the parameter it doesn't get away very easily and then at some particular parameters value, the locking is lost and then you have again quasi periodic behaviour. Again it gets locked to some other frequency. This is called mode locking or frequency locking. Very interesting phenomenon and completely nonlinear phenomenon, it cannot happen in a linear system. This mode locking is a very interesting nonlinear phenomenon. This is called mode locking or also frequency locking. Let me just repeat. If you keep on changing the parameter you would normally anticipate that you have to fix the parameter very accurately in order to get a rational frequency ratio but it's not so. Mysteriously you will find that for a large range of parameters the same ratio remains fixed that means they get locked. If you change it beyond a certain range, the locking may get broken. No, presently we are talking about only existence of two frequencies. I will come to those issues a little later. How does this physically happen? If you want to imagine how it physically happens, I can give you a sort of hand waving arguments.

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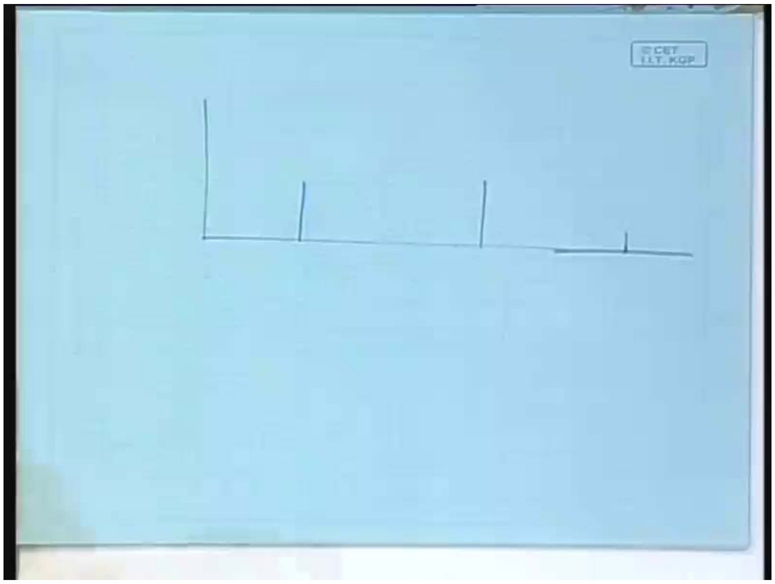

See if you have one frequency somewhere here and if you have another frequency somewhere here and then these two are in a rational ratio. Then there should be some kind of a harmonic somewhere here, harmonic of this one and there should be another harmonic of this one in which whose frequencies will match. As a result of which there should be a resonance between the two. That resonance cannot happen if the frequency ratio is incommensurate. If the frequency ratio becomes commensurate then there must be some particular frequency out there in the higher frequency range where both the components will have some harmonics there. There should be a resonance and the resonance itself is a nonlinear phenomenon. That resonance sort of holds on, it doesn't let it go very easily. Only when this is moved by significant amount does this move away, otherwise the resonance sort of locks hands and remains there. That is the intuitive way of seeing how it happens, sort of electrical way of seeing how it happens, the systems nonlinearity. This system has to be nonlinear in order to have any resonance.

Here we are talking about nonlinear system, the systems behaviour has to have some kind of a nonlinearity built into it and that produces the force. One side because of the nonlinearity must apply a force on the other frequency so that it gets locked. No wait. I am not talking about the linear system resonance there it happens only if you excite at that particular frequency. Here we are saying that no, we are not exciting at the particular frequency we are moving it, yet it is getting locked. That is a nonlinear phenomenon, it cannot happen unless there is a nonlinearity in the system but examples of such situations galore. For example you know that there are at least two even more actually but two very well-known situations within the solar system where it is happening. The earth moon system, you know that you see only one side of the moon. You don't see the other side of the moon. Why? The moon is rotating around it. If earth is rotating around its own axis and the moon is rotating around earth then obviously I should see, all sides of the moon because moon is rotating. We don't see, what does it mean? It is actually in the phase space, it is the motion on a torus with ratio 1:1 and it's a mode locked behaviour.

There is no guarantee that it was always mode locked. It might be at the time of dinosaurs, it got mode locked so that if the dinosaurs could see, they could have seen the opposite side of the moon but at the time may be it got locked but there is no guarantee that it was always mode locked. It got mode locked at some point of time and we are in a temporal face where it is still mode locked. Mercury for example same phase faces the sun and in that particular face, the temperature goes to something like 600 degrees. In the other side it's freezing. The closest thing to the sun, the other side that is in the shade it is freezing temperature because of that. Because there is only one side that is exposed to the sun. These are mode locking behaviours and such mode locked behaviours are very common in experimental engineering situations also.

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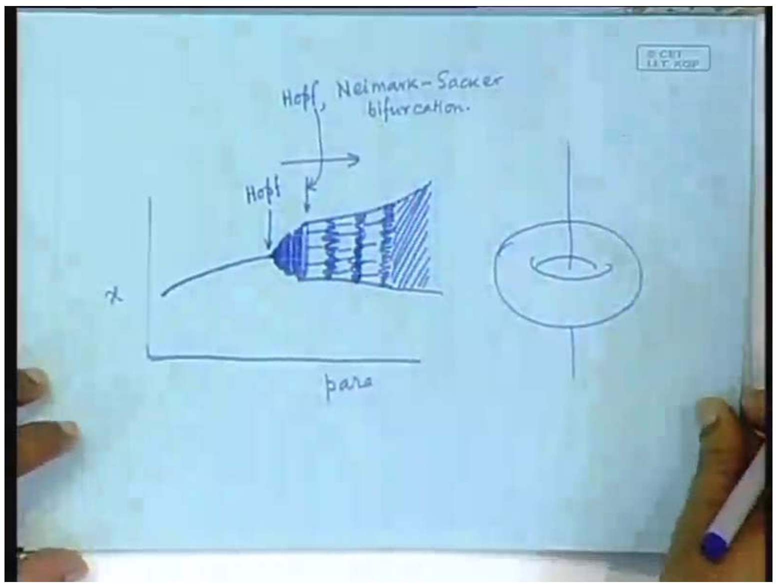


The total picture is that if you start from an initial condition say, if you look at the bifurcation diagram, suppose I am doing the wrong way of plotting the bifurcation diagram that means I am talking about a birth of an equilibrium and that equilibrium say I am changing the parameter. See I am changing the parameter and here is some kind of a variable that I am plotting. At that point of time that equilibrium started to exist, not the fixed point remember I am talking about the equilibrium point in the continuous time system. At that point a hopf bifurcation occurred, as a result of which what was born?  A limit cycle.

A limit cycle if I now draw then I can only draw the projection on the x component. What will you see? Its value will slowly increase so like this. I am talking about this orbit so these are all orbit like this and suppose at some point of time that again loses stability. That periodic orbit also loses stability, as a result of which what is born? A quasi-periodic orbit. Then that would be signified by some kind of a change, I am not the depicting because it is difficult to depict that. then within that range where it is a motion on a torus, as you are changing the parameter there would be a succession of mode locked behaviours, sandwiched between them would be quasi periodic behaviours, mode locked quasi period so mode lock quasi period.

You would see that there would be some range where you have quasi periodicity behaviour and some range where you have some kind of mode lock behaviour, again a mode lock behaviour again a quasi-periodic behaviour, again a mode lock behaviour these are not chaos though. Though I am drawing this way because all the points on the loop would be filled in quasi periodicity but again there would be some mode lock behaviour. Again it was quasi periodic behaviour, so the behaviour would be something like this.

Finally at some stage that gives rise to chaos. Here there would be a range where it gives rise to chaos, the torus breaks down. This is called the quasi periodicity root to chaos. We have already learnt about a few roots to chaos, we have learnt about the period doubling root to chaos and stuff like that, this is the quasi periodicity root to chaos. It has actual history. The history is that you must have heard of the names of the famous Russian physicists Landau (Refer Slide Time: 00:40:37) the famous book almost the bible in physics. This man Lev Landau, in early 40's he proposed a mechanism for generating turbulence. Turbulence means a chaotic behaviour. Now we understand that as a chaotic behaviour but how is it produced? His point was that as you change the parameters for example in case of producing turbulence, you increase the flow rate and beyond a certain flow rate you find turbulence. So increasing the flow rate that is the parameter.

Suppose you are increasing the parameter, changing the parameter then he says that beyond a certain parameter another frequency component comes in. beyond another parameter, if another frequency component comes in, another parameter another frequency component comes in. His proposition was that there is a progression of addition of frequency components, finally there is a situation where all the frequency components are there and that is turbulence. It is almost the same kind of situation as period doubling cascade. His proposition was that it is an accumulation of frequency components that finally gives rise to a chaotic behaviour. That preposition, it was I should say a conjuncture finally proved to be wrong. Now we understand that the first transition that means while from the laminar flow, the first transition accumulation of one frequency component is possible. Another frequency component is possible. You can see that addition of one frequency component is the first bifurcation, hopf bifurcation creating a limit cycle.

Second one happening on the Poincare section creating a closed loop. That is also an addition of frequency component but there after we have problem. When you have a periodic orbit, it is a one frequency behaviour. A torus, a two frequency behaviour. Naturally the question is, can there be a three frequency torus? The mathematical question (based) (00:43:06) on to that. People who are doing experiments they found that they somehow not finding it. Somehow in all experiments they are finding that after the two frequency torus, it somehow goes directly into chaos. Then when I say these experiments, I am talking about the time frames like 80's. There was a very long span of time when people believed that the Landau mechanism is true but people found that we are not getting it. In the 80's some people did experiments very carefully on oscillators where they carefully created one characteristic frequency of one part of the oscillator, another oscillator with another characteristic frequency and a third frequency that is an injection frequency. Deliberately created a situation where there would be three frequencies. They found that if the interaction between the two that means coupling between the two oscillators is weak then you do get three frequency behaviour.

But if the coupling is strong then the two frequency behaviour directly goes into chaotic behaviour but we have never found anything beyond three frequency behaviour. Three frequency torus I will not be able to draw because it is a conceptual thing, you cannot draw it on the sheet. It’s not possible to draw but essentially the concept is if you break up in frequencies spectrum, you will see three frequency components. Then came the theoretical break through where it was proved that the three frequency torus behaviour is structurally unstable in the sense that if you slightly part of that three frequency torus, slightly part of means either you change the parameter or you change the initial condition or whatever, slightly part of it where it goes into a chaotic behaviour.

Even though theoretically a three frequency behaviour is possible. That is why it is extremely difficult to observe because it is structurally unstable. The mathematical statement says that in the neighborhood of every three frequency torus, there lies a strange attracter at an infinitely small distance which essentially in physical terms means that even slight bit of random noise existing in any system will not allow it to stabilize in the three frequency behaviour. That is why you do not see the progression into three frequency, four frequency, five frequency and all that going into chaotic behaviour.

Now you understand that the essential mechanism of this quasi periodicity root to chaos is where you have a periodic behaviour, period one behaviour then a bifurcation leading to a torus behaviour and that torus somehow breaks down. There is some mechanism by which the torus itself breaks down and that leads to chaos. How can the torus break down? Torus is a very well defined geometrical structure. For mathematician, for a topologist the torus is a very well defined concept like if it is a donut, you can take a thread through so if it is a torus then you can take a thread like this (Refer Slide Time: 47:00). If you take like this then you cannot take it out without breaking the torus and stuff like that. There is very well defined mathematical concept by which a torus is defined.

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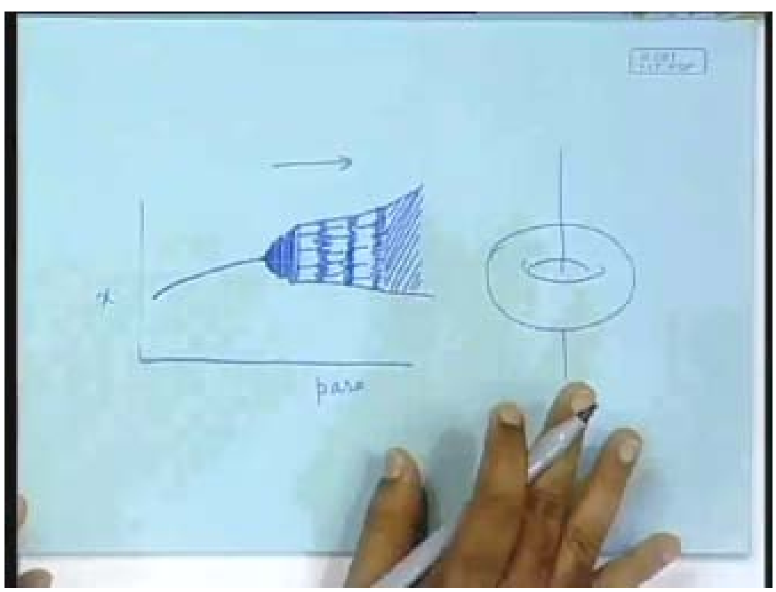

If that somehow breaks down that means some deformities appear in the torus so that it can no longer be called a mathematical torus, that is a transition from a torus behaviour to a chaotic behaviour that's another route to chaos. Before concluding today let me give you a name. We said that there are two bifurcation appearing on this. One at this point. What happened on this point, what is the name? Hopf bifurcation. What happened at this point? It is the similar thing to a hopf bifurcation, happening in discrete time. That is why some authors call it also hopf bifurcation. I mean this also called a hopf bifurcation, in some books you will find that this is also called a hopf bifurcation but more accurately this is called a hopf and this bifurcation is either called a hopf or discrete hopf or generalized hopf or there is another name for it, by the names of the people who actually discovered this mechanism is also called the Neimark Sacker bifurcation.

What is the characteristic of the Neimark Sacker bifurcation? In the discrete time you have got a fixed point, you locate the Jacobian. Locate the Eigen values and the Eigen values would be complex conjugate and exactly equal to one in magnitude then it is a Neimark Sacker bifurcation. Don't be confused by these two interchanging names in different literature. Same phenomenon, conceptually since it is the same as the hopf bifurcation in discrete time, it is also called hopf or generalized hopf and Neimark Sacker.

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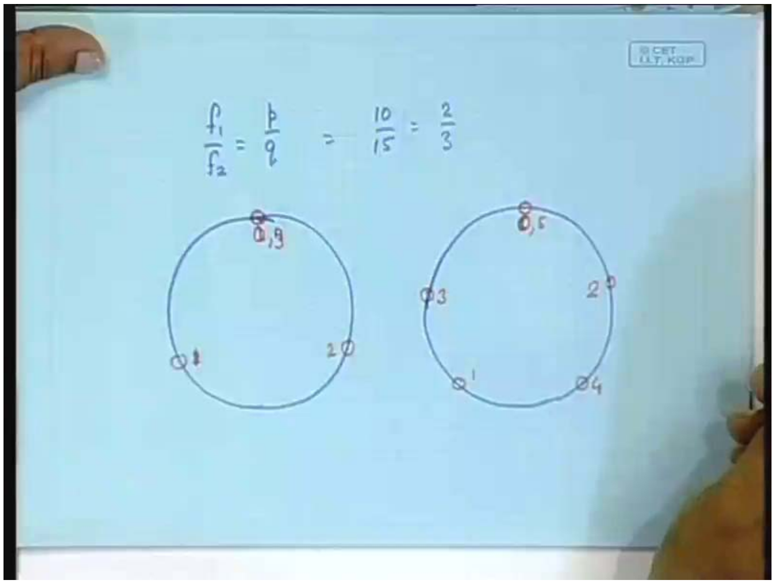

Let us explode the issue of the mode locking a little further. Suppose you have two frequencies $f_1$ and $f_2$, their ratio is commensurate and say that commensurate ratio is p by q where p and q are whole number. $F_1$ by $f_2$ can be expressed as p by q remember when I say p by q, I am not saying something like 10 by 15 is equal to 2 by 3 and then I would write 2 by 3. Whenever I say p by q in the further discussion, you understand that all the common divisions are taken out. So 2 by 3. What does 2 by 3 mean? 2 by 3 means by the time it goes twice around the big circle, it goes thrice around the small circle. So what does it mean, on the Poincare plane what will you see? Suppose this is the Poincare plane, there is a close loop on the Poincare plane. Suppose I start from this point.

After this what will happen? It will go into the big circle and after some time it will come back on this. Where will it come back? Two thirds of the circle later. If the ratio is 2 by 3, so it will come back here, if this is point number one this will be the point number two. Where will it intersect the Poincare section again? Again two thirds of a circle later, here. This point returns as the fourth point. This is what we will see on the Poincare plane. A 2 by 3 behaviour, $f_1$ by $f_2$ is 2 by 3 is actually seen as this. What about say 3 by 5 behaviour? Can you draw? $F_1$ by $f_2$ is 3 by 5, so in that case suppose you start from here. I should actually call it zero because this is starting point. This is 0, 1, 2 and here 3. Here it is zero then after that it will go into the big circle and it will come back on this Poincare plane. After how much angle? 3 by 5, three fifth of the circle say it is somewhere here. That is point number one.

Point number two is again three fifth of the circle, say somewhere here, two. Again three fifth of a circle somewhere here, three. Again three fifth of a circle somewhere here four. Even though things are happening, going out that way we are being able to see things simply by looking at the Poincare plane and we are being able to compute about the frequencies. Notice what I am mean? Meaning that the torus is there suppose I am not looking at a torus, I am only looking at the section. Therefore you tend to believe that the other frequency will now go out of my attention. This orbit will no longer be seen because I am seeing only this. No, it is still visible to us because the way it rotates. The way it rotates that tells us how it goes into the big circle. We will develop on this idea further in the next class.

Thank you.