Chaos Fractals and Dynamical Systems
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Lecture No # 23
Matrix Times Circle: Ellipse

Now I noticed a very important confusion in some of you which needs to be clarified at this stage.

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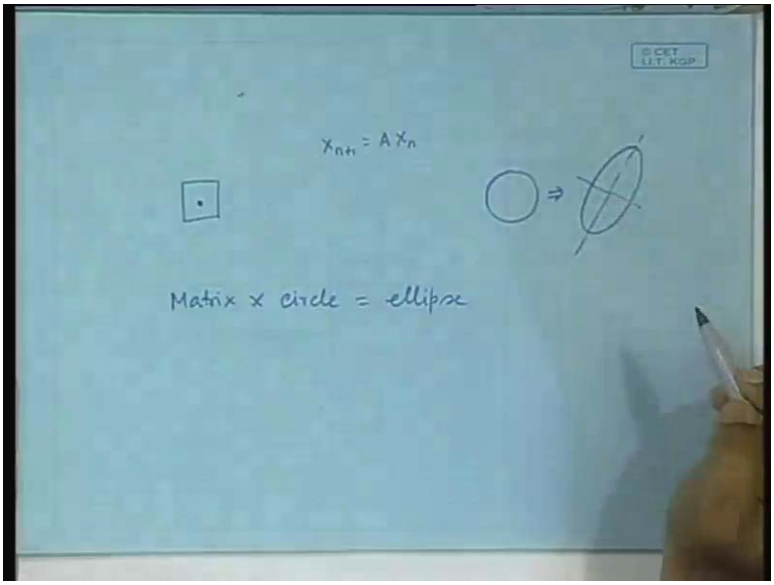

I said that if you have a fixed point and if you locally linearize it around that point, you get an equation of the form x_{n+1} is equal to AX_n. We said that if the Eigen values of the A matrix are real then depending on the magnitude of the Eigen values there will be some expanding directions, there will be some contracting directions. Now I notice that some of you are confused whether the Eigen vectors would be orthogonal to each other. In the exams you found that they are not orthogonal but then I had commented that any given region is tested in some direction, contracted in some direction so as a result if you start from a circle, what would be the result? It would be an ellipse.

I guess the source of the confusion is that the ellipse has a major axis and a minor axis and the major axis and the minor axis are orthogonal to each other. That is probably what led to the subconscious idea what we are looking for are orthogonal Eigen vectors. No, that's not true but it is still true that a matrix times a circle is equal to ellipse. While studying matrices we often do not realize this geometrical concept but it is true. Any square matrix operated on an area of a circle means operated on every point of the circle leads to another shape which is an ellipse. What is the problem? The issue is that here you are starting from a circle and here you are ending up in an ellipse.

There is obviously an expanding direction and a contracting direction, the major axis and the minor axis and obviously except for a few cases you do not expect the Eigen vectors to be orthogonal to each other. Only in symmetric cases they will be, else not. That is why we need to recall a theorem probably you have come across in mathematics classes but it will be necessary for us to recall this theorem.

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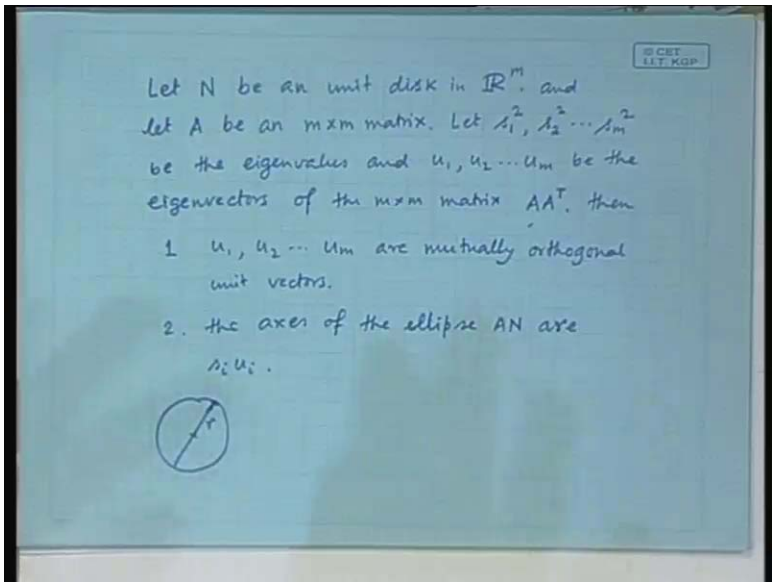

The theorem says let N is an unit disk, a circle in the space of real numbers of dimension m and let A be an m cross m matrix. Now this is the important thing. Let s_1 square, s_2 square, s_m square be the Eigen values and u_1, u_2 be Eigen vectors of the m cross m matrix AA transpose, this is important. Then u_1, u_2, u_m are mutually orthogonal and second point is the axis of the ellipse A operated on N are s_i u_i. Notice what it says. First what are the axes of the ellipse? They are the Eigen vectors of, not the original A matrix but the AA transpose. The moment you take AA transpose, the Eigen vectors of this matrix would be orthogonal to each other. (Conversation between professor and student: Refer Slide Time: 06:55- 07:04). Yes, but since A could be just any matrix, you need to get into that form so that you get orthogonal vectors and that is exactly what it says but in addition to that it says that the contraction or expansion along that direction would be square root of the Eigen values. The Eigen values are s_1 square, s_2 square and all that so suppose you got the Eigen values then whatever was the radius of the circle in a particular direction say from here to here it was say r. Suppose this is one of the Eigen vectors that means Eigen vectors of AA transpose. Then if I ask you how much would be the contractivity or expansion along that direction, it will be given by the square root of the Eigen value on that direction times r.

Essentially when we talk about the ellipse and when we talk in terms of a circle turning to an ellipse, essentially we are using the matrix AA transpose. What are the implications? Now is that clear that if you obtain the Eigen values Eigen vectors of the original A matrix they could be non-orthogonal, just any directions but the moment you take AA transpose they become orthogonal and they are the actual directions of the major and minor axis.

Let us illustrate that with one example.

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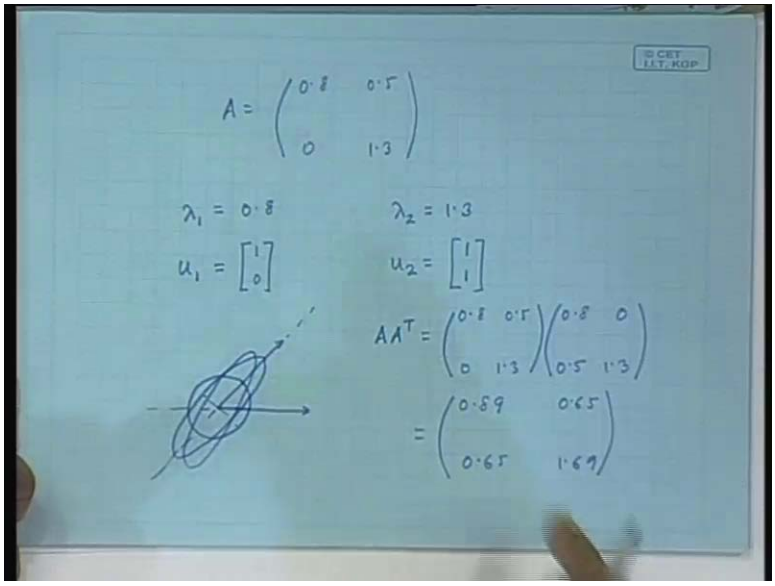

Suppose the A matrix is 0.8, 0.5, 0, 1.3. You can obtain the Eigen values and Eigen vectors. 0.8 and 1.3 so obviously this will be a saddle fixed point because there is one Eigen value outside the unit circle. What are the Eigen vectors for this one? Since you are able to obtain this, so I am not explicitly doing this here but you should be quite comfortable with these calculations. You might take minus one so the other one will be also minus one because this is just a direction. What are these? 1 0 will be along the x direction and 1 1 will be… (Refer Slide Time: 10:55) that these are the two directions and in this direction you have expanding behavior and in this direction you have contracting behavior. Now if you start from a circle, where will the next iterate of the circle lie? Next iterate of the circle will not lie along this direction obviously. Its major axis will not be along the ultimate expanding direction. If you keep on applying this map on the circle then ultimately it will expand along that direction but when you come to the question of the ellipse that you obtain in next iterate, it is not along this direction. It is along the Eigen vector of AA transpose. So obtain AA transpose.

This theorem has said that if you apply the matrix on a circle, the result will be an ellipse but that ellipse's major axis and the minor axis will be along the Eigen vectors of the matrix AA transpose and since the Eigen vectors of the matrix AA transpose are not the same as the Eigen vectors of A, therefore you cannot do this. You cannot say that it will turn into an ellipse looking like this, it will not. (Refer Slide Time: 13:14) it will not be because, the contraction is along this direction. The contraction is not along this direction, it is along this direction and therefore in actuality it will be something like this and so you need to know what is this direction and that is given by this matrix. Let us obtain it. How much? 0.89, 0.65, 0.65, 1.69. What are the Eigen values and Eigen vectors of this fellow?

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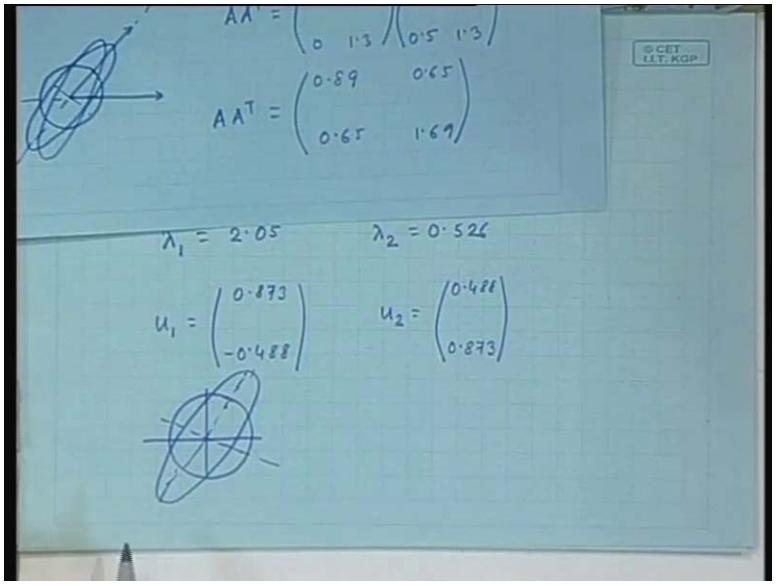

We start from here. Here is AA transpose, its Eigen values are 2.05 and 0.526 something like that. (Conversation between professor and student: Refer Slide Time: 14:30- 14:49) What about the Eigen vector? Now the Eigen vectors will be orthogonal to each other, so obtain the Eigen vectors. Why don't you bring calculators then this becomes simpler? Let me put the values and you check whether I am writing it correctly or not. These are obviously orthogonal to each other. If you start from the circle, it will turn into an ellipse with these Eigen vectors so this Eigen vector how do I draw? This is 0.8 something so say it will be up there and this way and down like this. This is one direction and this will be another direction and so this will be expanding along this and contracting along that. These are the major axis and minor axis. Now you might ask what is the relationship between this and the Eigen vector of the original A matrix? The idea is that in the first iterate it will fall along that but then if you ask, what if I apply the same original A matrix on this ellipse, you will get another ellipse.

Again if you apply that you get another ellipse and so on and so forth. Ultimately what will happen? It will turn into a thin elongated stuff. In what direction? Along the original direction. Is that clear now? There are not so intuitive things, counter intuitive things that result from this. For example if you have both real and equal Eigen values, what will be the behavior like? Then there is only one Eigen vector and if the Eigen value is less than unity, you expect it to contract along that direction.

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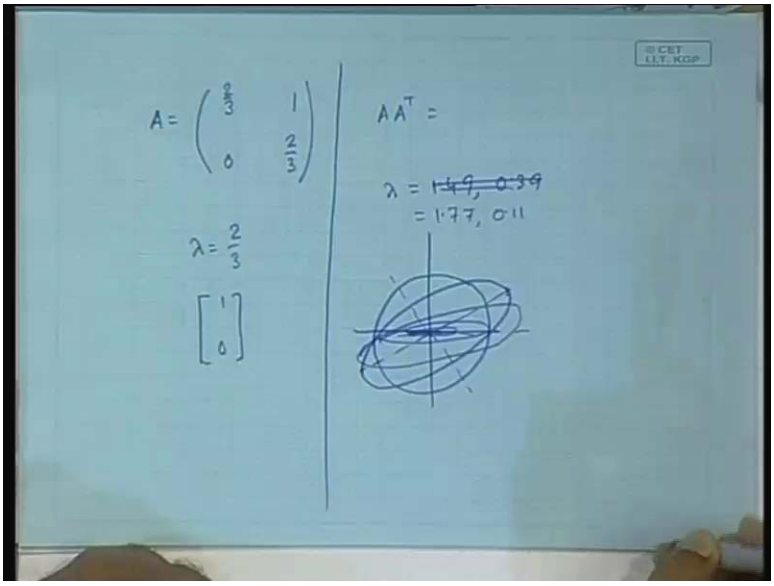

Now let us try to do this on a problem. Say A is equal to 2/3, 1, 0, 2/3. If you have this then what are the Eigen values and Eigen vectors? (Conversation between professor and student: Refer Slide Time: 19:11-19:28) 2 by 10, 2 by 3 and 2 by 3 repeat Eigen value and 1 0 but then you will not be allowed to conclude that in the next iterate itself, it will have the ellipse with a x axis, 1 0 is the x axis as the major axis, it will not happen. You will have to obtain AA transpose and it's Eigen value and Eigen vectors. You do that. This is what we had with A only. Now AA transpose, what do you have? The Eigen values and what do you have? The Eigen vectors. Are you getting something counter intuitive, the Eigen values. Yes, what is the Eigen value? 1.49, 1.39. See one is greater than one which means in the next iterate, a circle is going to expand in one direction, even though your Eigen value was less than one. It's a very counter intuitive situation that really happens and so it means that if you start from the circle, in the next iterate what are the Eigen vectors.

Suppose these are the Eigen vectors of this matrix. That means we were talking about the major and the minor axis. Then it is going to expand in one and contract in the next iterate but since the Eigen values of the original matrix are both less than one, the conclusion is definite that ultimately it is going to shrink but in the first iterate it doesn't shrink. It expands for some time and then shrinks. In the second iterate it will, what is the conclusion? Without actually doing the mathematics can you conclude how is the second iterate going to look like. It is going to look like an ellipse. Ultimately when every things dies down, what do you expect? Things to come to this axis because this is the Eigen vector of the original matrix but it is going shrink along that. What we expect is that slowly this will come down to x axis and then it's going to shrink. In the next iterate you expect something like this to happen (Refer Slide Time: 00:23:23). By logic that is what follows. So intermittently you may see an expansion. (Conversation between professor and student: Refer Slide Time: 23:44-23:57). What is that? Fine no problem, 1.77 and 0.11 still the conclusion remains intact. No, initially if you calculate the Eigen directions, it will not be the same as the Eigen direction that you calculated here.

Initially it will start from that Eigen direction but ultimately it has to convert on to this Eigen direction. That means slowly it will be more and more slim flips. Why slim? Because in the two direction there are two different contractivity factors, this one will contract fast. In this direction it is expanding. Further what will happen? It will become thinner and thinner and finally in that direction it will converge. Somewhat counter intuitive result but this is not very peculiar though because it is not always necessary then that this will happen, if the two Eigen values are equal. Suppose you slightly part of it. What will happen? The two Eigen values will not be equal but still this conclusion is very much strong you see. 1.77 is quite large, 0.11 is quite small.

If you slightly change it, this will slightly change but the conclusion will not change. That means even if the two Eigen values are both less than one, the ellipse may actually expand in some direction intermittently and then contract because we are trying to conclude something about the resulting ellipse, because without that ultimately what will happen? If you have only this, can you conclude anything about the ellipse that will result from one iteration of a circle. Starting from a circle after one iteration where does it land? Let me repeat the question for the audience's sake.

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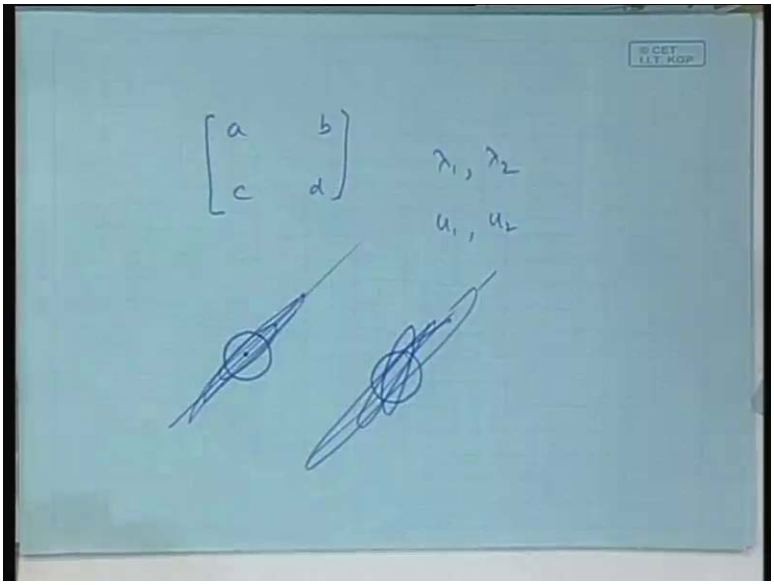

His question is that supposing you have got a matrix abcd and it has two Eigen values lambda$_1$, lambda$_2$ and naturally two Eigen vectors u_1, u_2. What do they say? They say that ultimately if this is a fixed point and if you start from a circle then suppose this is greater than one and this is less than one, expand along this direction ultimately. After many iteration ultimately it is going to look like a line and that line is along u_1 but that does not mean it will go like this. It will actually go like this, it will start from there, ultimately it has to go like this so it will go like this then it will converge. This particular behavior is to be captured only by looking at the matrix AA transpose. Other way what will happen? You will tend to conclude that right from the begin, it expands in the direction of the Eigen vector. No, it doesn't, except for the cases where the matrix itself is symmetric. (Refer Slide Time: 27:57) Then it's a bit more complicated that means the theorem is stated based on an ellipse.

What did we do? We concluded something from the theorem regarding first iterate from the circle and we also concluded something from our idea of dynamics that ultimately it is going to converge into that direction and from there by logic we concluded about the intermediate steps but really this theorem doesn't talk about the intermediate steps. It doesn't. What we have to do? You either need to numerically do this iterations and see how it changes, it will be good exercise to do that, very easy to do it in mat lab but normally the theorem talks about the first iteration. How does the ellipse do? Now if you have this idea that it is going to turn into an ellipse that idea can be used for various purposes, for example whenever we talk about the contractive direction, expanding direction we need to use these ideas but before that we have already concluded this was sort of a clarification that I needed to give but I will use these results later. Let us come to the issue of what we started off with in the last class, how do we characterize a chaotic orbit? Look at the screen.

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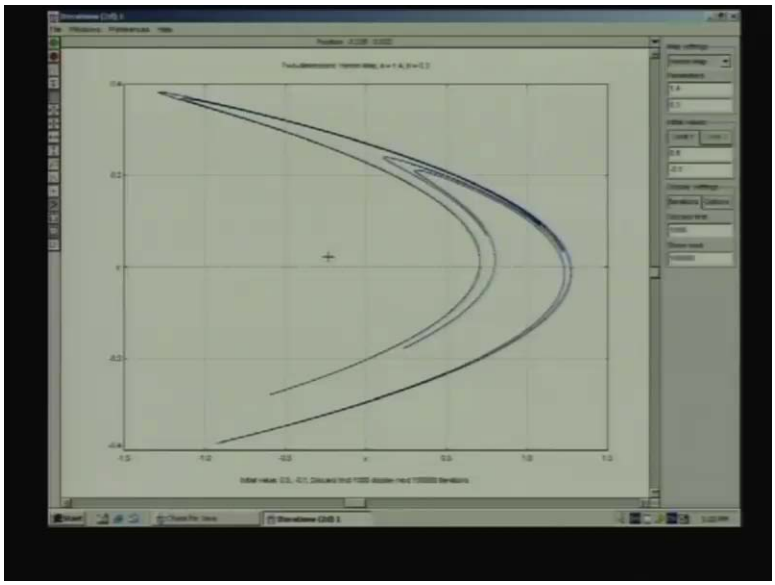

Here we have a chaotic orbit, you know that this is the attractor for the Henon map. Suppose in some way I want to quantify this characters given as a number. This is this much chaotic or some number that will distinguish it from a periodic orbit or other kinds of orbit. How do I obtain such a number? One very simple thing that one can we do that follows from what we studied in fractal geometry. This is a fractal and how are fractal objects characterized? By their dimension, so one simple way to characterize such an object is simply to obtain its fractal dimension and the fractal dimension is essentially, there is another name that goes with the concept of fractal dimension. It is called the box counting dimension because you really have to count boxes. So in that case what you have to do?

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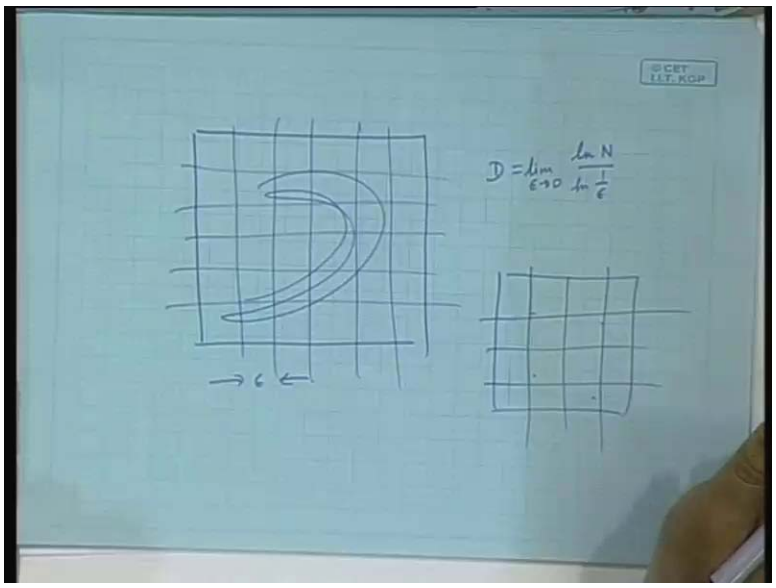

You have to overlay the orbits with boxes, say this is the orbit and you have to overlay with boxes and you have to count the number of boxes and then progressively make it smaller. That's what we did. Your size is then epsilon size and then progressively make epsilon smaller and what was the concept of the dimension? One by epsilon as limit, that was the concept of the dimension. So here also that will remain the concept of dimension. That means we can find out the dimension of a chaotic orbit. If this orbit happens to be periodic, what will be the dimension that you have obtained? Say it is a period four orbit means here there will be four points only. If you do the same procedure on that set, what will be the number you obtain? Ultimately it will go to zero. Epsilon is equal to 4, there are 4 points only. Imagine the situation you have four points only, initially there will be say all four filled up but then if you further keep on subdividing it, there will be many boxes that go out. Ultimately this four will be contained in an infinitesimally small number of boxes N tending to zero. As a result this D also tends to zero. While that will not happen, if you have a chaotic orbit.

The character of the chaotic orbit as I told you when I was sort of intuitively or by a hand waving argument, I did not regressively define ergodicity but I said that an ergotic orbit is one that visits many parts of the state space if that happens then the dimension will not be a integer dimension, zero or one or whatever. In this case for example if you do the same procedure for this map, how will you actually proceed?

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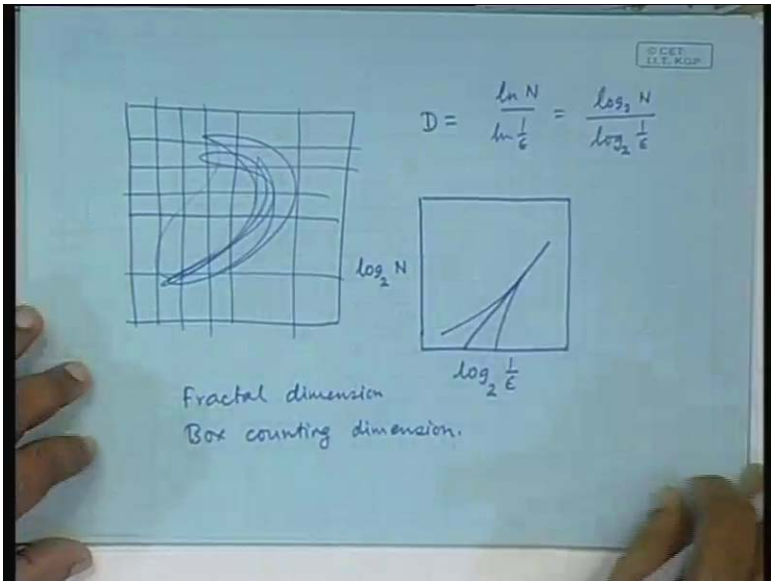

Suppose here is your chaotic orbit and you will first start from a box that includes the whole thing, divided into four parts, further divided into four more parts like this, further divide and so on and so forth. That is how you will you proceed and at every stage you will count the epsilon and you will also count the number of boxes that are required to cover the object N. Now when you do that, you notice that the epsilon is reducing by half at every stage. That is why even though the definition of dimension is given by ln N by ln 1 by epsilon. Often it becomes more convenient to express it the same, as log 2 of N by log 2 of 1 by epsilon actually the same thing but it becomes often practically convenient to write it in this part. When you actually have to do it, you often will have to express it in this form to make your life simpler because you are actually subdividing by two boxes.

Example for this specific map that you see on the computer screen, in this case if you do the same procedure you will get this kind of numbers. What actually is done is you draw a graph. In that graph the log 2 of 1 by epsilon is plotted in the x axis and the log 2 of N is plotted in the y axis. What do you expect? It will be a straight lines so it will be some kind of a straight line and what is the dimension? The slope is the dimension, it will not be one in this case. That is exactly why this is a factor. If it had been one, it will not be a factor, so the procedure of actually obtaining the fractal dimension is to plot a graph like this. So as you slowly reduce epsilon, you plot a graph and you obtain a straight line. If you do not get a straight line, you know that your procedure is not accurate enough as here. When can that happen? If your step sizes are yet too large or it has become too small, imagine that ultimately you have on the computer screen something that is generated by pixels and if you go on reducing the epsilon lower than the size of the pixels then what will happen? It will be meaningless. In this case you will get something that is away from the straight line that's why it is necessary to draw the straight line because then you know that I am on the right track, I can get this slope. You actually have to obtain the asymptote. If it goes like this or that, ultimately you have to take that part. Ideally you should take any infinite number of iterations, physically, computationally that's not possible so you have to take a large number that's all.

You would notice that there is a practical difficulty of this algorithm. The moment you imagine, how you would like to code it? Ultimately this has to be coded, you would not like to go on counting physically 1, 2, 3, 4, 5, 6, you cannot do that. Ultimately you will have to create some kind of bins, boxes and find out how many iterates fell into each box. For 2D that is not a big deal but there is no reason to imagine that a map will always be 2D there can be 3D maps. If you have a 3D map and if you want to subdivide each side into say 100, how many boxes would you need? It's a huge and that many boxes how would you really code it? Can you imagine? It is going to give a lot of trouble. When you actually want to code it because that will require this many positions to be created and those have to be implemented. Computationally that becomes inefficient. Imagine 4D maps, unmanageable. That is why for simple things we do use the fractal dimension as a measure of chaos, simple means at most 2D but if the dimension is larger then it becomes convenient to do something else. What's that something else? This was the fractal dimension equivalently called box counting dimension.

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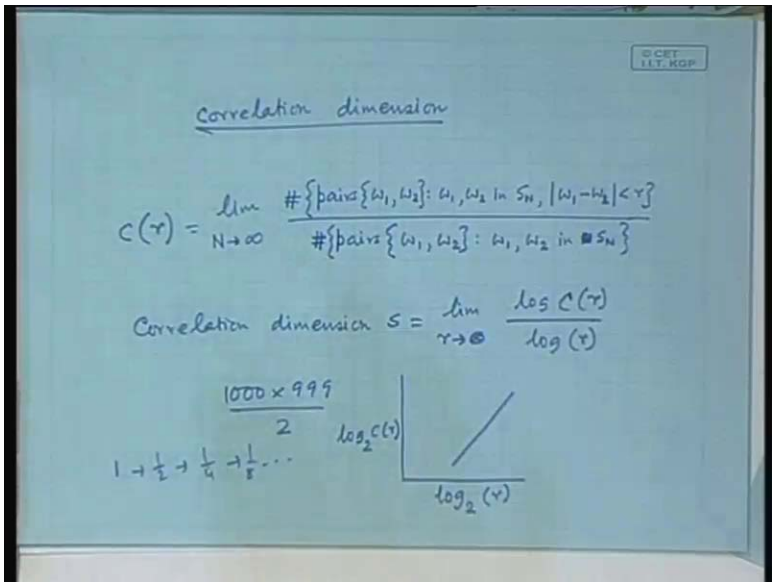

There is another measure called correlation dimension. Now what is that correlation dimension? Essentially this is a collection. If you look at the computer screen, you will see that this is a collection of points and each point is falling in different locations here. Each point is occupying a different coordinate but then if I ask you what is the distance between these points where my cursor is and another iterate, obviously that could be smaller than a specific given distance r or larger. If I ask you how many points are there in that chaotic orbit within a given distance r from a point. How many pairs are there that are situated within a certain distance? What actually are you measuring?

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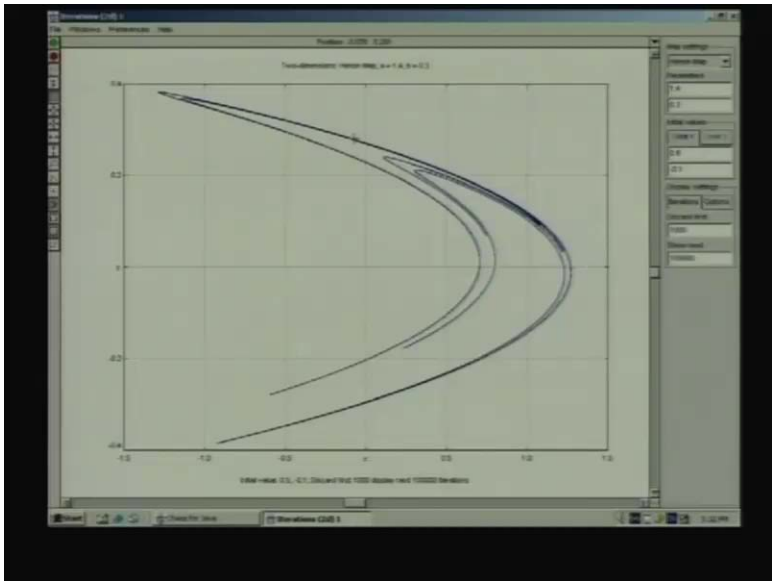

Yes, we are measuring correlation in some way but notice the conceptual difference with the box counting dimension. In the box counting dimension we are not distinguishing the density of orbits. If a box is filled, it is filled that's all. How many are falling in each box or how dense are parts of the orbit, those things are not really been accounted for. The moment I ask, how many pairs of iterates can I find in this whole orbit that are situated at distance less than r. If it is very dense in some parts, it will be measured by that. Now let me define the correlation C as a function of r, r is that distance which is limit N tending to infinity, number I hope you know that the hash to fill this number. The number of pairs w_1, w_2 this w's are not meaning (Refer Slide Time: 00:42:24) this is just iterates such that w_1, w_2 in the orbit S_N and w_1 - w_2 this is the distance which is less than r. Did you understand what I wrote? I will repeat, let me write the denominator also. Number of pairs, so in the denominator the total number of pairs. Pairs w_1 w_2 where w_1 w_2 are in that attractor. In the numerator those pairs of w_1, w_2 which are in the attracter but which are also whose mutual distance is less than r, n is the number of iterations.

Now obviously if you make r very large then this will be more or less equal to one because if r is very large then the numerator and the denominator will become equal, r is very small say zero then obviously it will be zero. It is the function that depends on r. Then the correlation dimension S is given as limit r tending to 0, log C is a function of r divided by log(r) but this number will actually converge and it so happens that it converges to a value that is reasonably close to the fractal dimension, not exactly equal to fractal dimension. It is also a measure of the character of the chaotic orbit but it also incorporates within it, the idea of the density of points. In some parts it is more dense, in some parts it is less dense and that will make a difference in the correlation dimension. While in case of the box counting, it only cease which parts are covered which parts are not. In this case how will you actually measure it? Notice here there is no question of box counting and therefore there is no question of creating a 1000 to the power 3 number of boxes that computation problem is completely overcome. What do you have to do? Start from a point, how many total points will be there, how many total pairs will be there? If there are total 1000 number of points in that orbit, suppose in this orbit there are 1000. How many pairs are there?

There will be 1000 points, from each point, how many points distance can be measured? Not itself so times 999, one less. This many pairs will be there. Start from one and find out how many are less than r so collect them into this, go to the next one find how many? It doesn't require that much of memory, you simply go on measuring this and putting it in a bin that counts only the numerator that's all. Computationally therefore this is far more simpler and since it yields a value that is indicative of the character of the chaotic orbit, this is a good measure. Here also the procedure will make a graph, if you are reducing r by 2 that means initially r was covering the whole thing then you make half then you make half of that, then you make half of that, it will be convenient to express it in a log 2. So $\log_2$ of r and here it is $\log_2$ of C(r) and what do we anticipate. It will again be a straight line and the slope will give that's all. That is another measure of chaos that means how chaotic it is or the character of a chaotic orbit as you have already seen that as you change the parameter, the chaotic orbit doesn't really remain the same. It remains chaotic but it doesn't remain the same.

For some parameters value it was two piece chaotic orbit, for some parameter value it became one piece, for some parameter value it was a very small but chaotic orbit, still those things can be distinguished, differentiated and characterized by means of the correlation dimension which is easier to compute than (Refer Slide Time: 00:48:48). (Conversation between professor and student: Refer Slide Time: 00:48:49-00:49:14). Yes, we are talking about the ratio. This ratio is actually, both the numerator as well as the denominator are changing. Obviously you are not taking to zero that's a fictitious situation. You are slowly starting from say one then you are changing to half, then you are changing to one fourth then you are changing to one eighth and so on and so forth. You will go at most up to some 8^{th} or 9 steps like this and calculate the slope that's it. That is computationally more tractable, that matters. If you are asked to actually characterize a chaotic orbit, this is a more tractable thing to do.

You might imagine a distinction as, everywhere in the attracter it has equal density then they will become equal but if they don't, then there will be some difference between the (Refer Slide Time: 00:50:25). Apart from these two, one is box counting dimension another is the correlation dimension. Apart from these two there is another measure of dimension which is often used but that is related to Lyapunov exponent. Lyapunov exponent is something already done but in relationship to continuous time systems. Now we have to do it in relationship to the discrete time system, it is not exactly the same thing but since time is coming to a close. I suppose it will be good to cover that later in the next class. Is there any question on what I did today? The number of points that you have taken on the orbit and then you are going on applying the procedure so there is some number to start with. Normally no. What happens is that if you see that it is asymptoting to a straight line then you will know that we are true. (Conversation between professor and student: 51:53-52:10). No, after all you are calculating the distance to all points and then finding how many are falling within r but what are you doing. You are starting from a point and going on calculating, you are not requiring that much of memory. That's what the confusion and advantages. That's all for today, we will continue in the next class.