Chaos Fractals & Dynamical Systems
Prof. S. Banerjee
Department of Electrical Engineering
Indian Institute of Technology, Kharagpur
Lecture No. # 24

Lyapunov Exponent

The last day we learnt about two measures of dimension, one the box counting dimension and two the correlation dimensions. These are things to characterize a chaotic orbit also to distinguish between a non-chaotic orbit with a chaotic orbit but then we have already learnt while we were talking about the continuous time dynamical systems that one way to characterize chaos is through the exponential separation of nearby trajectories that means sensitive dependence on initial condition. The two measures that I have already talked about is they do not quantify exactly the sensitive dependence on initial condition. They sort of quantify how distributed are the points of the orbit but then there is another measure of chaotic orbit that quantifies the sensitive dependence on initial condition and you have learnt that is called Lyapunov exponent but what was central concept then?

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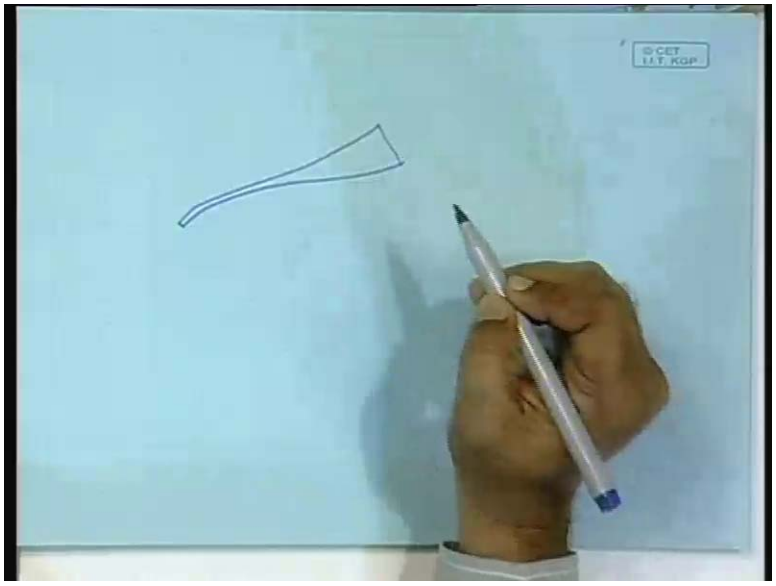

That if you take two points very close to each other then if the system is chaotic then one would evolve like this and the other would exponentially diverge on it. The two will evolve in two different ways, we will not keep hand to hand with each other and the separation therefore will increase. That's what we learnt at that time. Now we are trying to put the same idea in discrete time. Earlier it was a continuous time concept, now we are trying to put the same idea in discrete time. In the discrete time also then we will have to talk about two very nearby points and how their separation explodes. Of course here also we need to track of the fact that even though their separation diverges, really the separation between them do not go to infinity.

Why? Because overall the orbit is within a definite boundary because the chaotic orbit is always bounded. If both the points are bounded then mutual distance also cannot go infinitely. Obviously we have to keep track of the issue that their mutual separation is increasing exponentially but at the same time that cannot go to infinity so that there are also folds.

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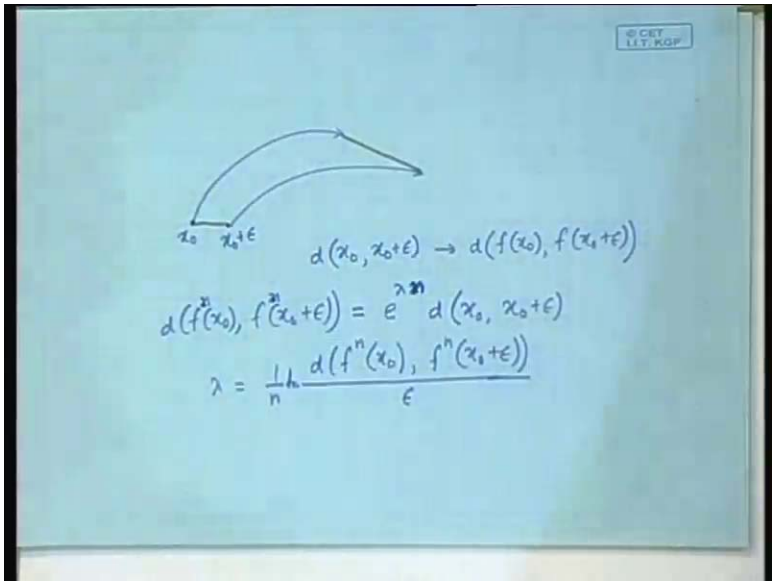

Now when we are consider maps essentially we will again talk about a point say x_0 and another point say x_0 plus some small number epsilon and then in the next iterate if you apply the map to both these points then they will map to two different places and their separation is expected to increase. Here was a separation between the two points and that is what it becomes when you apply it for one iterate. If you keep on applying that this separation, will it keep on increasing? No obviously not, because after sometime it cannot indefinitely increase of course. We can say that if you have the distance between x_0 and x_0 plus epsilon to begin with and that becomes the distance between these two, $f(x_0)$ and $f(x_0 + \text{epsilon})$. This distance maps to this distance and we wanted to say that this d of $f(x_0)$, $f(x_0 + \text{epsilon})$, this fellow is the initial separation times e to the power lambda. It is an exponential separation, we were talking about exponential increase in separation, so e to the power lambda times d to the power d (x_0, x_0 + epsilon). If you take say two iterates then you might say twice iterated and if it is n iterated we would say… (Refer Slide Time: 06:17). Essentially then you get your lambda as, how do you extract lambda from here? This is n, 1 by n, it is the d of $f^n(x_0)$, $f^n(x_0+$ epsilon) divided by this fellow. This fellow is nothing but epsilon. The ln is here. This would be the concept of the Lyapunov exponent.

Conversation between professor and student: Refer Slide Time: 07:24- 07:30). We are assuming that the two points, their separation increases and that increases exponentially. You might question why do you assume that it is exponentially increasing. It might increase any other way but it is just an assumption. We assume that it is exponentially increasing if that is so then the exponential (Refer Slide Time: 00:07:59). Now you would realize that even though the essential concept is that it is not so easy to actually measure it.

Why? Because this distance if you keep on iterating then the distance is no longer really going apart, so you need to do something else to it. Essentially what we do is since we have to take the nth iterate of x_0 as well as the moved one x_0 + epsilon, nth iterate and essentially what is happening? We are looking at how the nearby regions expand. So from one point you take one step and you see how it expands then you take the next step, you see how it expands then you take the next steps you see how it expands. Then the rate of expansion average over the whole orbit is nothing but this. What you are really looking for is the average rate of expansion. The direction of expansion is different in every place, as you go on from one point to the other the direction of expansion is different. You can look at it this way.

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That suppose an orbit is going like this and so on and so forth. That means starting from x_0 it is going through these set of iterates. Then if we locally linearise around that and obtain the Jacobian matrix then that Jacobian matrix its Eigen values will tell me how a particular direction will expand. Go to the next point, do the same thing you will get again a Jacobian matrix but the Eigen directions in all those places will be different so that they will be moving but nevertheless at every place you can find the expanding direction. Average about the whole, you get the Lyapunov number. Essentially since you can say that the derivative, here what am I doing? Here I am taking derivative of the nth iterate. For example we have taken the second iterate of some map, the third iterate of some map and taken the derivative, it is like that. Taking the derivative divided by the initial separation, taking the derivatives at the two places but the chain rule we can say that the derivative of the nth iterate is nothing but the derivative at this point times the derivative at this point, times the derivative at this point and so on and so forth. Then we hence divide by n to get the average dual (Refer Slide Time: 00:11:31) procedure. In general we define the Lyapunov number on the average rate of expansion in that way.

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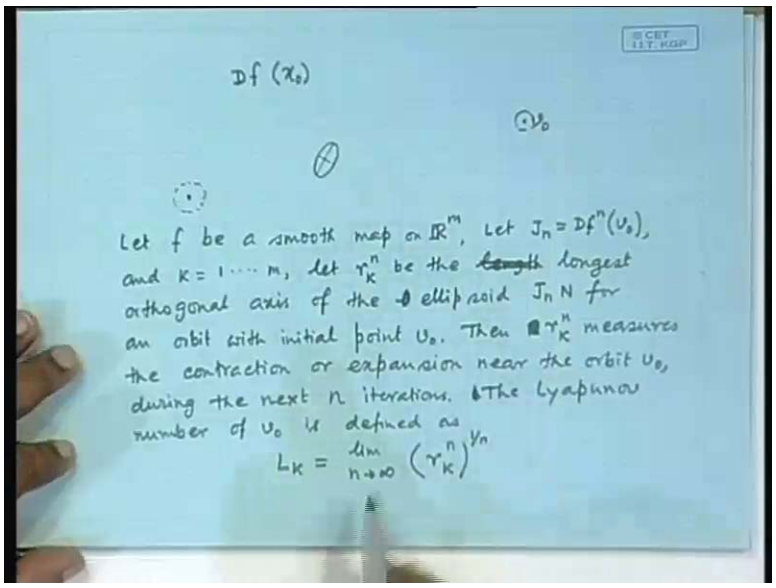

Now let us bring back the concept that if you have a point and you have obtained the Jacobian matrix at that point then essentially at this particular point, you would be writing that x_0 is the point say, applied on it you have to say that the Jacobian that means it's often written as Df (x_0), Df is the Jacobian of x_0. If you apply this and if you apply not only to the point but also to a circle around it, what I will get? You get an ellipse we have already said that. You actually get an ellipse, somewhere else it falls as an ellipse. The point maps to a point somewhere here but the region around it maps as an ellipse and we have already seen that the major axis of this ellipse and the minor axis of this ellipse are given by the Eigen vectors of this matrix times its transpose, we have already seen that. You might argue that we can keep on obtaining this and multiplying this. That might be a procedure provided you are able to obtain the sequence of points, take Jacobian around them and apply this procedure. It's possible.

Now if you can do that then the definition of Lyapunov number and exponent would be as follows. Write down this definition because that will be followed in the later classes. Let f be a smooth map on R^m dimensional space real space, what is the meaning of the term smooth map? It is differentiable everywhere that's all, smooth means differentiable everywhere. Let J_n … (Refer Slide Time: 14:40). These are the Jacobian matrices and K varies from 1 to m and let r_k^n be the longest orthogonal axis of the ellipsoid. Normally it will be ellipsoid, if it is 2 D then we will say it is an ellipse but normally it will be an ellipsoid, m dimensional ellipsoid. J_n N for an orbit with initial point V_0. So V_0 is the initial point, we have applied the Jacobian, we have obtained the Jacobian of the nth iterate of that is called J_n,

After we have obtained that then the longest orthogonal axis is r_k^n, what is k and what is n? K is the dimension, it measures the dimension. If it is a three dimensional system then obviously there will be three directions in which it will measure. The ellipsoid will have three axis and this is the longest axis of that then r_k^n measures the contraction or expansion near the orbit V_0. Notice what we have done.

We have started from a point V_0, we wanted to find out how it expands or contracts. The region around it expands or contracts after n iterates. So what we did? We obtained the nth iterate of the map, obtain Jacobian of that map and from there we obtained the major axis and the minor axis of the ellipse and we call this the major axis. If this is the major axis that will measure the expansion of contraction. Essentially at this stage avoiding the question of changing of the dimension because we have already taken the n nth iterate so that contains within it, all the changes and stuff. So that will measure the contraction or expansion near that orbit starting from V_0 in the next n iterations. Then the Lyapunov number of the point V_0 is defined as limit. If in n iterations it increases by so much, in one and half iterates how much does it increase? To the power one by n. That's why this term is needed. This sort of gives my average rate of expansion but that is not Lyapunov exponent because the Lyapunov exponent is an exponent, so e to the power exponential factor has to come in. That is why this is not called the Lyapunov exponent, it's called the Lyapunov number. This is the Lyapunov exponent. Conversation between professor and student: Refer Slide Time: 20:28). We are replacing the chain of rule here. As yet no. As yet we have said that suppose we are able to somehow derive the nth iterate of the map and then the Lyapunov number is given by this and then we will say that n goes to infinity.

Obviously you might ask how do you really derive the nth iterate and then let n go to infinity? We will treat those issues later but this is the definition. Conceptually you need to understand it. This is the Lyapunov number provided this limit exists. Here around V_0 we have taken a circle or sphere n dimensional sphere that is called J_n^N. If you apply J_n on N then you will get a ellipsoid. No, capital N is the n dimensional sphere around V_0. So around V_0 you need directions in all the directions and then whatever that defines is the starting wall. It might be as well smaller than the initial sphere, in that case we don't have chaos but nevertheless we have the number. It might be contraction, it might be expansion. If it expands then only we will say the system is chaotic but in a general system, it could as well be contractive, it will be stable that's all.

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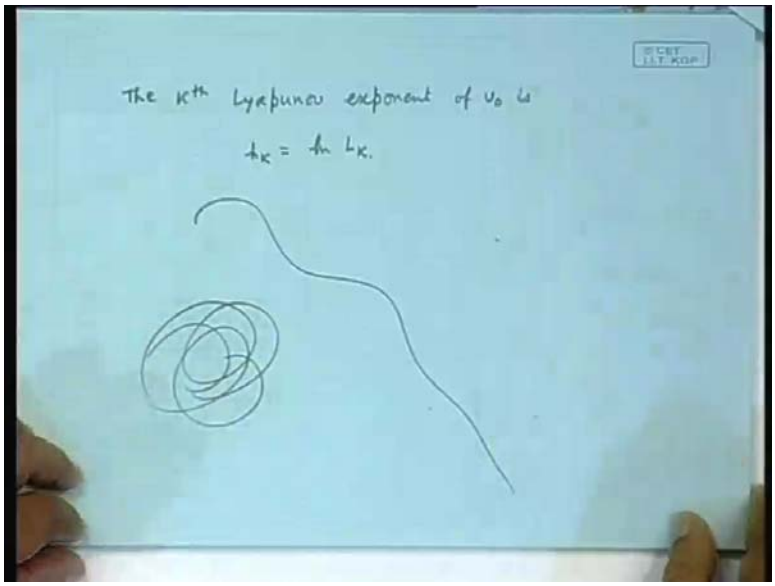

Now then the k^{th}, now exponent of V_0 we will say h_k is equal to ln of L_k. The Lyapunov exponent is nothing but the natural logarithm of the Lyapunov number but as yet we have not addressed the question that he was asking. How actually do we do it? How actually we do it is a somewhat different ball game, I will come to that but essential point should be understood. The concept is that that we take n iterations and then we have a map, so if that is applied on some point V_0, some ball around V_0 we get a ellipsoid, the longest axis of that ellipsoid is essentially given by expansion or contraction or whatever. If it is smaller, its contraction. If it is longer then expansion. So the extent by which it expands is the Lyapunov number, take the natural logarithm of that, you will get the Lyapunov exponent.

Yes, here is the question. We have started from the point v_0, if it is a typical initial condition then it is the typical initial condition. What does it mean I will come to that. Then it is the Lyapunov exponent of the system. But consider that there is a chaotic orbit and there is another stable periodic orbit with a basin boundaries separating then what is the typical initial condition? Start from here, you get something, start from here you get something else. Naturally we have to say that starting from here I got this but supposing there is a unique chaotic attractor in the system then every point is as good. No, not really because you know that within the chaotic orbit there are a larger number of unstable periodic orbits. If you are stationed on the unstable periodic orbit, you will forever remain on the unstable periodic orbit and that fellow is periodic. There are those atypical initial conditions within the system, if you start from there you don't really get the right Lyapunov exponent.

So you have to start from a typical initial condition to get the proper Lyapunov exponent. That is why it is dependent on v_0. Starting from an initial condition somewhere here, you will measure something else. You might say that okay, if I know the systems definition then I might as well compute this and say I might as well obtain the long axis, short axis by simply obtaining the Eigen vectors and Eigen values of AA transpose. We have already seen that, we can do that but that would be a little bit cumbersome procedure. Nth iterate, it will be even more cumbersome. So actually it is not done that way, conceptually it is that but actually it is not done that way. But since we can keep on applying the chain rule and apply at each point and obtain. Therefore there are some simple systems on which you can apply this concept.

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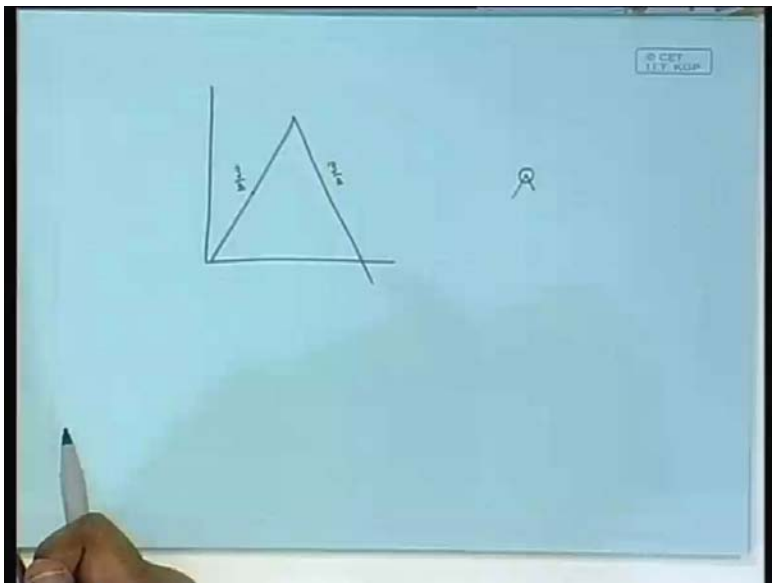

For example if I tell you what will be the Lyapunov exponent of this map. Can you say? Imagine that this loop is 3 by 2 and this loop is 3 by 2. It's not smooth but smoothness is not all that important. Smoothness becomes important only in situation where I take a point and then if you take an air ball, they map differently. But suppose you take initial condition somewhere here then can you say what will be the Lyapunov exponent of this fellow? At every iterate it will expand by 3 by 2, wherever it falls so even if you say it is nth iterate, the slope should always be this. For such systems using that concept it is possible to obtain the Lyapunov exponent straight away but in general when you have to really compute it, that doesn't really help much. In that case we take little different procedure.

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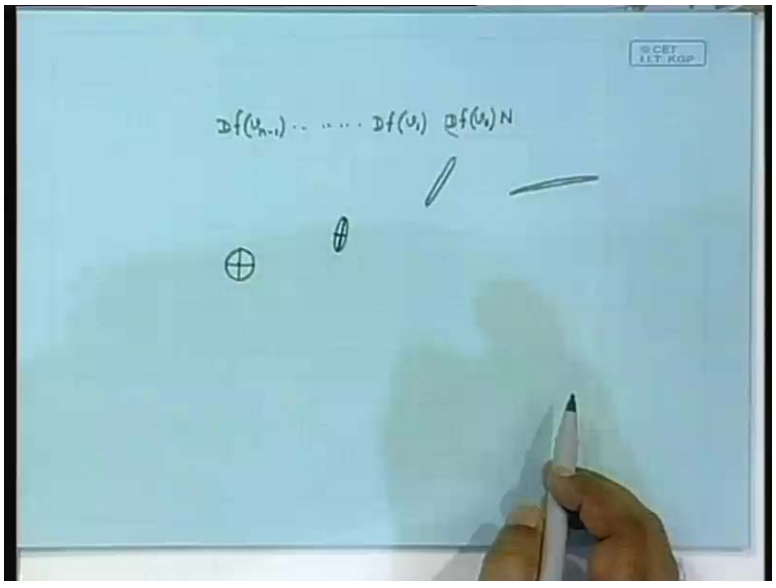

We have to essentially start from a unit ball N and around the initial condition and then we have to keep on those matrices. Essentially what we will do? We will say Df (v_0) applied on N first then we have to apply Df (V_1) so on and so forth Df (V_{n-1}). All these matrices applied on N should ultimately give me the matrix that we want and its major axis minor axis has to be obtained. That is the essential problem. If you start from a standard error ball and then in the first one it lands somewhere here, in the second one say it lands like this and the third say it goes. So it sort of goes on expanding, if there is an expanding directions in N iterations you can expect it to a very long and very thin something, still ellipsoid but nevertheless very long, very thin something. Still we are not allowing to fold. It will of course fold, so that is another problem we have to keep track of.

If you do not consider folding, it will become a very long ellipsoid and if you allow folding then it will fold so that gives us some problem in actual computation. What we will do? We will keep track of the change in one iteration. In one iteration we don't expect any folding. So a circle, a sphere, a spheroid will go into an ellipsoid and there can you obtain the major and minor axis? Of course you simply apply, you have the system definition in your hand that means you have the system model. At this point you obtain the Jacobian, you got a matrix and that matrices, the Eigen vectors and Eigen values will give you the expand and contracting directions. If you want to exactly obtain this spheroid, all that you need to do is to obtain the Eigen values of AA transpose. But if you want to simply obtain that from this by one application of v_0 obviously you get it get an ellipse but you do not directly get the major and minor axis. That we have already talked about.

So something is to be done in order to orient our coordinate say this was the coordinate. Now the coordinate has to transform in such a way that one coordinate is this, another coordinate is that then we can keep track of how the expanding coordinate changes to this one. What we will essentially do? How many of you have attended some course on function analysis, Gram-Schmidt ortho normalization. You have done? Has anybody else done? No. Conversation between professor and student: Refer Slide Time: 32:40) That is the issue then because I was anticipating people have done but then done but don't remember. The average over the class, that is the concept. Then then will do this?

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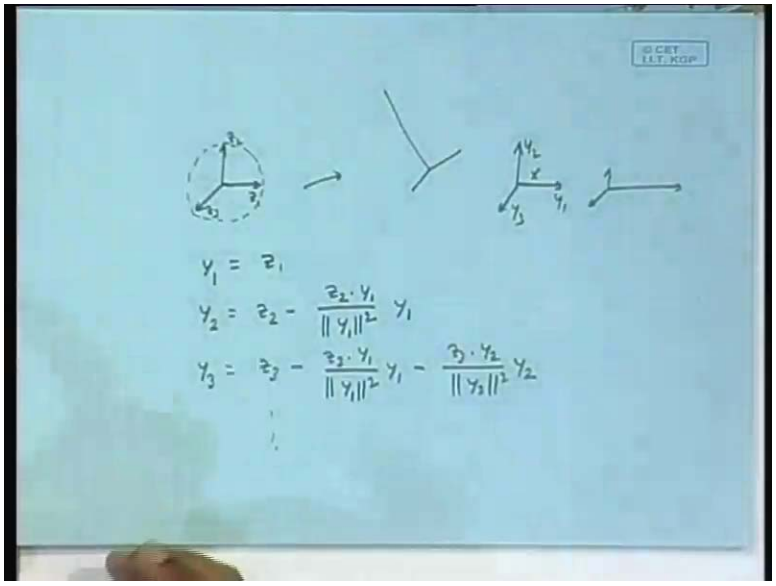

Suppose I have defined a sphere with conjugate vectors, let's say z_1, z_2, z_3 unit vectors defined by that is my sphere. Each one then will either expand or contract, ultimately giving a coordinate system that will not be exactly this. It will be something like this as a result of the transformation. We need to find out where is the major direction and then we need to reorthogonalize so that we get back the sphere to take the next step. The procedure for that is essentially, we define a new coordinate system here y's that means y_1, y_2, y_3 such that y_1 is z_1 that means the first coordinate we keep as it is. The other ones have moved in angle, so we need to somehow bring them back to orthogonal directions. So the way it is done is, y_2 is $z_2 - z_2$ dot y_1 by y_1, this is the norm square times y_1. The y_3 is equal to z_3 minus, here it is z_3, y_1 by the norm of y_1 square times y_1 minus z_3 times y_2 norm of y_2 square times y_2 and so on and so forth. What exactly are we doing?

We are saying first that whatever we took as the first coordinate let it be retained. Second coordinate now it has moved and therefore it will now have a component, a projection on the first coordinate and we need to subtract it out so that we turn the axis back to orthonormal coordinates that is exactly what it is doing. From z_2 we are subtracting the amount that is actually the projection on to that. As a result of which you get again orthonormal coordinates. The third coordinate z_3, now this will have some projection on the plane from y_1 and y_2 and this fellow has to be orthogonal to y_1 and y_2. Therefore whatever is the projection on that will subtract so on and so forth. That is called Gram Schmidt orthogonalization procedure. If you start from a ball then after every iteration we need to apply this orthogonalization procedure so that we always keep the axis orthogonal to each other. Else what will happen? Axis will become unmanageable, some will expand, some will contract and become distorted in every possible directions. This is how we tackle the problem of turning.

At every stage we are reothogonalizing the coordinates. Then the issue is that every stage then supposes it was a z_1, z_2, z_3 and now we have orthogonalised and we have turned into… But now y_1, y_2, y_3 are not of the same size (Refer Slide Time: 37:35). Why? Because some directions have expanded, some directions have contracted so essentially you will not get this. You will get something like this. You still have a set of orthogonal coordinates and you can keep on applying it but that will be inconvenient. Why? Because then this will lead to very small numbers and this will lead to very large numbers after n iterations and as you know in computer science it is a very established fact that if you are having to deal with numbers of entirely different orders of magnitude in a same computational procedure then it is going to likely end up in errors.

We always like to operate do all sort of numerical operation with same or close to each other, size of the numbers. Here we have a problem then. If in one iteration it is going to look like this, after say thirty iterations one is going to be very long and the number that you have to have to handle may be ten to the power of four or something and the other one maybe ten to the power minus three or something, very inconvenient to handle in a computational procedure. What really done is that after every stage you not only orthogonalise but also renormalize. That means you divide again to bring it in a circle shape or sphere shape.

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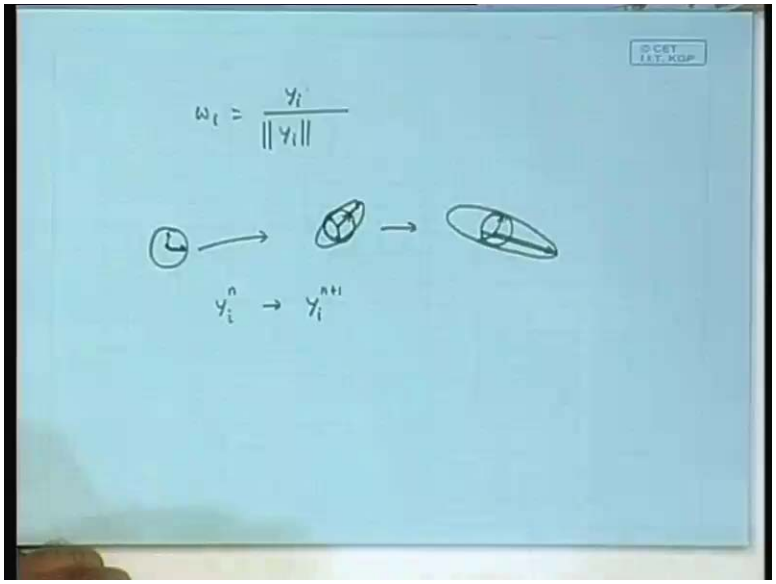

In order to do the renormalization we say that there will be new set of coordinates w_i which is y_i divided by the norm of… so again becomes a unit vector but the directions are set according to this. What are we doing? We are saying that we start from a point with a ball around it, after first iteration it became a… Naturally we had a long axis, I am drawing in two D so that it becomes clear to you, but it could as well be 3 D or 4 D. In this case the two coordinate systems are like this but that turn into this. Then we say no, if it becomes like this in the first iteration we have a measure of Lyapunov exponent that is this axis divided by this axis. But in order to take the next step we have to again define a circle around it. In order to do that whatever was the actual size, we divide by the magnitude to get again a unit vector. Now unit the vector w_i is this vector and the orthogonal to it and the orthogonal to it. Next iterate it might again turn like this so that here

you will obtain this unit vector, again you will obtain, again you will obtain. Having done that, you would say what measures then the growth in one step? It was initially y_i^n that went into, here it is y_i^n. There has been a growth or decay in the size as we have measured here. Growth in the decay in the size and that measured the expansion or contraction in that particular iterate.

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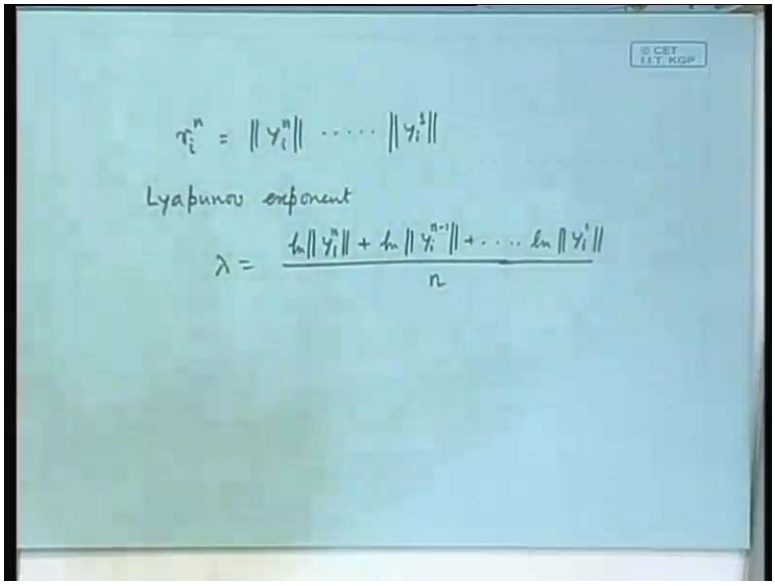

Ultimately we can say the Lyapunov number is given by simply the average. In the first iteration whatever was the expansion, second iteration whatever was the expansion, third iteration whatever was the expansion that is the total thing. There is a Lyapunov number, exponent is… Firstly you have to take the ln, norm of n plus. All these put together divided by n. That is a computationally realizable, easily realizable procedure for obtaining the Lyapunov exponent. What exactly did we do? We started from initial condition, took one step. At that point we locally linearise it, took the Jacobian obtain the Eigen vectors and thereby you obtain the expanding and contracting direction. When we have obtained the expanding contracting directions, we take the maximum expanding direction and then maximum expanding direction is the longest that means it is the big axis of that and then renormalize it and change all the other axis as orthogonal to that axis. The maximum expanding direction is retained as the principal, axis one.

Again you make a square, again you take the next iterate. Do the same procedure. In every particular iterate, particular jump you have some expansion of that principal axis. This taken logarithm, average about the whole thing gives you the Lyapunov exponent, essential procedure.
You might say that there are different situations where, I may not have the map in hand. For example if you have the Henon map in hand, there you have the exponent for the map and therefore at every point wherever it lands, you can take the Jacobian, you can do this procedure no problem. Therefore I would advise you to really try that once but supposing you have got a continuous time dynamical system which goes on like this and you have taken a Poincare section and you are obtaining the crossings. You have got only collection of points, you really don't have a map. How do you take the Jacobian? You cannot do that, you cannot take the Jacobian

really. In that case the same procedure has to be followed. By the way while I was talking about the continuous time dynamical system and the Lyapunov exponent, did I tell you how to obtain that from experimental data? Let me just repeat that once again.

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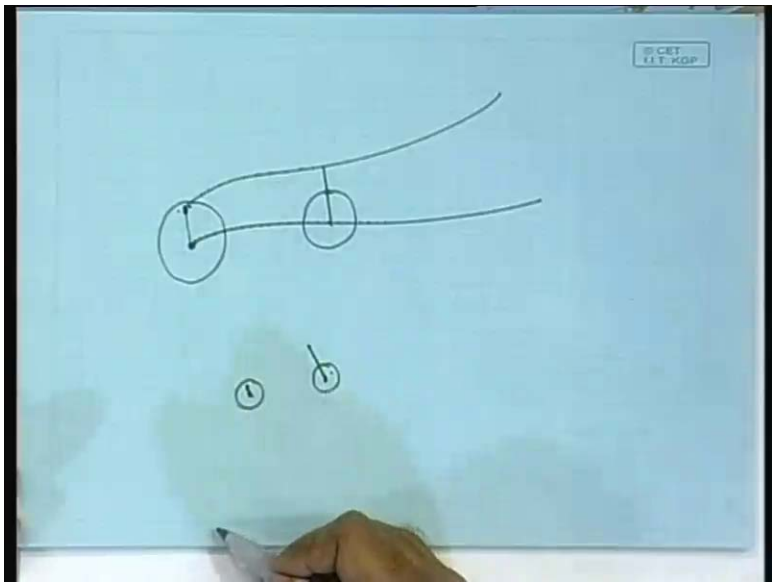

In continuous time dynamical system what are the procedure? Starting from one initial condition, you have got one trajectory. From another initial condition you have got another trajectory but if you have the data set then you do not have this. Any question? If you have the equations then you can start from a different initial condition. See how it will works. If you have data set only, you don't. Then what do you do? Simple, after all if it is a chaotic orbit it is winding on and if the orbit is ergotic, it will eventually visit a close neighborhood of that initial condition. That means while going on, some time it will come here or someplace within a close neighborhood of this. The moment you have detected it, you know how it evolved next because you already have the evolution of that. So you have very closely spaced data evolution results here, you also have the closely spaced data evolution results. So you know how it will works.

After some time again you stop it, renormalize it to this distance and then again scan the data set to find another point which is within this circle and go on. In case of the map also, you do essentially the same. Starting from an initial condition, you will go on evolving in case of that kind of systems where you do not have the map explicitly. That means starting from initial condition go on evolving you get a point. Again ultimately you get a sequence of points, large number of points, a data set. In that data set what are you looking for? You are looking for starting from an initial condition I have to define a error ball, you cannot define a error ball but you can define a point on the error ball within the error ball. So starting from initial condition simply sort the data set to find out another point that lies within that one and then you know where it jump next, you know where this fellow will jump next. You can find the distance. From this distance and this distance, you can find at least the initial estimate of the Lyapunov exponent.

Then again stop here and again define an error ball, sort the data set to find another point on this error ball, go on doing this procedure. That is how the Lyapunov exponent is actually defined. Now notice what will be the Lyapunov exponent be, if the system is periodic. If the system is periodic what will the Lyapunov exponent be? Lyapunov number, its natural logarithm is a Lyapunov exponent. What do you expect the Lyapunov exponent to be in case of a periodic orbit? Negative, the Lyapunov exponent will be negative, the Lyapunov number would be… Lyapunov number the extent by which it stretches or squeezes that should be less than one and as a result, the Lyapunov exponent will be negative. In a chaotic system what do you expect? In a chaotic system there has to be stretching in some direction that is the signature of chaos otherwise it cannot have the sensitive dependence on initial condition. In order to detect what is the necessity? That at least one Lyapunov exponent should be positive so that is the ultimate signature of chaos. If it is chaotic at least one Lyapunov exponent should be positive.

Normally the kind of system that engineers have to handle, they are all dissipative systems. There is always dissipation. Dissipation means if you start from error ball, after some time it's size will reduce but then its size wise its volume will reduce but at least in one direction it will be expanding that is the character of chaotic orbit. There are very rare situations where there are two directions in which the Lyapunov exponent will be positive, very rare but there exist such systems where two directions are positive. They are called hyper chaotic systems. In a periodic system, periodic orbit the Lyapunov exponent will be negative. Largest Lyapunov exponent will be negative.

Essentially what really matters is not all the Lyapunov exponents but the largest Lyapunov exponent, if the largest Lyapunov exponent is negative then you have contracting system and in contracting system means you have periodic orbit. If the largest Lyapunov exponent is positive, it is a chaotic system or an unstable system. Unstable system means it goes to infinity but nevertheless if you measure, do this procedure then it will yield a Lyapunov exponent that is large but it goes to infinity means unbounded. Therefore we leave it out of our attention right now. What if the Lyapunov exponent is zero, what kind of situation is that? Tthe Lyapunov exponent zero means two initial conditions will always remain the same distance apart. That is the physical meaning of the concept Lyapunov exponent zero, same distance apart. Now while talking about quasi periodicity, I have already done that, this is evolution on a torus.

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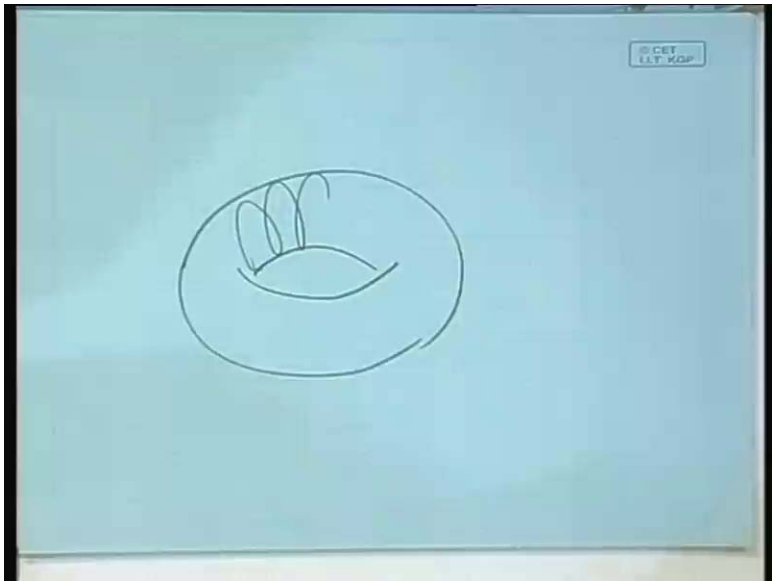

You have a torus orbit and if it starts from another initial condition, it also goes the same way and their relative distance does not increase. That is the situation where you have zero Lyapunov exponent, at least one Lyapunov exponent being identically zero means it is a movement of torus. I will stop now, we will continue from here in the next class.