Chaos Fractals and Dynamical System
Prof. S. Banerjee
Department of Electrical Engineering
Indian Institute of Technology, Kharagpur
Lecture No. # 02
Vector Fields of Nonlinear Systems

In the last class, we had talked about the essential method of solving differential equations and we said that even if we forget the exact method, you cannot exactly solve equation. Fine. But nevertheless the qualitative character of the orbit in the state space should be understood. For that, the route that we have taken was that in the whole state space which could be normally non-linear, we identified the points that are equilibrium points. These are given by x dot = 0 where x is the state vector.

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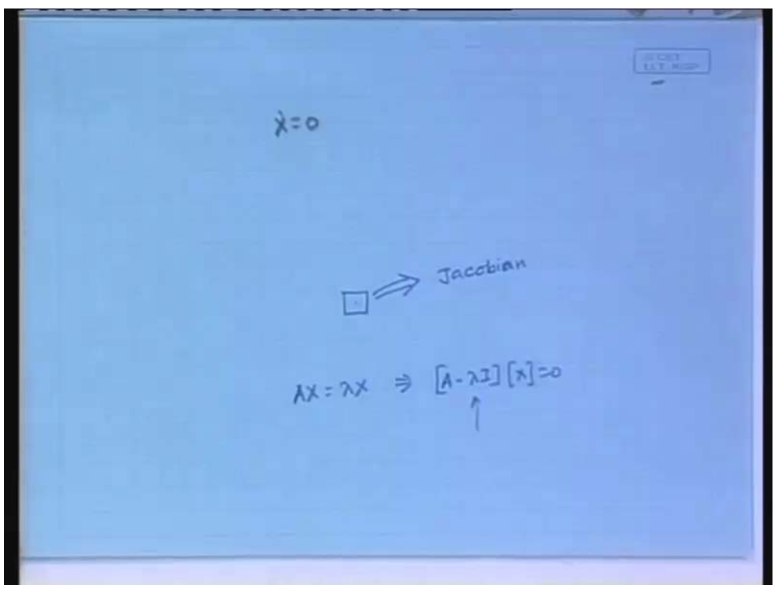

Solving the right hand side we had obtain there and from there we said that fine we are now identified that this is equilibrium point and we will look at only the local neighborhood of that and in that neighborhood, we can locally linearize. It so we used the Jacobian to linearize that. So here the Jacobian would be representing the local linear behavior. So we obtain a linear set of differential equations and then if we concentrate on that neighborhood, the essential method of solving the equation would be to identify the eigenvalues and eigenvectors. We are also saying that since the eigenvalues are obtained by an equation of this form, A times x is lambda x which yielded |(A- lambda I)X| = 0. That yielded the eigenvalues

Now in that I said that the Eigenvalues then would be the result of a quadratic equation in a two dimensional system, a cubic equation in a three dimensional system, a fourth order equation in a four dimensional system and so on and so forth. If you take the simplest quadratic equation that has various possible types of Eigenvalues, the Eigenvalues could be negative, positive, real complex all these possibility exists. In each case we wanted to understand the qualitative character of the orbit. The exact solution I said that you study from the differential equation text books. I assume that you have already come knowing that if you have forgotten that just read any differential equation text books I am not going into that but for our purpose it will be necessary to understand the qualitative character of the orbit in this state space. In doing that we had come to two cases. One if the Eigenvalues are real and negative, if the eigenvalues are real and positive and eigenvalues are real one positive one negative. These are the three cases we tackled.

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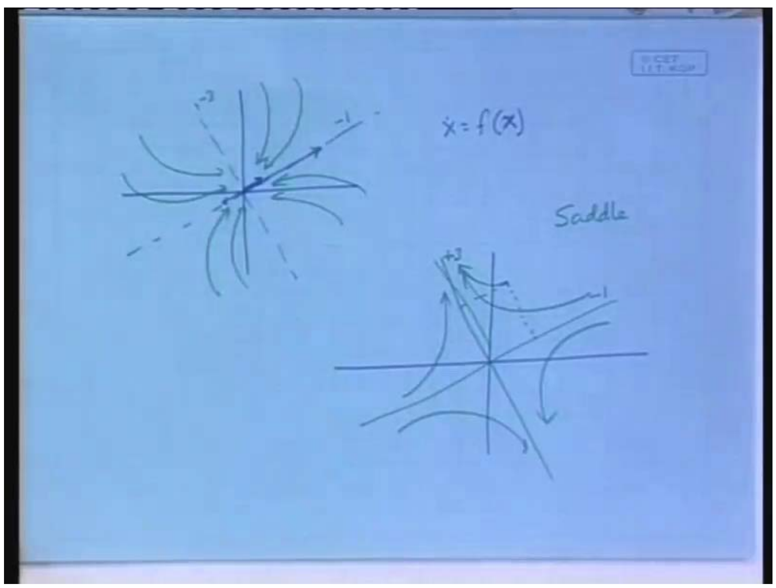

Just to briefly recapitulate what we said, in the state space there will be certain direction; the Eigen directions and if you take a vector along that direction, then your x dot = f(x). That was the the equation that we started from. So this would be a linear equation in the right-hand side. If this is the Eigen direction, then x dot will be in the same direction and this vector will be this vector that means x vector times the eigenvalue will be the x dot vector. If the eigenvalue is positive then the x dot vector will point in the same direction as the x vector. As a result x will keep on increasing in this direction. It will go on. That represents an unstable system. If it is a negative eigenvalue, then the x dot vector will point to the opposite direction and therefore the distance from the origin will exponentially reduce and ultimately it will converge on to the equilibrium point. That’s what you said. following that argument we concluded that if there are two Eigen directions in this system, then if the eigenvalues are negative, say -1 and -3 in both the directions, then the the behavior would this (Refer Slide Time: 06:06). Thus we drew what is known as the vector field. The character of the vector field will be like this.

If we have the eigenvalues positive that means +1 and +3, it will be just the opposite. Arrow direction will be opposite but the the character will be same. It will be unstable equilibrium point and then we went on to consider the case where the eigenvalues are one positive another negative. That means again let us draw. Suppose this is one eigendirection and this is another Eigen direction and suppose here it is -1. Here it is +3. Then the behavior would be again starting from any initial condition. The essential point is that you are decomposing this vector into the components along the Eigen directions. So here is one component. Here is another component and this component will reduce this component and this component will increase. So this fellow will go like this. similarly any initial condition here will go like that, any initial condition here will go like that, any initial condition here will go like that and that is the character of the vector field here (Refer Slide Time: 07:22). We say that that goes by the name of a saddle. So we start from there. Remember this saddle point is very important for our future discussions. So the saddle equilibrium points will have to be understood quite well. Apart from these, what other behaviors can there be? What are the types of eigenvalues?

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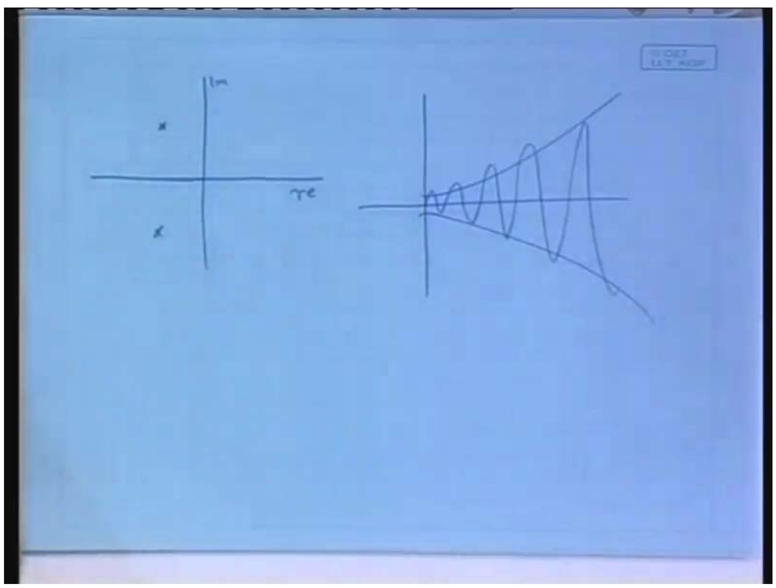

Well if you draw the complex plane, real and imaginary, then we had considered this line real negative we had consider this line real positive and now suppose there are two eigenvalues here and here. Of course you know that eigenvalues if they are complex they also always have to be complex conjugate. They cannot be just one here another here it will always at the complex conjugate only. If they are here then what will be the character again we will consider the two components two parts this is the real part and this is the imaginary part. Real part will cause a e to the power lambda t type of behavior. So in this lambda real part of the lambda is negative then it will be a e to the power minus something t it will result in the decay.

Decay of what decay of whatever is the result of imaginary parts. Now imaginary part will result in e to the power j omega t kind of behavior which is nothing but a sinusoidal behavior. So we conclude that if it is purely imaginary then you have a pure sinusoidal behavior. If it is imaginary with a negative real part, then it will be a sinusoid but damped sinusoid. It will decay. And if it is a complex with positive real part it will be like this (Refer Slide Time: 09:47). This will be an exponential envelope. This is all we need to know really as far as the linear neighborhood is concerned. By the way, the frequency of this oscillation depends on what? The imaginary part. So the undamped component depends on the imaginary part damped component will be slightly changed. Depending on this can we now work out a problem?

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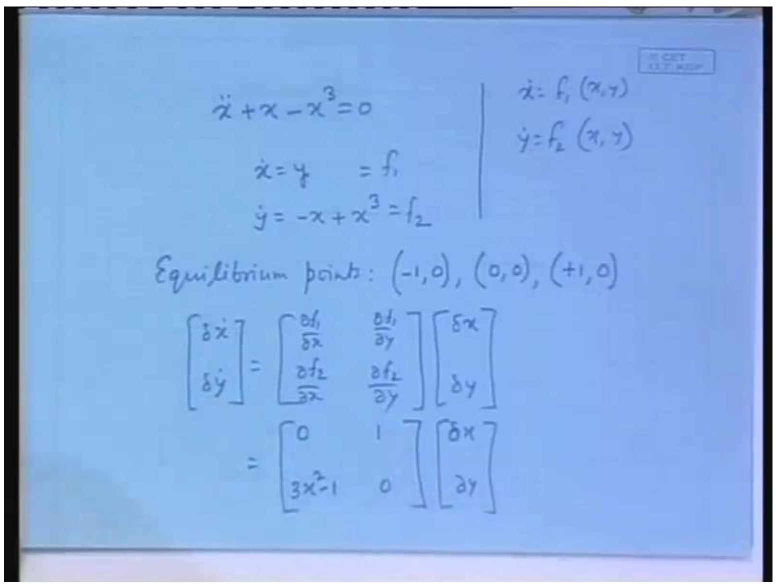

Let’s see suppose there is a there is a system whose differential equations are x double dot + x – x^3 = 0. How will its behavior be? I want to deduce that by looking at the local neighborhoods of the equilibriums points. So in order to approach this problem. How will I deduce the equilibrium points? Remember the first step is to obtain first order differential equations of this. So first order differential equation in order to obtain it just say x dot = y and then y dot is equal to, bring this to the left-hand side. y dot is - x + x^3. So this is the state of differential equations. Now we need to obtain the equilibrium points. equilibrium points are where this x dot is set to zero, y dot is set to zero x dot is set to zero gives y is equal to zero and y dot set to zero gives x^3 – x = 0. There are three solutions. What are they? So the equilibrium points are: (-1, 0), (0, 0) and (+1, 0). The next step would be is to try to understand what is the character of the vector field in the neighborhood of each of this equilibrium points. Can you deduce that? Quiet easily.

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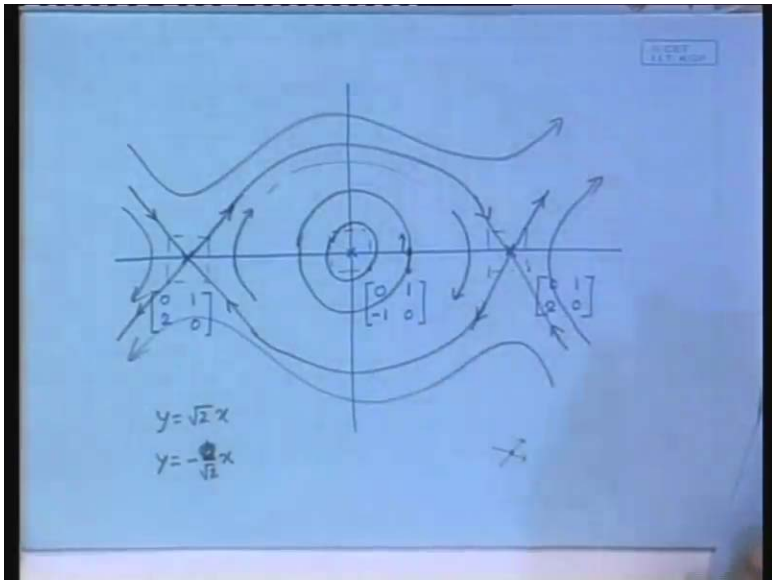

So now let me draw. Here is the whole of the state space. Here is the point (0, 0).here is the point (-1, 0) and here is the point (+1, 0). The next step is to look at the local neighborhood of this point this point and this point (Refer Slide Time: 13:19). How will you do that? We will say that now we are considering the deviation from the equilibrium point. So deviation of delta x dot and delta y dot will be given as a Jacobian matrix delta x delta y. so here is your f1 and here is your f2. So what is this particular component? They are 0, 1, 3 x^2 – 1, 0 (Refer Slide Time: 14:31).

Now we need to evaluate this. Not just anywhere but at the three equilibrium points. So these three equilibrium points are (-1, 0), (0,0) & (+1,0). Here it is x dot. In the right hand side just you asking what function of x and y and that function is not a function of x. it happens to be just to function of y. so if you if this is this is the f one in general you write that x dot is equal to f one of xy and y dot is another function of xy. so this one is answering the question: how does x change with time ‘x dot’ and that is given by this specific position in the state space x and y. now it’s so happens for this particular system that x dot does not depend on x but depends on y. so it’s become a zero. At these three locations we will need to substitute minus one zero and one in place of the x values to obtain three different Jacobian matrixes at these three locations. So for this (Refer Slide Time: 16:43) particular location you have (0, 1, 2, 0), at this location what we have is (0, 1, -1, 0) and at this location, we have (0, 1, 2, 0) (Refer Slide Time: 17:00). Now the next step is to obtain the Eigenvalue and the Eigenvectors. Let us start with this particular point where the location is (-1, 0). This is the Eigenvalue. This is the Jacobian matrix. What do you get as the Eigenvalues? Yes +/- root two. So what kind of equilibrium point is it? It’s a saddle. a saddle means I have to identify the Eigen directions and then I can now put arrows. Can you find out the Eigen directions? That means which is the eigenvector associated with the eigenvalue plus root two and which is the eigenvector associated with the minus root two eigenvalue.

All of you remember how to calculate eigenvectors and eigenvalues. So in general I will draw them as this but you have to calculate and tell me what the slopes are. y = root 2 x. the eigenvector associated with the + root two eigenvalue is y is equal to root 2 x. that's what you said. now y is equal to root two x plus root two eigenvalue means it is a outgoing direction y is equal to root two x means it is in this quadrant (Refer Slide Time: 19:53). The other eigenvector is y is equal to –root 2 x. So what is the angle? y is minus something x, so it is this (Refer Slide Time: 20:48) vector and then it is associated with - root 2 eigenvalue. So these are the incoming eigenvectors. What will happen here since the Jacobian matrix is the same so it is not necessary to calculate it all over again we can safely draw the directions and put the similar arrows. Now let us come to the middle one.

What are the eigenvalues? These are purely imaginary eigenvalues. plus minus j. there is no fun in calculating the eigenvectors because they will not be real eigenvectors. Eigenvectors will also come to the complex conjugate eigenvectors. They will be useful in obtaining the actual solution. So if you want to obtain the actual solution in writing, you need that. But here today we are not concentrating on that. We are concentrating on developing an idea. What the orbit looks like and for that it will suffice for us to know that if the the eigenvalues are purely imaginary, there is no contraction. There is no dissipation. The sinusoid will not die down and so in this state space, how will it look like? So they will look circles. Suppose you start from this initial condition. You will get a circle. Suppose I start from this initial condition. You will get another circle so on and so forth. But you cannot really carry on this argument at infinite term because this argument holds in the close neighborhood of this one. It will be circle only in the close neighborhood. as you go far and far, it will no longer remain circles but then I will ask you, with this much information obtained out of looking at the local linear neighborhood, can we infer what will the character of the vector field be in the rest of this state space? This much we have been able to obtain simply by calculating the eigenvalue and eigenvector. Well on that there are two ideas about vectors fields which will be useful one vectors field behaves something like magnetic lines of force. In the sense that they don't ever intersect. Do they intersect? They do at the poles but elsewhere they don't. You cannot have a magnetic line of force intersecting. You cannot have the vector fields also intersecting. What does it mean? Suppose at this point if there are two vector field lines intersecting then if the initial condition is here, it will not know where to go.

The moment you have defined the differential equation, at every point there is a unique vector means they cannot intersect. There is another idea that is unless there is some very strong reason to believe so; the vector fields will be smooth. That means they will not suddenly twist and turn everywhere. Let's see the application. We have seen that here this will be rotational. Can you tell me whether it will be clockwise or anti-clockwise. The equations don't say that. The equations don't say that. Eigenvalues don't say whether it is a clockwise or anti clockwise. They only say that it is a rotational. You can have perfectly sinusoid with both clockwise rotation as well as anti-clockwise rotation. So what will be the direction of rotation? If you say clockwise, justify why clockwise because as you go far and far it should smoothly go into this one and here you can see a movement like this and that gives a rotation. There is another way to infer whether it is clockwise or anti clockwise. Just choose the point somewhere here. This is a point with x coordinate existing and y coordinate zero. Put it into the original set of equation. x coordinate positive y coordinate zero. You can now obtain what is the vector. say x coordinate is point one and y coordinate zero x dot is zero and x coordinate is point one you can calculate this you will get this as a negative number because of this. Which means that here the vector will be pointing like this (Refer Slide Time: 26:40). Once you do that you don't really have to do this exercise for every point because of that theorem that they do not turn and twice. They more or less smoothly change.

So you cannot have just one vector here pointing this way, another vector here pointing that way. No. it's not possible. So combining these two ideas is it now possible to sort of intuitively construct what can be the behavior of the system. In this example here it is bound to be like this. Now you can see that there is one eigenvector's deviation going. Where will it go? it will come back on to this and this one will come back on to that. In between you will have one type of behavior. Towards the center it will be more or less perfectly circular which means the time domain behavior would be almost perfectly sinusoidal as the oscillation increases, the amplitude oscillation increases. It will be distorted sinusoid. Now as you go further and further, if you across this point, it will go to infinity. Starting for any initial condition here, it will go to infinity starting for any initial condition. There is no other way.

Can you see that? Starting from here there is no other way. It has to go like this and this has no way that it can enter this whirlpool. So see starting from the only view of the local linear neighborhoods. We are really solving any equation. It was possible to infer things of interest about the overall character of the vector field. What ideas did you really make use of? We really made use of the idea that we can identify the behavior in these local linear neighborhoods by simply obtaining the Jacobian matrix and its eigenvalues and eigenvectors. we will then draw the local vector fields and then extend it with those two ideas that they cannot sharply turn that means their changes have to be smooth changes and two vector fields line cannot cross except the way you had the magnetic field lines crossing at the poles, they can also cross at the equilibrium points where actually the magnitude of the vector is zero. At those points they may intersect. So you see in the whole of the control theory you take a look only at these points. You know that here there is a system whose equilibrium point is like this. Its behavior is like this. If we have a small perturbation it will it will behave sinusoidally. We know that and that is what the whole control theory will be based on and in that small neighborhood you can apply all the techniques that you have learnt in Contour theory. Laplace transforms root-locus and stuff like that. By looking at the local linear neighborhood we have no idea that they are actually very very different kinds of dynamics possible in this system.

What different kinds? For example starting within this island it will always be oscillatory behavior. Anywhere outside this island it will collapse. It will go to infinity. Start from here it will run to infinity that means the systems is will become unstable the that means that here the system can be can have a stable oscillatory behavior only within a certain range. These lines separate two different types of behaviors of the system. Same system may collapse. Same system may nicely work like an oscillator. And if there is some way there is some exertion due to noise, perturbation or somebody hitting with a hammer, etc., the system will collapse. The overall picture of the vector fields tells you that. So are you now comfortable with idea of vector field?

So the idea is that if you look at the vector field, you can have a clear idea what can happen to the system. Often for that it is not necessary to actually solve the differential equation. In general you will be able to solve the differential equations only in this closed neighborhood. Elsewhere you will not be able to solve because they are nonlinear differential equation. They will have to be solvable only by numerical values. With this amount of ideas, let us simply work out. I am giving you the task of working out the simple pendulum now.

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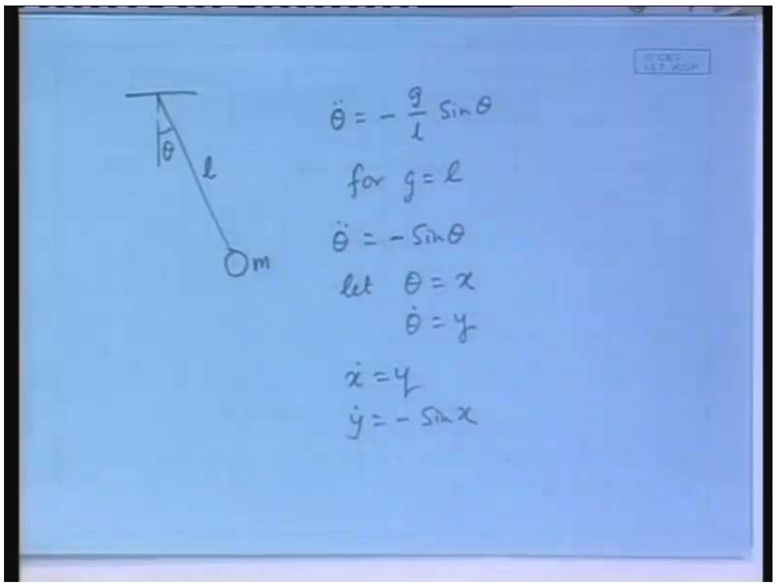

First obtain its dynamical equation in terms of theta. Of course you will not do it in xyz coordinates. That will be foolish. Here is a length ‘l’; here is a mass ‘m’. How will you proceed? Also assume a bit of air friction. That will make it deviate slightly from what you did in school. Can you write down the differential equations? If there is no air friction, it will be theta double dot = - (g/l) sin theta. The only problem that you did in school was that sin theta was approximated by theta. Let us assume there is no friction. Let’s start with this. Theta double dot is –(g/l) sin theta. now let us make your life even simpler. you can always have a pendulum 9.8 meters of length. –(g/l) will be unity. So for g = l, you have theta double dot = - sin theta. Of course we will need to write these in terms of first order differential equations. So we will say let theta = x and theta dot = y. then the equations are x dot = y and y dot = – sin x. I don’t want to put sin x = x as yet. Now what are the equilibrium points? There will be an infinite number of equilibrium points. y = 0 of course. But we will assume sin x = 0 value for 0, pi, 2 pi, 4 pi so on and so forth. So there will be infinite number of equilibrium points. But let’s draw a few of them.

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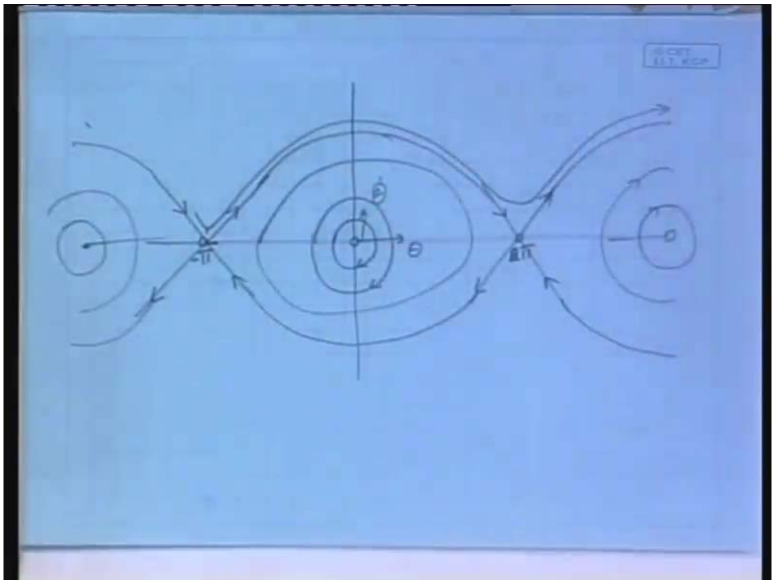

This is zero. This is say pi here it is - pi (Refer Slide Time: 37:32). Imagine the pendulum zero is hanging downwards. Pi is up. - pi is in the opposite direction. So + pi and - pi are really the same and zero and twice pi are also the same. Also it is essentially the same thing carried over and over but mathematically it yields an infinite number of equilibriums. But for our purpose, it will suffice to keep ourselves in this range. What are these equilibrium points? For example, what is this equilibrium point? You will have to then locally linearize it, obtain the eigenvalues and the eigenvectors and then only tell. You cannot really just say like that. You have to say after having calculated that. So what is this equilibrium point? When I say what is this equilibrium point, I guess I am expecting names. While doing this exposition, I have given only one name. So I need to give more names. If both the eigenvalues are negative, then you know that all vectors fields converge. Then that is more or less like a sink. If you open the sink in your bath tub, every waters goes in. if eigenvalues are both positive then it's a source. So two names. If both the eigenvalues are negative, it is a sink. If both eigenvalues are positive, it's a source. If one eigenvalue is on the negative side, it is a saddle.

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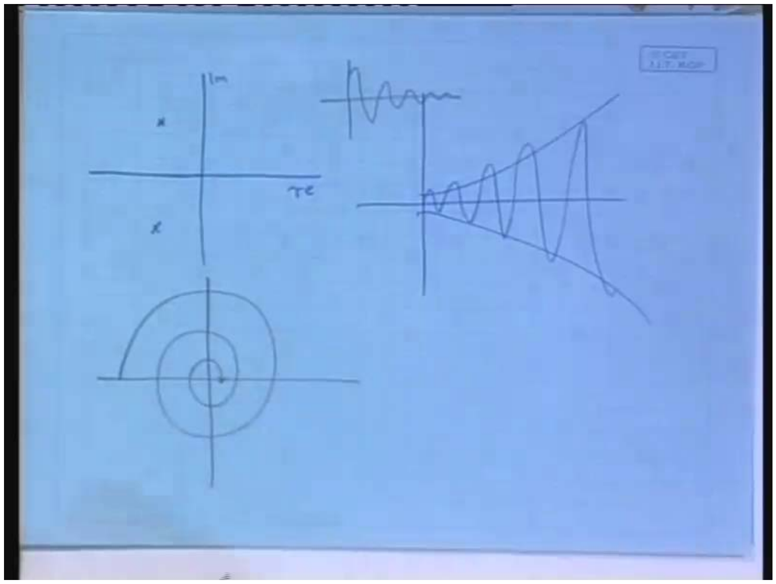

If you have complex conjugative negative real part in the state space, the behavior would be of what type? In the time domain it will be like this. In the state space it will an incoming spiral. it will be a spiral sink. Similarly this type where the eigenvalues are having a real positive real parts it will be spiral source. So these are the names. Based on these names call them. What are they? What is it? In this equation you will have to take Jacobian. You will have to evaluate and put it. You get what here +/- 1. So it will have 45 degree lines as the eigenvectors again. Almost the same kind of behavior as we have just shown. And here, it will be a circular behavior. Now we can easily put the arrows here because we know that it has to go like this. So again for the pendulum, the behavior would be something like this. The reason I did this is notice that its behavior is more or less the same as the one that we just evaluated. Only thing is that this pattern is repeated at infinity. That means this fellow goes and this fellow goes and here there will be again a circular behavior. Again sometime here, there will be another circular behavior so on and so forth. It will go on in this state space.

But now can you relate this with the physical behavior of the pendulum? Normally you just move it and oscillate it. Where do you see it on the vector field? Around (0, 0) it goes on oscillating. So an oscillatory orbit visually what is oscillatory will be seen as a closed loop in the vector field. In the state space, it is just a closed loop. So after sometime it becomes sort of a distorted closed loop. What happens if I start here or what happens if I start slightly up? You notice that there is a specialty here. if you are exactly here you don’t know where to go but if you are slightly away from it in this direction, it will still go on rotating because see that from the vector field if you are slightly off this way, then it will have to go like this. It cannot ever do anything else. Physically what is happing here? Now what are these? Here is a theta direction and here is your theta dot direction. This is - pi up there. Theta direction means slightly off and at zero velocity. If it start what will happen? It will still be oscillatory behavior.

But give a velocity push, it will go on rotating and the rotation rotational motion will translate into the state space as not a rotational motion but as a motion like this (Refer Slide Time: 45:48). So I will be getting a field for the nonlinear behavior of such a simple system. In school, we had only taken a close look at this point where you approximated sin x by x. essentially that will be local linearization. so by locally linearizing you have taken look at that but over all behavior of this simple pendulum is not that simple and that becomes clear only when you take this sinusoidal account and work out the whole behavior of the state spaces. You don't have much time left so let's talk about something else. this might give you the impression that this method that I was talking about; the method was identify the equilibrium points, work out the behavior at equilibrium points and then join the vector fields and go by a sort of common sense. That will suffice in understanding any nonlinear dynamical system. It will give that kind of impression. Well for the next five minutes, I have to dispel that myth but you cannot really do that. That works only in some rather simple systems. In the main run of non-linear systems, you cannot deduce the non linear behavior by looking at the linear behaviors. We are looking at the linear behavior specifically. Here is a linear behavior, here there is a linear behavior, here there is a linear behavior, join them and you get the whole thing. The whole is not always just the summation of the parts. More things can happen. Well in order to understand that let us look at a situation. I will write down the equations there. Then it will be easier to understand that.

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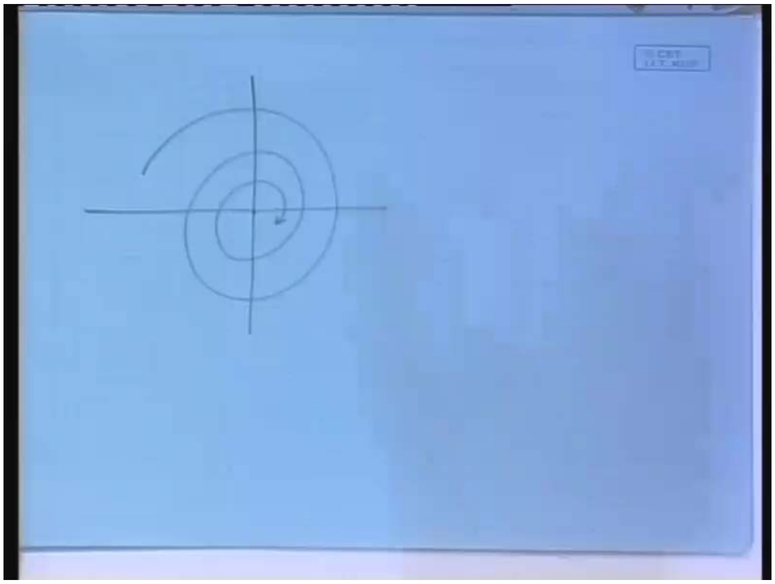

Suppose there is a system whose vector field is like this (Refer Slide Time: 48:19). What will you say about the equilibrium point? It's an incoming spiral behavior. The equilibrium point will be called a spiral sink. Suppose there is a system with a spiral sink type of behavior. Any system has some parameters. For example electrical circuit has parameters like the inductance value, the capacitance value, the voltage applied, the resistance.

If you have a rheostat, changing it means you are changing its parameters. Similarly you have the pendulum with in which the length of the chord is a parameter. Air friction is a parameter. By the way if we add air friction, what will happens to this fellow?

All these things no longer remain the same. They will become an incoming spiral. Try to construct it by drawing in coming spiral in these places. You will notice that you can no longer bring this to here. It will come inwards.

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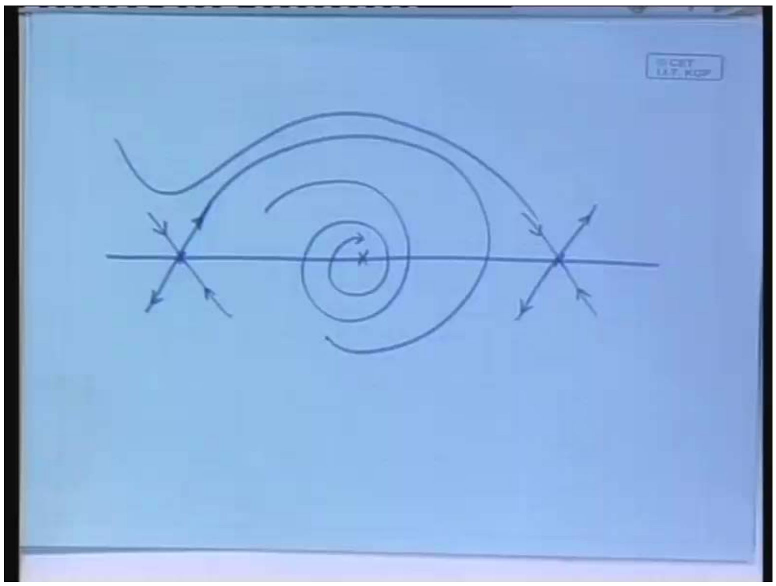

So before we go on to the other things, let us just just illustrate how we will it look. If we have a pendulum with friction, suppose these are the three equilibrium points and here I know that the behavior is like so and here I know the behavior is a saddle with Eigenvectors going like this, Eigenvectors going like that and therefore this one will go like this. It will not go there. But from here it will go here (Refer Slide Time: 50:27). So that if you start from that kind of a position which I said, “up but with a push in the velocity”. Then also it will not simply go on rotating but after sometime it will enter and that’s it. So this becomes clear the moment. You try to construct it by hand, you understand the velocity for the pendulum, physically the system is simple but for more complicated system this gives a lot of insight. This kind of approach of drawing the vector field and in general it is possible to draw the vector fields for a two dimensional system. If you are using “Mathematica” or Maple one MATLAB, all these programs allow you to draw the vector fields. There are comments by which if you give the system equation, it will draw the vector field for you but that vector will not look like this.

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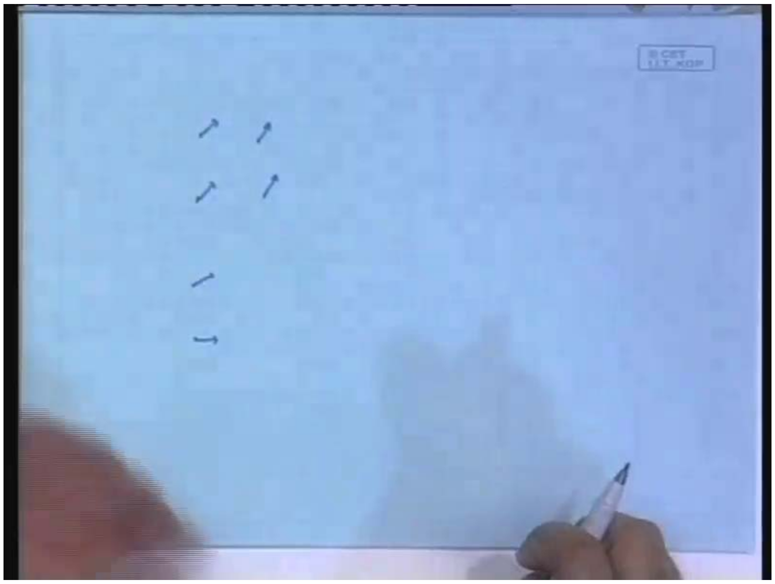

That vector field will look like a series of points and a field of arrows but from there, it is more or less clear how the behavior is. It will be instructive. Therefore for you to learn those comments in MATLAB or Maple or Mathematica which ever program you are comfortable with, for this course, you will have to write programs in one of these high end software programming languages. But I will not command which one you will use and you can use anything that you are comfortable with. We were talking about the situation where you have an incoming spiral orbit.

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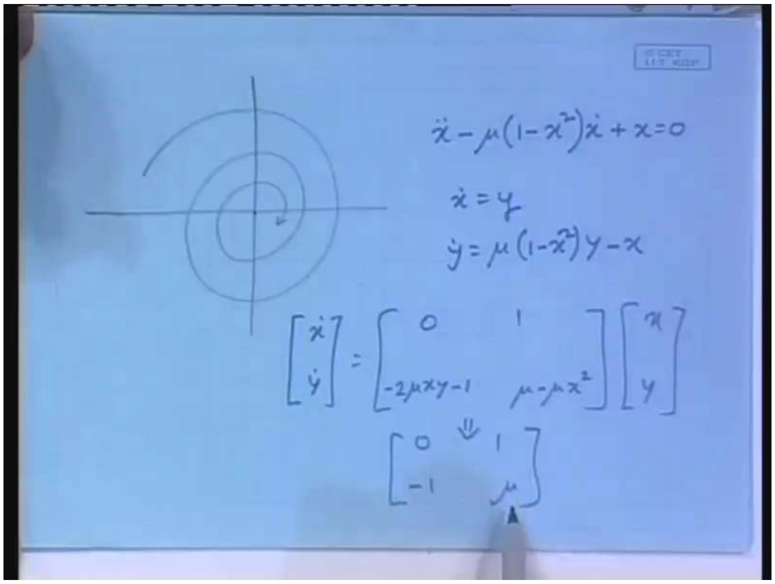

If you have an incoming spiral orbit, let's take a system where it will happen really. x double dot minus mu ($1 - x^2$) x dot + x = 0. Can you calculate with the behavior of the system? With the tools that you have learnt you should be able to. First will decompose it into second order system. So we will say x dot is equal to y and y dot is equal to mu one minus x square into y minus x. we said x double dot is y dot. The rest of it is taken to the right hand side. Where are the equilibrium points? (0, 0). x dot zero means y zero and then y zero means the whole thing is zero. Therefore x is zero. Now can you find out the Jacobean? Now notice that mu is a parameter. The way I said your voltage is a parameter in a system, the length of the coordinate is a parameter, and the friction of the wind is a parameter of the system. At the bottom of the the pendulum if you have the magnet that will change the parameter of the system and so on and so forth. It changes the gravity. So mu is such a parameter. Nevertheless it appears in the equation and therefore it should be able to obtain the local linear behavior including this mu as a function of this mu. You will get from if you take the deviations from here. x dot y dot in the local linear form will be zero one. The Jacobean will contain other things and then if you put (0, 0), then only you get those things out. Therefore minus twice mu xy - 1 and here it is mu minus mu x^2. Right now this takes the form 0,1, -1. Now the rest you take as an exercise that you will do before coming to the next class. For mu values negative and for mu values positive, you calculate the eigenvalues. Therefore as you change mu as a parameter, the vector field will change this way. in order to try to understand how the vector fields changes, as they change the parameter, you simply imagine that you are changing mu across from a negative value to a positive value and tell us in the next class what will be the behavior of the system as that happens and then that will be the starting point of another derivation. That's all for today.