Chaos Fractals and Dynamical Systems
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Lecture No # 20
Stable and Unstable Manifolds

In the last class we were talking about the saddle fixed points and we had seen that there can be Eigen vectors which would be the stable and the unstable Eigen vectors locally but if you look at the rest of the system that means if you consider the state space outside the immediate neighborhood of this fixed point, you will find that these lines are all bend so in all sorts of possible ways they can bend and we are said that this is a stable manifold and this will be the unstable manifold.

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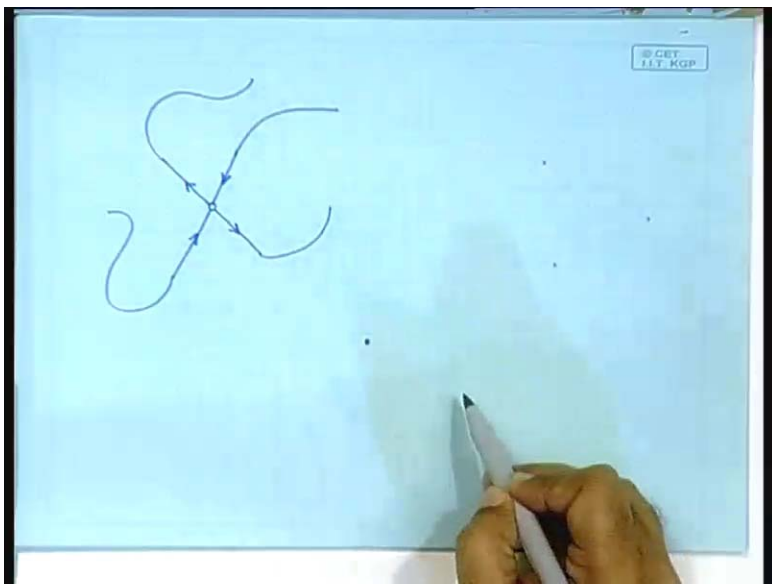

Now let us consider a few properties firstly we know that in a dynamical system we can expect attractors. Attractors means starting from anywhere, in the state space if any initial condition starts from anywhere in the state space you will ultimately be attracted to those collection of points. It can be a period one orbit, period one attractor or period two attractor even chaotic attractor whatever it is, the point is that it is a collection, it's a set to which initial conditions are attracted but there is no reason to believe that in a nonlinear system there can be only one attractor. At the same time there can be more than one. For example it is possible that here is one fixed point attractor and here is another period three attractor both are stable which means that both will have their own basin of attraction that means there will be a collection of initial conditions which will be attracted to this one, there will be another collection of initial condition which will be attracted to that one and they would be divided by a basin boundary. So let us now consider this kind of situations, in a dynamical system there are all these possibilities.

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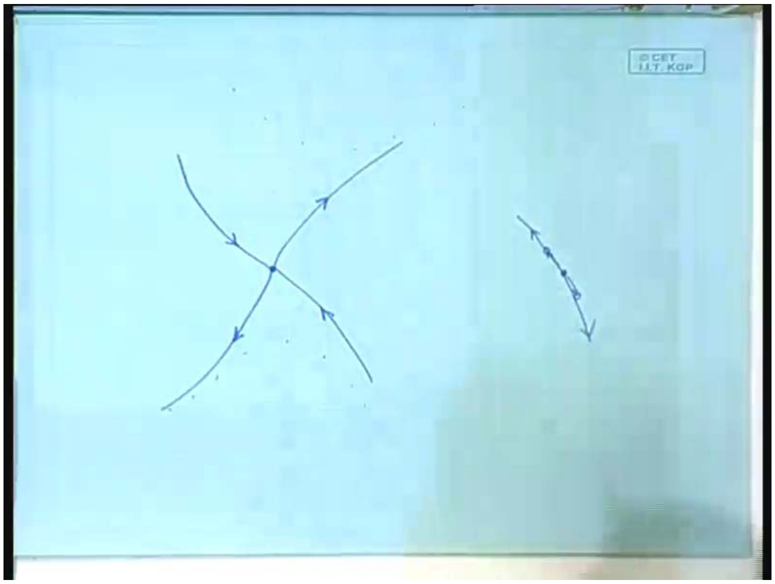

Now if you consider there is one saddle fixed point with some kind of an unstable manifold and a stable manifold. Then notice that started from any initial condition say here the orbits will be progressively attracted to what? Suppose it is here, how will it proceed? It will progressively proceed like this, started from any initial condition here, it will proceed like that. So that is it is not difficult to see that the orbits are attracted to the unstable manifolds. So in initial conditions around here will be attracted the unstable manifolds, now that's leads to a conclusion that if there is any attractor that must be situated on the unstable manifold of a saddle fixed point. For example it may be so that a periodic orbit was stable for sometime and then that became unstable as a result of which period two orbit appeared, a period doubling bifurcation happened and then now a period two orbit which is stable.

The immediate conclusion would be that those two points of the period two orbit must be situated on the unstable manifold of that saddle fixed point. What am I referring to? For example in the last class or the class before that we had considered the Henon map and we had seen that at a specific parameter value, it loses stability. The fixed point loses stability and it becomes period two orbit. So it is something like this that earlier let me draw here. Earlier this point was stable and then now it has become unstable and as a result of which there is a period two orbit which have become stable now and this fellow has become unstable.

Immediately the conclusion from this consideration would be that these two fixed points must be situated on the unstable manifold. So this must be the unstable manifold direction and as you know further changing of the initial condition will lead to further series of period doublings and finally it will land up into a chaotic orbit and then the conclusion would be that even the chaotic orbit must be situated on the unstable manifold. So if you find any system in which there is any attractor that must be on the unstable manifold of a saddle orbit provided there is a saddle fixed point existing.

If there is no saddle fixed point existing obviously that won't be true but else it has to be on the… Then notice another consideration. What is a property of the stable manifold? The unstable manifold is an attractor that we have concluded. What is a property of this stable manifold? Just consider two fixed points, two initial condition at the two sides of the stable manifold. What happens then? Their further iterate it will go like this which means they would be rippled from the stable manifold. Initial conditions at the two sides of a stable manifold would be ripple from each other will go in different directions. So the property of a stable manifold is like that of a rippler. It ripples and where exactly do we need that property?

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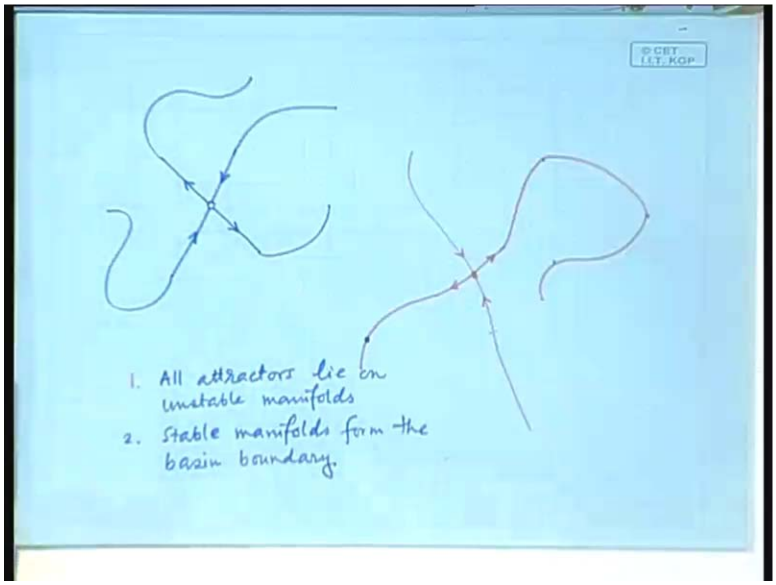

We need that property in a situation like this where you have one fixed point another orbit both existing stable at the same time. So that you immediately need some kind of a separation between them, a basin boundary and the basin boundary must have that property that two initial condition at the two sides of that must be rippled in two different directions. If that means so then it requires the property that is already contained in the stable manifold. Does it stand to reason therefore that the basin boundary must be nothing but the stable manifold of a fixed point which means immediately that there must be a saddle fixed point sitting on the basin boundary.

Notice the line of arguments I am talking in each sentence but these are all individual theorems. So there must be a fixed point sitting, wherever you find two attracters existing at the same time separated by a basin boundary. Here is a basin of attraction of one attracter; here is the basin of attraction of another attracter separated by a basin boundary. Immediately you have to understand that there must be, even if you do not know where it is, the logic should tell you that there must be a saddle fixed point sitting there and it should have the stable manifold here and the unstable manifold there. The consideration that I just said that all the stable attracters must be sitting on the unstable manifold means that this fellow must be sitting on the unstable manifold and these guys also must be sitting on the unstable manifold. Must be, there is no other way.

So you can see that from logic we can infer a lot of things about this character of the state space, simply by logic. We did not know where this fellow is but from logic we infer that there must be one. We did not know actually how the stable and unstable manifold will go but from logic we infer that there must be going through those points and if there is a whole complicated chaotic orbit, the unstable manifold must go through all of them, the whole chaotic orbit. These are two very important conclusions. These are the prime roles played by the saddle fixed points. That is why saddle fixed points are the center piece of complicated dynamics. So wherever you find very complicated dynamics, you know it must be the handy work of some saddle fixed point. So first conclusion let me write down clearly so that you it goes into your head. One is that stable manifolds form the basin boundary. Now let us understand some of the other interesting thing that the stable and unstable manifold do. Just recall in the last class when we considered the Hénon map we found that there are two fixed points one, fixed points sittings somewhere here.

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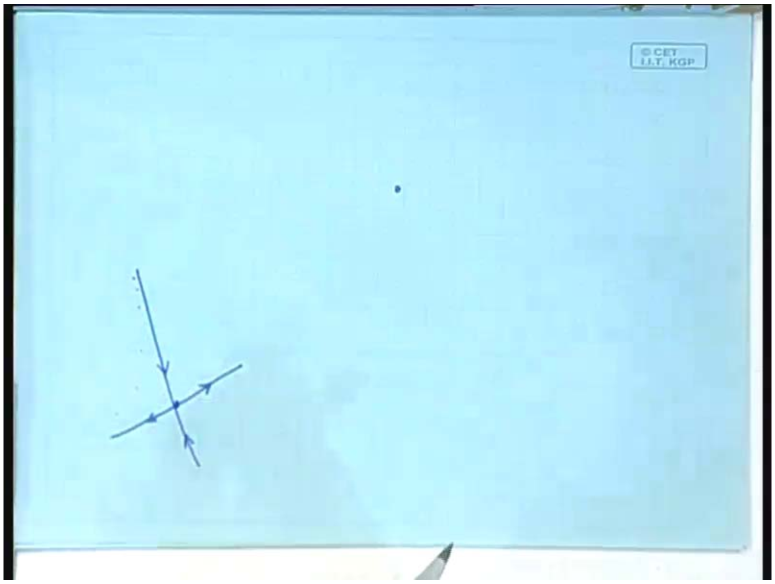

Both are on the 45 degree line so I will draw on two different places and this fixed points was the type of a saddle. This fixed point was, it was an attracting fixed point. We have also concluded that beyond a certain parameter value, as you change the parameter value this one also loses stability. Just recall, if you do not recall just open the page and recall yourself, we have done that. So this one was at some parameter value, both of them were born and when they were born they were born through saddle node bifurcation and this fellow was a saddle that fellow was a node and that node also became a saddle beyond a certain parameter value. So beyond a certain parameter value, you would expect two saddles to exists, this and that and this fellow was attracter earlier then it became a saddle and this fellow was throughout a saddle (Refer Slide Time: 00:13:50 min).

A saddle means it will have stable and unstable manifolds. So it's actually the stable manifold and the unstable manifolds are something like this. So this is the stable manifolds and this is the unstable manifold. Consider a situation where this fellow is a stable fixed point. Can you infer that where does this fellow go? It has to go there. Where does this fellow go? To infinity means that in this state space there will be some region that is the basin of attraction of this attracter and there will be the rest of the region which are the basin of attraction of the attracted at infinity. So starting from any initial condition say here it will go to infinity. So even though this fixed point is stable it is not true that all initial conditions are attracted to this. There is a basin and the basin boundary is created by the stable manifold of this fellow. Now beyond a certain parameter value this fellow becomes unstable, this fellow becomes unstable and you have the period two orbit then the period four orbit and so on and so forth. You have seen the bifurcation diagram of this system on the computer, so you know that follow the same sequence of period doubling cascade but at some point of time it will also become chaotic.

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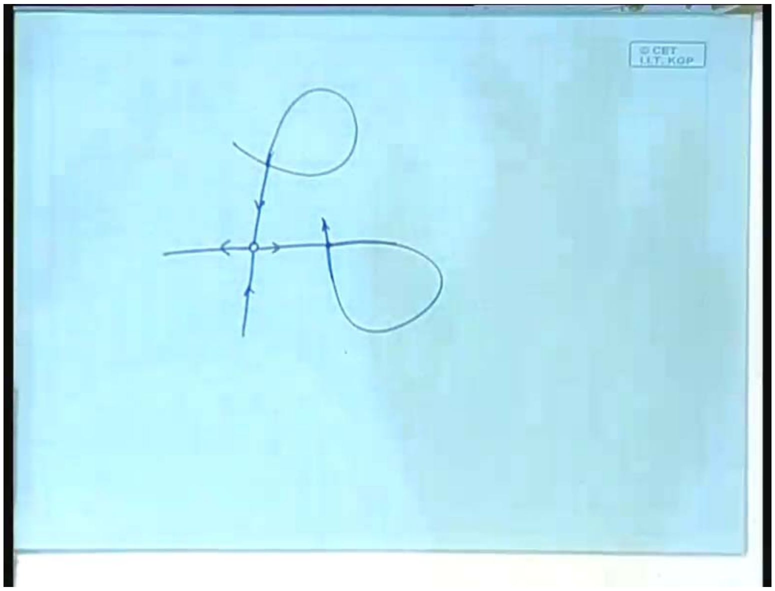

Notice the important thing. Suppose you have a saddle fix point here and suppose here is the stable manifold and here is the unstable manifold and as I told you that beyond a close neighborhood it will be bend, it will go anyway. So suppose in this fellow going like this, can it intersect itself? No, why not? What is the logic, why is this not permitted? Try to free yourself from the ideas that you learned from the continuous time dynamical system, this is the map. So the kind of logic that you are probably going on in your head are actually taken from the continuous time dynamical system at the same point they cannot be two vectors but here it is a map. Let us do it with the attracter, is it possible? Both are not possible because if we have a point here, in the next iterate it should go along outwards from here and because it is here it should go outward from here. So at the same time we cannot do both. So that will not be permitted both these situation would not actually happen, this is not physically possible but that to happen.

So even though they wonder any which way they cannot really cross themselves, that is a limitation but then can this happen? There is no concrete logic that would prevent this. There is no concrete logic you can provide that will say that no, the stable and unstable manifolds cannot cross but they can cross.

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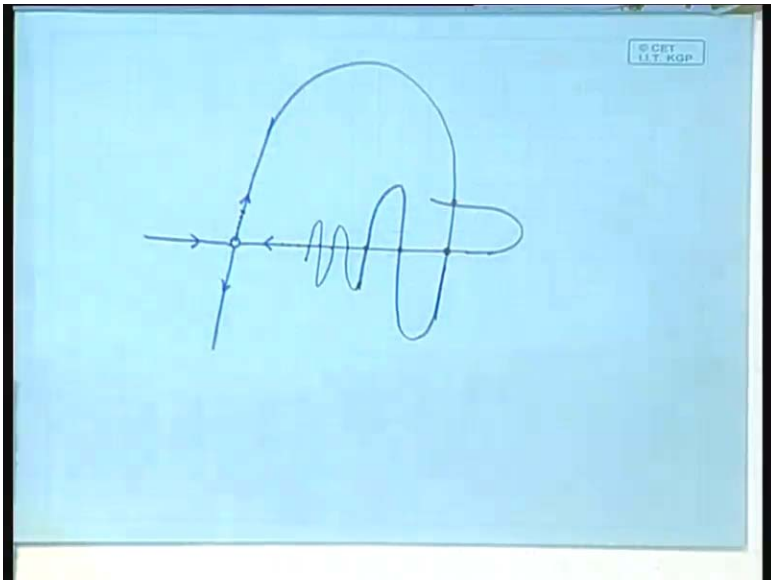

Now notice a very interesting thing that happens immediately, notice this particular point. Now this point is both on the stable manifold as well as on the unstable manifold. Now if you take this point and apply that map, it should map somewhere and since it is the stable manifold it must be along that direction it must approach that. So suppose it maps here, it must be on the on the stable manifold the next iterate but it's also on the unstable manifold. How can that be possible? No, the only way that is possible is that unstable manifold also crosses. That is only way it is possible. It cannot map to itself because if it maps to itself, it becomes fixed point. Fixed point was here, the only way that you can have this point mapping to this point is that this particular point where it maps to that must also be at point of intersection between the stable and unstable manifold.

Next point says here that must again be and if there are an infinity number of points, it goes like this. So a single inter section between this stable manifold and the unstable manifold, immediately implies as a theorem that there are infinity of intersections. A single intersection means an infinite number of intersections. Notice I am not doing any mathematics, just geometric logic. Is the proof convincing? Wherever it maps to this particular point, where ever it maps to, that must be a point again both on the stable manifold as well as the unstable manifold. So the unstable manifold must go like this that's it. Not only that where did this come from? It must have come from some place that means it must have been somewhere else in the previous iterate which means that supposing it was here because this is the unstable manifold, it goes in that direction suppose it was here.

What does it means? It means that this fellow has to turn around and intersect so on and so forth backwards. So both the forward iterates as well as the backward iterates should undergo an infinite number of intersections. Do you understand the incredible amount of complexity that give rise to? These manifolds so far they seem to be very genuine stuff, only bend and twist this must that much but now the moment there is one intersection, there has to be an infinite complexity in the structure of these manifolds. There is no amplitude actually these are all schematically drawn. I have schematically drawn. The point is that how do we actually plot this stable and unstable manifold? As I told you take the Eigen vector, take a point on that Eigen vector and see where they map and then take many points between these two points and see where they map also and that way you can trace the whole thing. So I have schematically drawn, it might actually be like this or small amount of but whatever it is, there must be infinite number of intersection that's guarantied. Now if this fixed point, the fellow which is at the centre of this whole business is this one which means the one that was earlier stable, now it has become unstable, the one that is at the centre of the thing that is ultimately going into chaos.

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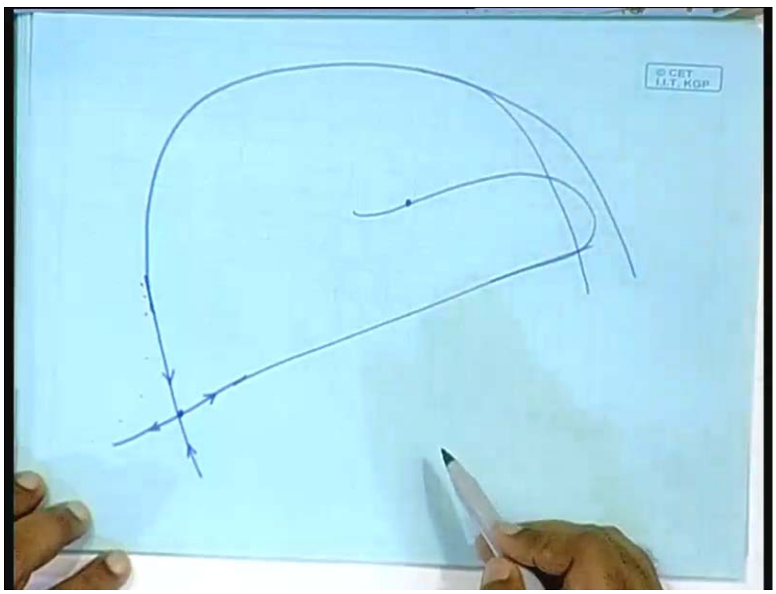

What it will lead to? The moment you have this immediately you know that there are infinite number of points. What kind of attractor can there be in which there are infinite number of points? A chaotic attractor. So if there is a homoclinic intersection notice here the homoclinic intersection has a different connotation from the homoclinic intersection that you came across in continuous time dynamical systems. Here a homoclinic intersection has very specific definite meaning that a homoclinic intersection immediately implies the existence of a chaotic orbit, existence of a snail horseshoe. Just consider a neighborhood of this and check out what happens in future iterates, you will immediately find out that has to go through snail horse shoe kind of stretching and folding. So the point is that if there is an intersection between the stable and unstable manifold that leads to chaotic orbits.

What about this fellow? The fellow who was sitting not in the orbit but on the basin boundary, there was another fellow sitting here. There was another fellow sitting here which was not on the on the orbit, it was on the basin boundary that was also a saddle and its stable and unstable manifolds can also intersect. When that happens what will be result? suppose this stable manifolds goes like this say it ultimately has go here so let me draw it at least politically correct diagram because this has to go otherwise somebody might raise the question and this fellow say it goes like this. So at some point time it may intersect. If they do what will happen? Immediately there should be an infinite number of intersections between this, means the basin boundary becomes infinitely complicated. In other words it becomes fractal, so this is the mechanism by which we have fractal basin boundaries. Let us illustrate that using a program that we have it is called dynamics, it is written by James Burok University of Maryland. Let me show from the start, so that you can see how it works.

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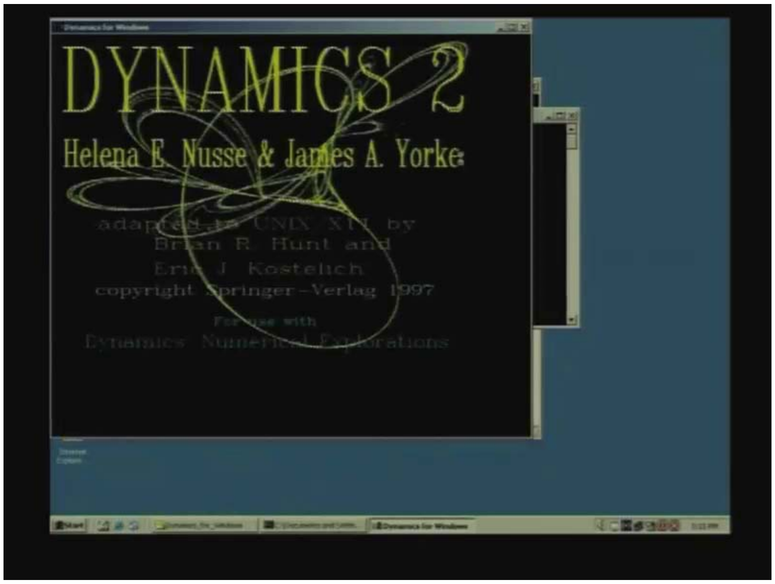

This is dynamics written by Helena E Nusse and James A Yorke and there are few menus available and if you type out the name written here, it will take that particular map. In this case you can see that h stands for the Henon map, so h is a Henon map. Here it writes Henon map. Probably it's writing too small, so you might not be able to see nevertheless. So you remember the Henon map, you open that page. How did I write it?

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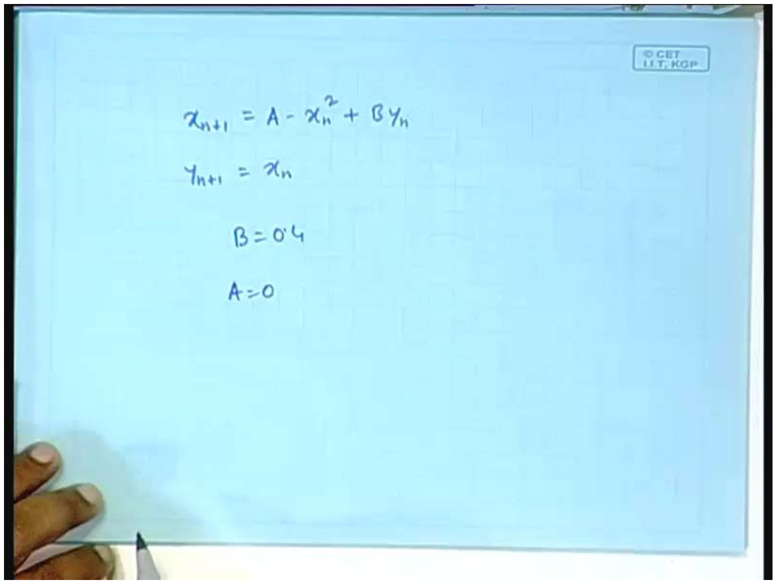

X_{n+1} is equal to A - x_n square plus B y_n and y_{n+1} is equal to x_n and we have done the exercise the last day where we had obtained the two positions of the fixed points. For what parameter value? B is equal to 0.4 but A is equal to; we calculate it for some parameter values, minus 0.09 before minus 0.09 there was no fixed point, beyond minus 0.09 say A is equal to 0, what should we conclude? What should be the behavior? Two fixed points.

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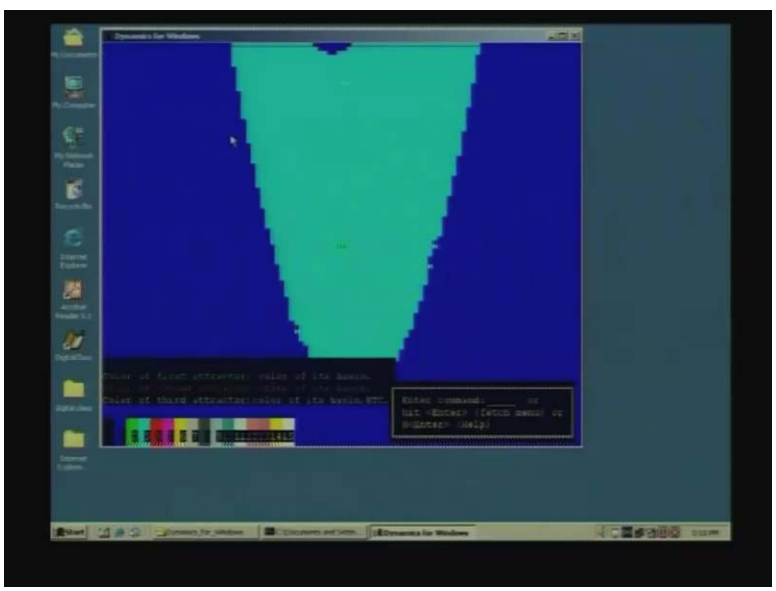

I will see the parameter menu c one, I will set it as minus zero. Now in this program the variable parameter is called row so I will set row as zero. Here it plots the basin of attraction. Now you notice I will increase the basin resolution but if I make it too high then it will be a problem, we will not be able to see. Yes, it's reasonably good. Notice that still there is a basin of attraction for this fixed point situated here and there must be another. The moment you see a basin boundary you're immediate instinct should tell there must be another fixed point sitting on the basin boundary. You knew that there was another fixed point saddle but you don't know where it is, looking at the picture you must know that okay this must be somewhere. In order to calculate where, this program has an interesting facility that is if you press II then, can you see two crosses? One cross is here small cross, a big cross is here. So actually my position is there and it is automatically showing where is my next iterate, it's called the image of that point.

If I now move this point, see the other one is moving, can you see? So that they are approaching each other, so I know that there is a fixed point sitting quite there. Can you see? When they coincide you know that the fixed point is there. That is a nice way of graphically calculating the fixed point. Now one fixed point is there I know, where is the other fixed point? It must be somewhere on the basin boundary and both the fixed points are on the 45 degree line. So logically I expect it somewhere here right on the 45 degree line, so let's go there and I go down further this way, further down, yes we are close. So we had calculated and found that is there. So I'll initialize it here, now I can go close to the point, it is here.

What I did was after coming close to that point, one can actually look at the point by a simple Newton reference solver. So I gave that as the initial guess and the Newton reference solution gave me the position of the fixed point. This quasi Newton method of 10 steps, if you write the quasi Newton steps, it also gives the Eigen values. You can see one Eigen value is one point four seven which is an unstable direction and this is minus 0.27. That's stable direction and the Eigen vectors are also given. You can easily calculate it that way. Now we are here, we had actually guessed our position, so we traverse this way but nevertheless we are here. At this point I want to calculate the unstable manifold to the right? If you want that will go here. So unstable manifold to the right ur.

See it has started from here and it has gone exactly to the fixed point that confirms our initial theoretical structure that is must go actually to this point. The one to the left should go to infinity and this is actually the stable manifold and stable manifold turns and goes this way. What was the parameter value at which this fixed point became unstable? 0.27, so let us choose a parameter value slightly bigger than that say 0.28. So row 0.28. Can you see now there are two points. In order to convince ourselves that these are really the two points, let us start from this initial condition and let me change the color. Now these points probably are not visible. I started from this point, first iterate was here, the second iterate was here, third iterate was here, fourth iterate was here. It was actually going to this fixed point and then going this way and finally it merged on to this. So here are two fixed points it has now become period two orbit. If we increase a parameter further it was 0.28 let us increase it to 0.3, still period two. Can you see these two points? No, let me reduce the basin resolution. Now can you see? So here are two points, quite clear. By the way how is this program plotting this basin of attraction?

You can easily write this program, what it is doing is it has taken a box on the state space, starts from one point in the box, goes on iterating it and if it exceeds a certain circle of radius, there some pre assigned radius, a very large radius in some iterates it says that is actually going to infinity. Though it is true that you might question how do you know that it will not come back and stuff liked that but nevertheless it is reasonable estimate to say that beyond a certain number of iterates, if you find that it has gone beyond certain big circle it is going to infinity. So if that happens, that particular initial condition is given a color. Go to the next one, go to the next one and come here, it is actually remaining fixed, remaining stable so it is given a different color that and where it ultimately stabilizes those things are also plotted with a different color. You can easily write this program, it’s not a big deal.

Now let us increase the parameter further say 0.4, still its period two. Row 0.5, still its period two. You can find out where the period two orbit will become unstable but the calculation will become bit untidy. So let us do it by this way, 0.6 still period two. Now let us do a relatively quicker thing. Draw the bifurcation diagram so that from there I’ll know what is the parameter necessary in order to get something more. So BIFR is the bifurcation range, we set from 0 to 2, large range.

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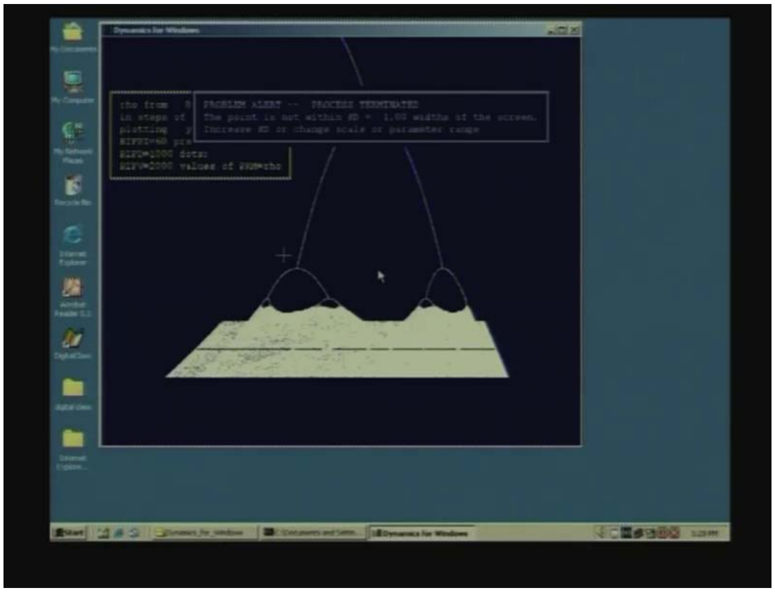

Is it visible now? Anyway here that it has gone into chaos is more or less visible so I can see that the range has to be less 0 to 1.5. Probably it is white on block that is why you are in problem. Is it okay let’s see. Is it better visible now? No, let me increase the resolution. Bifr, no bifv. Bifurcation values and probably these points are now better visible. Is it? Yes. Now you can see that it remain period one till some value and then it came here. At this point the period two orbit became unstable. So in order to get a period three orbit, this whole range is zero to 1.5. I guess I need to go to about one, not less than that, so row one. So now you have four pieces already, four points at 0.9 beyond that what you expect? Beyond that it will go through very quick period doublings and very soon it will land into chaos. So if you see first period doubling, second period doubling and then chaos but notice what is what it is situation here? It should be say you are choosing a parameter somewhere here, it should have one piece, two piece, three piece and four piece chaotic orbit.

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So if I choose the row somewhere one, it left four pieces of chaotic orbit but what was our conclusion then? All these four pieces must be on the unstable manifold, all the fixed point that must be sitting here. Let me check. So these are the four pieces. Let's check, if they are really sitting on the unstable manifolds because we always need to convince ourselves that we are really on the right track. So I have pressed i i which means that I expect the fixed point to be somewhere here because it has to go on the 45 degree line, So it goes and finally see they are coming close. I have to go little up, I think yes we are reasonably close, very close. It has found the Eigen values as minus 1.7 and 0.23, it is a flip saddle. So we are here. Now I want to plot, which one do you want to plot? The unstable manifold to the left or the right?

Let me plot the one that is to the left UL. See starting from here, the unstable manifold has gone this way, gone this way, turned around, gone this way and as a result it does contain this piece and that piece and if you draw the one that goes to the right, you will find that. We can also plot the one that is to the right but you can easily guess that it goes this way and you can also guess that as you increase the parameter further there must be a situation where it just intersects rectangle. It has come very close, it is now a chaotic orbit you can see that, if you want to plot exactly the chaotic orbit looks like this and this is the basin boundary means the fixed point must be sitting somewhere here and this is the stable manifold. It has gone round and it is going like this and here is a very close situation which is close to the…What will happen when they intersect?

Some parts of this orbit will fall in the basins of attraction of the attractor at infinity. When this fellow goes this way what will happen? Some part of the basin boundary will fall there. Let see what happens? Very close, almost touching. Next, now row is 1.2, I will make it say 1.25 see what happens? No attractor. So there was an attractor so far, suddenly it vanished. Why did it vanish? Because a part of the attractor intersected that and it went into the basin of attraction of the attractor at infinity. So every initial condition ultimately goes to the infinity and therefore the attractor that was so fast stable, suddenly loses stability. This in stability notice that it's not the instability related to an Eigen value.

You cannot credit this instability simply by looking at the immediate neighborhood of any initial condition any fixed point. This instability cannot be inferred from any linear system stability but it still become unstable. This is a completely new class of instability that is called a crisis. This is called boundary crisis where the attractor intersected the basin boundary, attractor itself intersected the basin boundary, it is called a boundary crisis. So this is one of the mechanism by which a system may collapse and it's the handy work of the stable and unstable manifolds, it's just that they intersected. Is the concept of boundary crisis clear?

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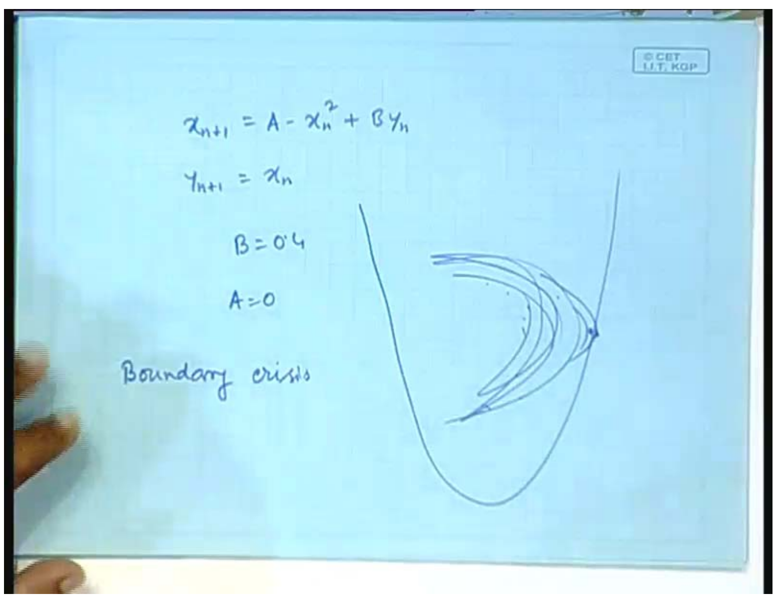

Boundary crisis is created when the attractor itself, a part of the attractor goes in the basin of attraction of another attractor but notice here the situation was that there was an attractor something like this and there was a basin boundary going like this and it intersected. only this part has now intersected and gone to this side which means that while doing the iterates on this, if an initial condition somehow falls in this part then only it will go to infinity. Otherwise it will keep on oscillating here, as a result in a system that has under gone a boundary crisis you can expect a very long transient because only when an iterate falls here then only it will go to infinity else it will go on.

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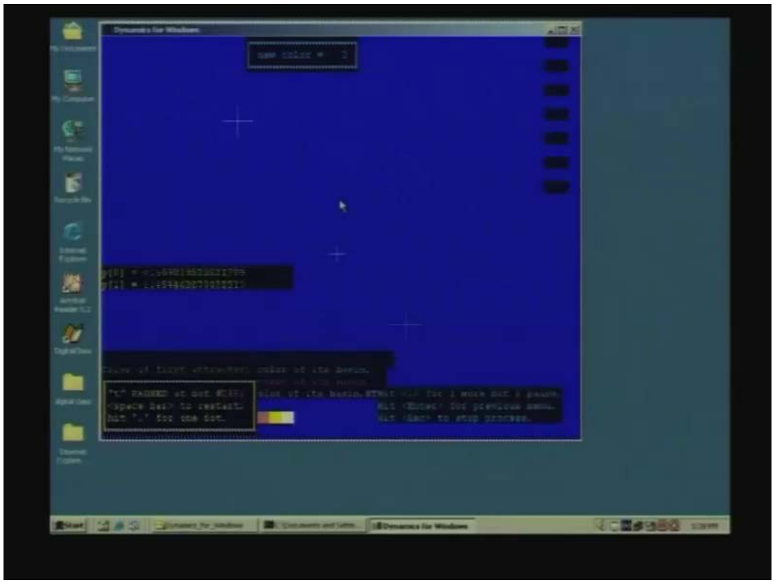

Let's start from some initial condition in this situation say here. Look at the big cross, I am doing iterates it is very much there inside, it is not going out. I am giving the iterates, no problem it's still there. It will vanish the moment one point goes here but before that it will still remain very much stable. Let me change the color so that is clearly visible. See now I am doing it fast, now it has suddenly gone out. So it has actually gone out in 296 iterate and there are situations where the transient may pass it from millions of iterate. for a million iterate you will find nothing is happening it's still there but then a particular iterate falls in a particular point and goes to infinity that is possible. So this is an instability that can happen only in non linear dynamical systems and in situation like this.

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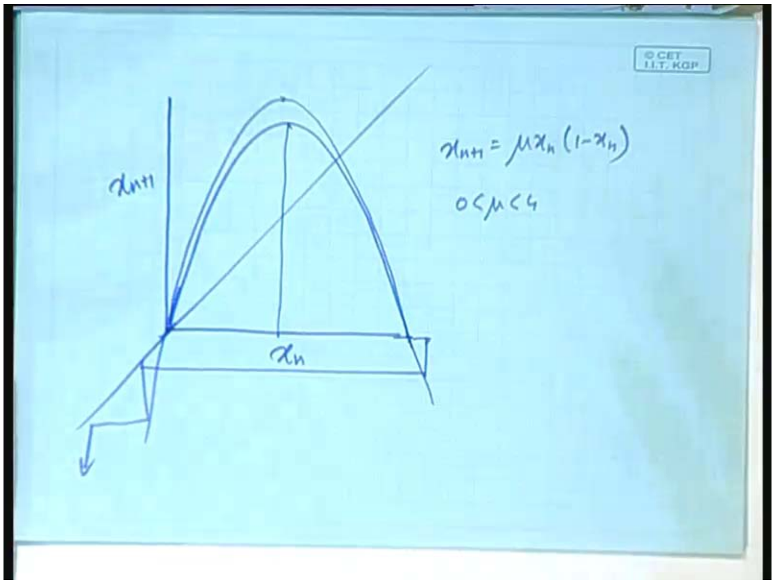

Let us recall the situation that we had in the logistic map, x_n versus x_{n+1}. We had done it x_{n+1} is equal to mu x_n (1 - x_n), this was a map and we had gone mu, why four? If you put 4 here, where does the middle point map to? 4 0.5 0.5, you get 1 which means this point maps to this point. If mu is greater than 4 it will be larger, as a result this point will map to a point that is farther this way and as a result of which in the next iterate map here and the next iterate it will go the 45 degree line which is here. So what actually happened in this system also? Though at the time of talking about it i did not make it clear but there is a basin of attraction of the attractor, wherever the attractor is the basin of attraction is region between zero and one. If you start from an initial condition that is outside this region, it goes to minus infinity. Now it so happens that beyond this particular value of mu, a part of the chaotic attractor especially the point that is at the top that goes, that maps to a point that is outside this range and therefore it goes to infinity. So this is also a boundary crisis.

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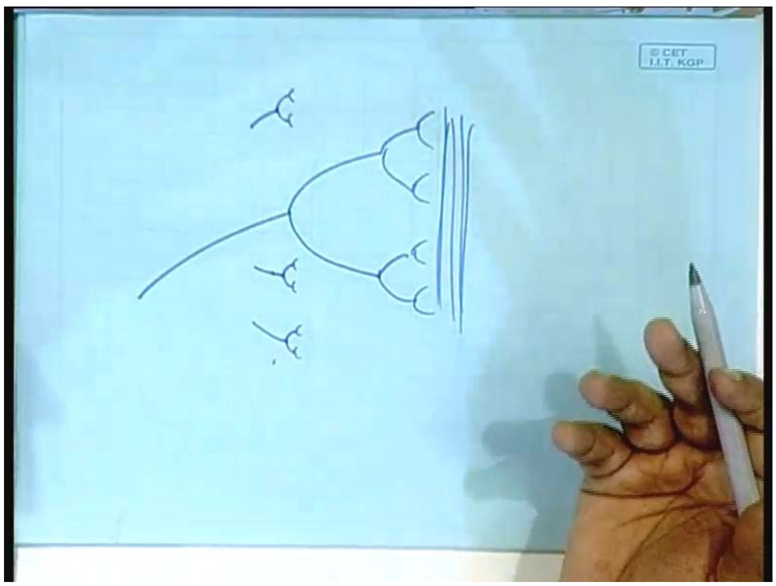

So what we have talked about here in this case is known as the… Often you will find that while if you draw a bifurcation diagram of a system, it goes on like this and finally it goes to chaos but why it does so a coexisting periodic orbit may be born. For example suppose a coexisting period three orbit was born. How can it be happen? Through a saddle node. Now as you change the parameter further this will also undergo period doublings and stuff like that. These are pretty common situation where you have a big bifurcation structure along with it's some smaller once, coexisting attractors coming into existence but also going an out of existence. They are there in this parameter range, they should have their own basins of attraction. At some parameter value the basin of attraction of this one may touch this attractor. As a result of which what will happen? Any initial condition that is here will ultimately go to this attractor. As a result of which this attractor will vanish that is also another boundary crisis. So boundary crisis are the mechanism by which a system ultimately loses stability globally. Now here we are not talking about the local stability, here we are talking about global stability. We will continue with that in the next class.