Chaos Fractals and Dynamical System
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Lecture No # 17
Interactive Function Systems

So for we have taken three classes on fractals and this is the fourth one in which we will mainly try to understand how to generate fractals by a method that is not the same as the Mandelbrot and Julia sets and in order to do that we had proceeded by the function analysis approach in which we had defined the space of all possible images, all possible compact subsets of the R_2 space. We have shown that space is complete, if we take steps then all my steps will fall on the elements of the space and then we proceeded on to define how to take the steps and we say that the simplest possible way of taking a step form point to a point is by a affined transformation which is given by x_{n+1} y_{n+1} is equal to a b c d x_n y_n plus e f.

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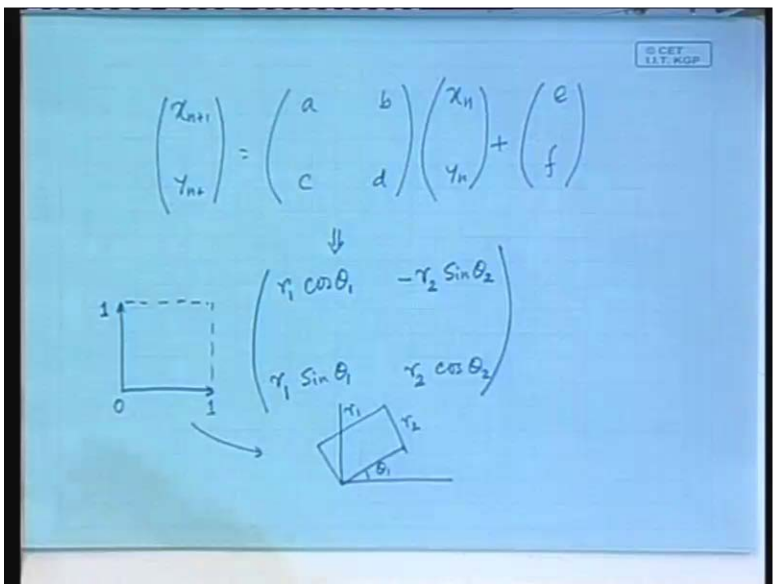

It is not difficult to see that these two fellows define translation and these four fellows a b c d they define rotation, flipping and stuff like that. In order to visualize what these four people do, it is easier to represent it in terms of r theta coordinates. So this thing can also be written as r_1 cos theta$_1$ minus r_2 sin theta$_2$ r_1 sin theta$_1$ and r_2 cos. What are these angles? It is not difficult to see.

Suppose you have a 2 D coordinate system in xy and we have defined a square, zero to one, zero to one. Now if you apply this transformation on to all the points of the square, what do you expect it to happen? This square will be deformed somewhere and it's a linear transformation and since it is a linear transformation, lines will still remain lines. So we expect the square to become rhombus that kind of stuff and so in order to facilitate drawing, let me draw it here. You expect it to become something like this.

Obviously there would be two of those size, they would have the sizes of r_1 and r_2 that is how we interpret the angle by theta one and the other one I cannot represent here anyway we can visualize that. The point is that the abcd matrix essentially does things with rooted at the same point origin and then this two fellows give the x coordinateship as well as the y coordinateship. So that is how you can define the steps.

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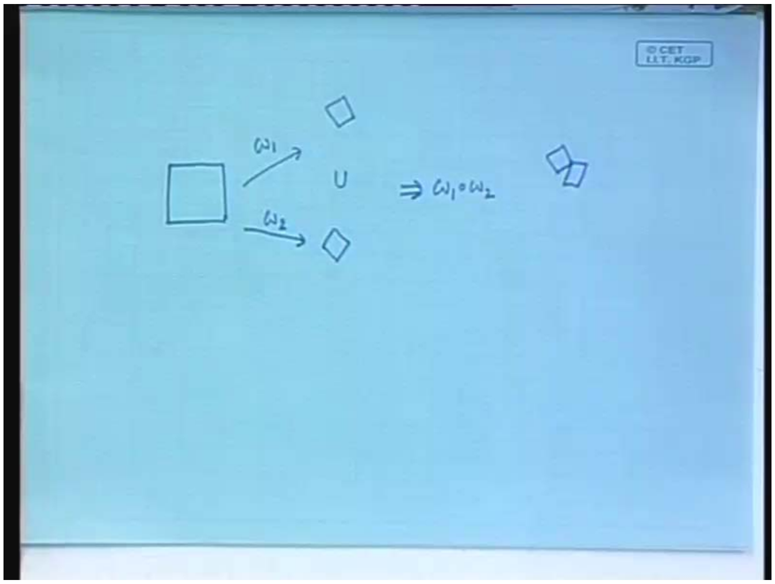

I also said in the last class that suppose you have some kind of a figure to start with and you defined one transformation say it makes it like this and we have another transformation say this one is w_1. You have got another affine transform that makes it say like this. Then you might say ultimately I have the w_1 which is the union of this two sets, so it will be this way we can add a little bit of interest to things that we ultimately construct. So if this is the result of the application of this transform on the original set and this is the product of the application of this transformer on the original set then the ultimate thing is the union of this and that. So that is the essential idea, how we actually applied and what we derive out of it, get out of it that we will come a little later but first let us concentrate on the character of this transform. In the last class you have seen that we are homing on to something that is we will start from any image and we will define some kind of transform that will transform that image but will then apply the same idea that we did in case of maps. We will keep on applying it.

Ultimately if certain conditions are satisfied then that will converge on to something. Something will be any ways that is the idea. So incase of the maps when you talking about maps, we also had fixed points and the convergent on to the fixed points. We are following the same kind of idea but then we have to do something in order to ensure that they do converge. We should not really define some kind of a transformation that will keep on moving away, ultimately it is not converging on anything that is not what we want, we want convergence and for convergence the crucial concept is contractivity.

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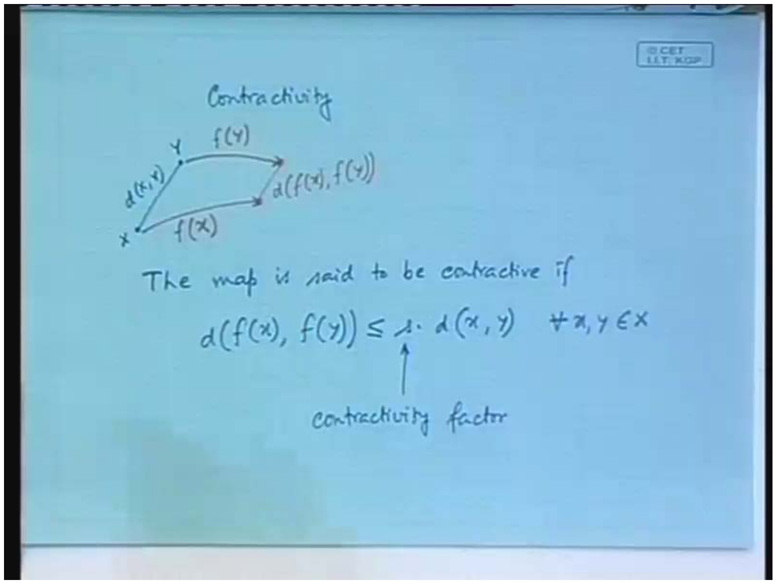

Contractivity is simply as follows that if you have one point x and another point y and you know the distance between them d of x y and then you define a function f of y and f of x. it lands here, it lands here, here also you will have a distance d of f (x) f (y). I am a writing the sufficient size that you can see. Now if this distance is smaller than this distance then the transformation will be safe in the contractivity, common sense things. So it is contractive or it is said to be a contraction mapping, if it is less than equal to some s times d for every xy a member of the sets. This s is called the contractivity factor. So once I define the contractivity factor, if we have the mapping that we did earlier this one, what will give the contractivity factor, what will ensure that it is contractive? The determinate should be less than one, simple stuff. So if you simply ensure that the determinate is less than one we are through, the mapping is contractive. Now if the mapping is contractive that means we have define something that is contractive then quite long time back some very interesting and important theorems were proved that applied to contractive factor. Naturally immediately those theorems will become applicable to what we are doing. One of that, one of the very important theorems crucial theorems in a functional analysis is called Banach’s contraction mapping theorem.

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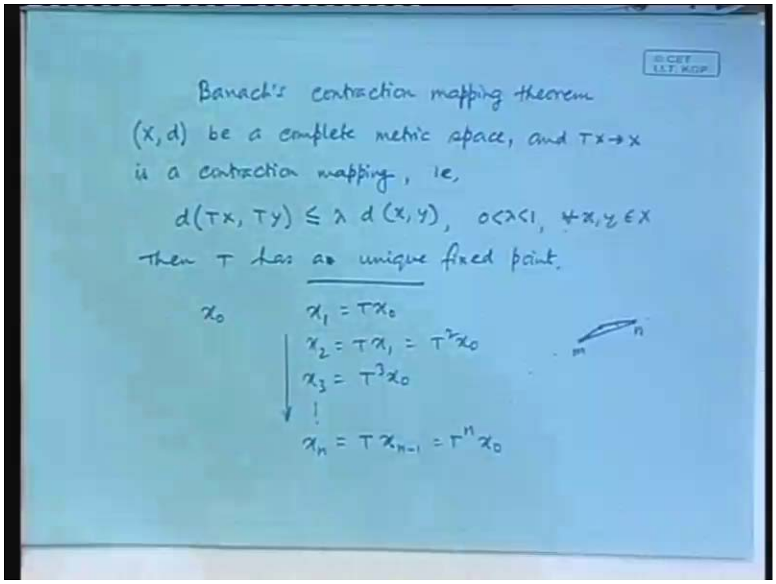

How many of you have done crucial functional analysis in maths department? None, in that case I have to do this function. That's fine no problem only thing is that while we do this, you switch the mode of thinking to a mathematical mode of thinking. When we say x then that x could be a point then x could be a line, the x could be a figure, x could be a function depending on what we have defined. So if you say let x comma d be a complete matrix space essentially we will mean that I am not now taking into account what that particular thing is. We are generalizing concept, we are generalizing and saying that x is a space in which I have defined the distance between two elements, that's it.

Complete matrix space means all the points in space are defined. So this switch in the line up thinking is necessary in the sense that that we engineers are often use to thinking in terms of concrete things and mathematician talking in terms of the generalize abstract concepts. So when we talk about this we will presently talk about the abstract concept, you easily see that if in your switching mode of the thought, if you keep thinking in terms of points that yes, x and y at two points. Fine, no problem. in fact I find it easy to understand all these things just by visualizing this x and y as two points but keeping in the back of my mind that these are all general concepts they will equally applied two a set of functions, it is a set of images.

So the Banach's contraction mapping theorems says that if (x, d) be a complete matrix space. What is matrix space, where d is defined. What is complete? That all crucial equations converge on to elements that are members of the set and T applied on x to x is a contraction mapping. So T applied on x also falls in the space x that's why T x, x that is contraction mapping would mean that the distance between Tx Ty will be less than equal to some number, contraction contractivity factor times let call it lambda times d (x, y) where lambda is between 0 and 1 for every x, y member of the set x. See these notations you should get used to, this is the notations for every, this is a member of the set.

Now the statement of the Banach's contraction that means theorem that if this is true then T has a unique fixed point. If this is satisfied then there is a fixed point, it is unique and the serial application, iterated application of the function T will always lead to a convergence sequence and that convergence sequence will converge on to a fixed point so this is actually three components of the statement. Repeated application of T defines say a convergence sequence. The convergence sequence converges on to the particular fixed point x and the fixed point is unique. See fixed point is unique is a very strong statement because we have already seen that there are maps in which the points fixed points are not unique. So these are so called reasonably strong statements. Let's prove it. This is the only proof that will do here, the other proof will be referred to some book because if we keep on doing the proof, it will become mathematics course which I don't want, we will actually like to do things more than to do the course but this one I will prove because this is half a century old proof, Banach's was a foolish mathematician which he learnt guide in the German concentration camp so there are historical interest in his life also.

So let's start, let's say we start from a point x_0, apply the transformation T to arrive at x_1, apply the transformation T again to arrive at x_2 and so on and so forth. So we can write x_1 is equal to Tx_0. Now x_0 is arbitrary anywhere, x_2 is Tx_1, x_3 is similarly and so on and so forth. x_n is Tx_{n-1} is equal to T to the power n of x_0. So this is the how we are defined a sequence and then the next step is to prove that this sequence is a crucial sequence. We have already learnt the definition of crucial sequence in the last class. The distance between any two elements of that sequence m and n, there will always be a number such that the distance is less than that. So that is how crucial sequence is, so we will try to prove that.

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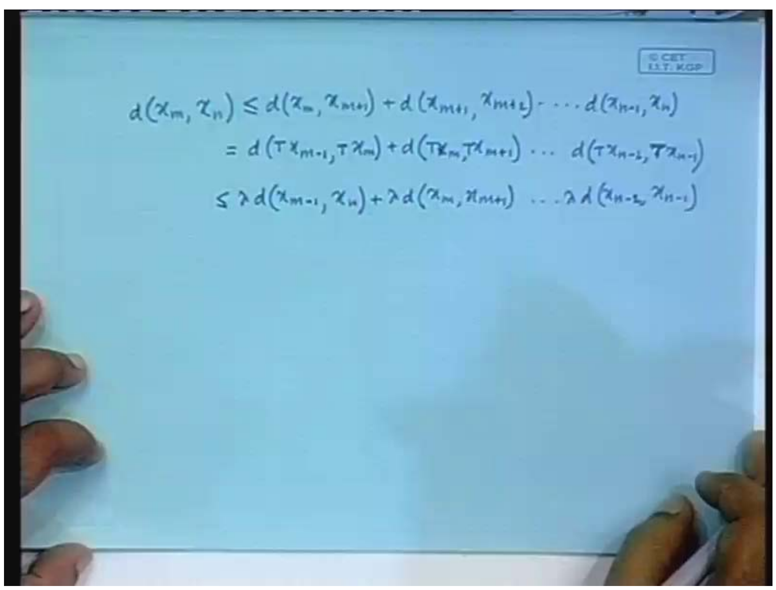

So we are interested in the distance between x_m and x_n because the mapping is contractive, this will definitely be less than equal to the distance between; not because of this property but because of the triangle inequality. We can write that x_m and x_{n+1} plus d x_{m+1}, x_{m+2} so on and so forth. Finally we will reach d x_{n-1}, x_n.

So in that sequence there are say two points m and say n then we are saying that the distance between this is definitely less than this plus this plus this and we are treating it that way. It is a triangle inequality. Now this we can further write as, this number in the right hand side we can equate to d of T of x_{m-1}, T applied on x_{m-1} yields x_n, T of x_m plus d of same way T of x_m x_m plus one… d T of x_{n-2} T x_{n-1}. So we have just taken one step back so what is wholly about it? Nothing, only we can now write that. Now this should be less than equal to lambda times d of xm minus one x_m plus lambda times d of x_m, x_{m+1} to lambda times d of x_{n-2} x_{n-1}. You can write that way. This is because of the contractivity factor, lambda. Now notice I will come that to this one little later, let me write somewhere else.

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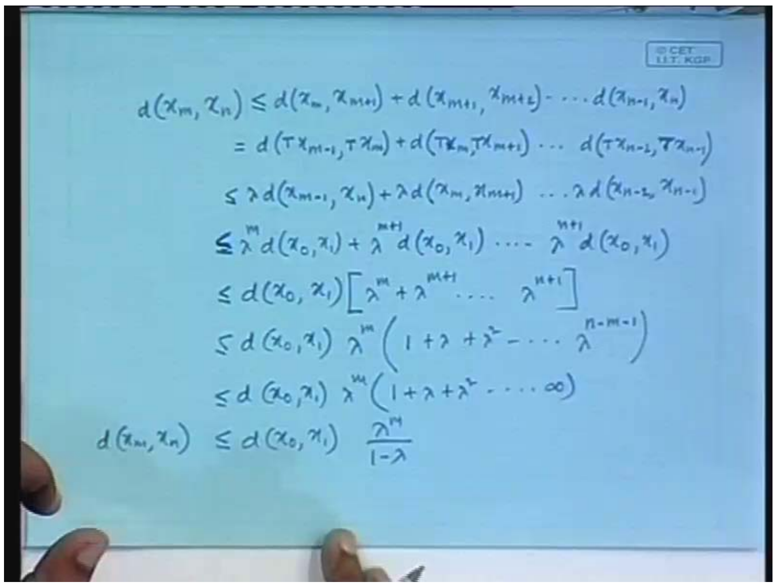

Now notice that the distance between x_m and x_{m-1}, one step is equal to the distance between Tx_{m-1}, Tx_{m-2} which is less than lambda times distance between x_{m-1} and x_{m-2}. So this way we are just going one step back. All these are just application of this step into various faces. Similarly in proceeding the same way, we can write d of x_m, x_{m-1} already we have obtained this, we can write this is less than equal to this lambda square another step back x_{m-2} and x_{m-3}. So this is equal to this is less than this, in the next step we can go one step further back so we get this. It's not difficult to see that this way we can go further and further back. Ultimately end in where? The first step the distance between x_0 and x_1 (Refer Slide Time: 23:40).

Now let us come back to here with that logic if you go by that logic then all this terms can be related to the first distance, distance between x_0 and x_1. So this will be equal to lambda to the power m, the distance between x_0 to x_1 plus lambda to the power less than equal to x. This term is equal to, but ultimately this is less than equal to; this term is equal to next term; less than equal to the right thing to write it. So m+1 d x_0 x_1 go on, you ultimately get lambda to the power n plus one d of x_0. So you can now take it common, d of x_0 x_1. What do you have left? It is going on increasing, n is one step and n is some step later. So it is going through that and ending here. This is nothing but less than equal to d (x_0, x_1) we will take lambda to power m common, we have one plus lambda plus lambda square plus, where does it end? Lambda to the power n minus m minus one. This term is not very important because in the next step we will say this must be less than, if up to this point it is less than then we can easily add up to infinity and say that it is still less than that.

Now this one you can shorten it and write it as, very important stage. Let me write that again, here x_m the distance between two steps, two things and we are related to this. Now you see if these things go to infinity what happens here? Lambda is a term that is less than one, if m goes to infinity then obviously this goes to zero so the distances shorten progressively ultimately that must become a crucial sequence. So this is nothing but a crucial sequence. So the point that has been made yet is that if you can define a contraction mapping T, repeated application of the contraction mapping will define a crucial sequence which means that it will converge. Now if it converges then next step is to prove, suppose if converges on to the fixed point x star. We will use this particular sheet later, I will keep it here.

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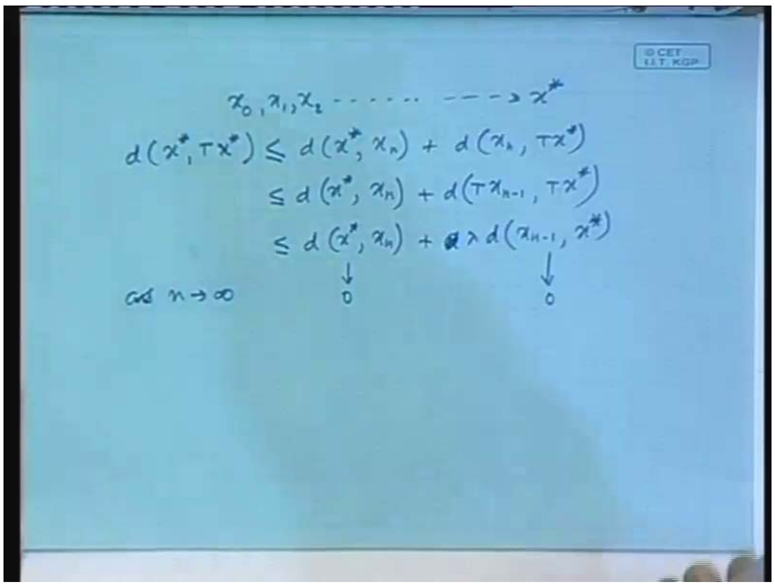

There are two more elements we need to prove. one is that where it converges suppose if the sequence x_0, x_1, x_2, so on and so forth ultimately converges to x star and we need to prove that it is a fixed point. Fixed point means x star is equal to Tx star we need to prove that. Now that comes from distance between x star Tx star must be less than equal to distance between x star and some elements x_n plus triangle in inequality x_n, Tx star. We are using the triangle inequality. So this is nothing but less than equal to d (x star, x_n) plus d of just take one step back Tx_{n-1} Tx star. Now we can again write this as d (x star, x_n) plus lambda we will take out d of x_{n-1} x star. Now as n tends to infinity what happens to this one? By the earlier proof this goes to 0 and this one also goes to 0 (Refer Slide Time: 29:47) and therefore this thing goes to zero and therefore this distance between x star and Tx star will be 0. So that proves that it converges on to the fixed point. Lastly we need to prove that the fixed point is unique because ultimately what we are going to do, use these theorems ultimately to do things and we need to make sure that we ultimately do converge on to things that we want, it has to be unique.

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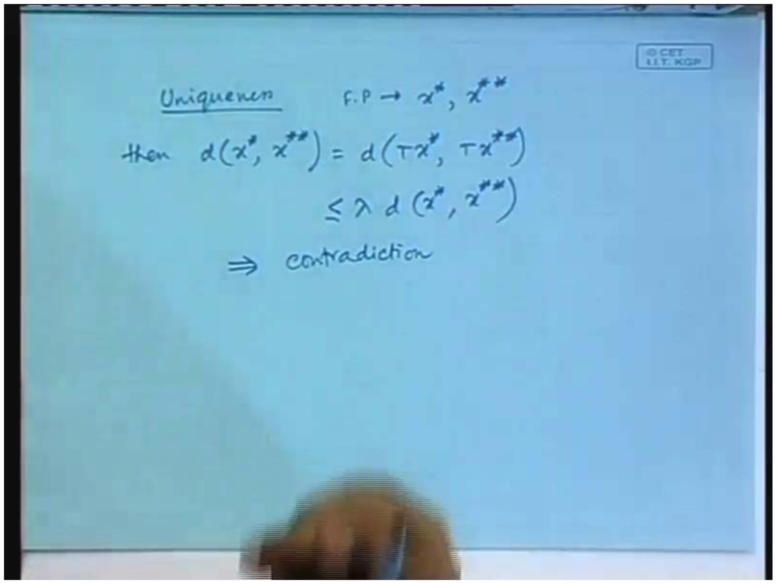

So proof of uniqueness, in this proof you start from the assumption that there are two fixed points. So fixed points are x star and x double star. Suppose there are two fixed points and will lead to a contradiction then d of x star to x double star, what is this distance? Equal to d of Tx star because it is a fixed point, x star is equal to Tx star and Tx double star, this is also fixed point. So this must be less than lambda d of x star x double star. Obviously this is a contradiction because the number cannot be smaller than the same number. Obviously this is a contradiction. So this leads to the conclusion that x star must be same as x double star, there cannot be two different points. So if you have a contraction mapping, the Banach contraction mapping theorem proves that if you repeatedly apply it will always converge, it will converge on to the fixed point and the fixed point is unique.

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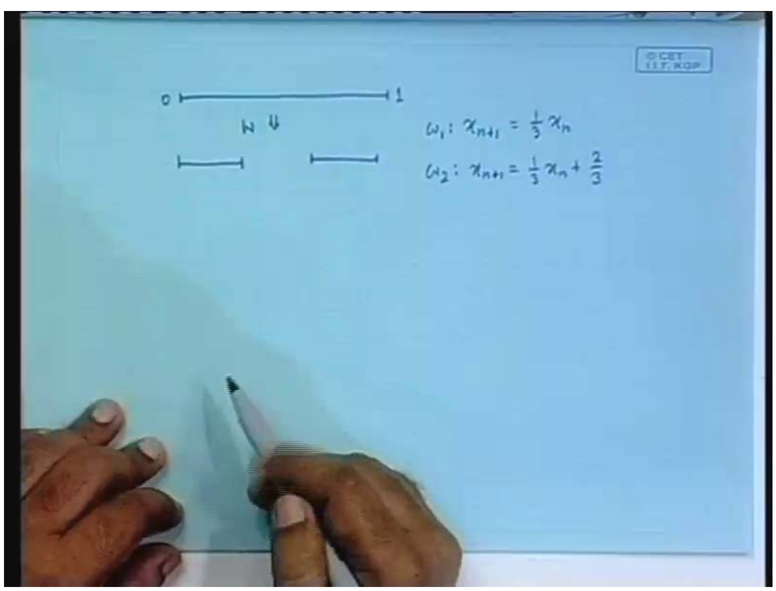

Now how to apply that? Let us first try to do that on a one dimensional map. Suppose we start from the distance between zero and one. If I now say that x_{n+1} is equal to one third of x_n is my map, is my w_1, what do you get out of this? In the next iteration it gives this much. If you keep on applying it, it ultimately converges to a point. Now let us define another w_2 as x_{n+1} is equal to one third x_n plus this other contraction mapping, this is contractive one third. So these are both contraction mapping. Now if I say ultimately my mapping is the union of the two, what happens? This and that put together will now be my capital W, so capital W is this step which is a union of W one and W two. Repeat this procedure, what do we have? The cantor set, yes we have already done that. So if you repeat this procedure it converges on to the cantor set.

So here we have a contraction mapping define, a contraction mapping in which both these are contractive. Ultimately that converges on to a point. If you start from the cantor set and apply this, it again converges it again lands on the cantor set. The cantor set itself is the fixed point. Now let us go forward. Now suppose I have in the two D space, define two of the W's, two of the contraction mappings and putting them together, union of them it is defining my ultimate mapping. What will be the resulting contractivity factor? Here both of them had the same contractivity factor, one third but we might have a situation where you have both of them having different contractivity factor. So suppose I am working now on the two D space. Let's look at for another angle.

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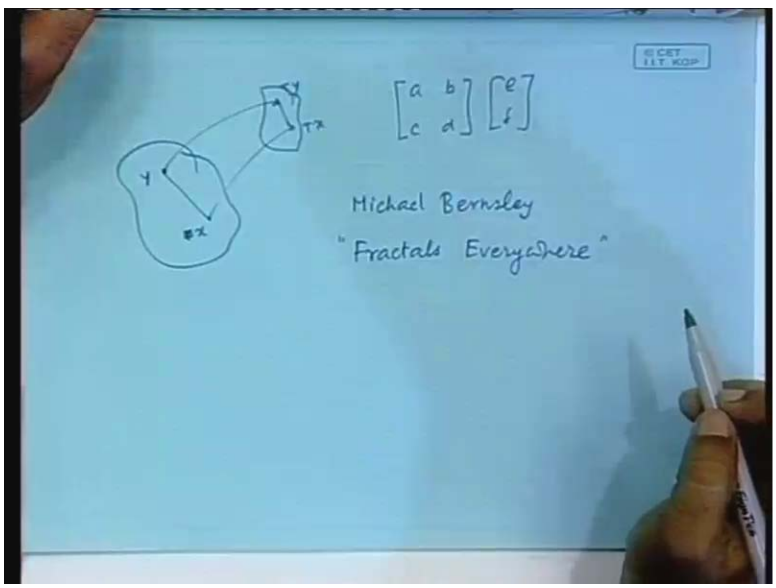

If a point maps to a point, another point maps to a point and I have defined a contractivity factor between these two as s. So x and y, T of x, T of y this contractivity factor we have defined as s. If this particular, this same map is applied to a set yielding another set what will be the contractivity factor? Same, you get the point. So if I have an abcdef defined that means abcdxy plus e f, this defines how a point maps to a point. This could be applied to a set yielding another set. The question then is for the set how much is the contracivity factor? It is the same so this is the another theorem which I will not going into the detailed proof, you can intuitively understand it. That's why I am not going into the detailed proof. Better I will refer you to the book where the detailed proof are obtained. This is Michel Bernsley “Fractals everywhere” is a very good book from which the detailed proof you can get, if you want to get the proof.

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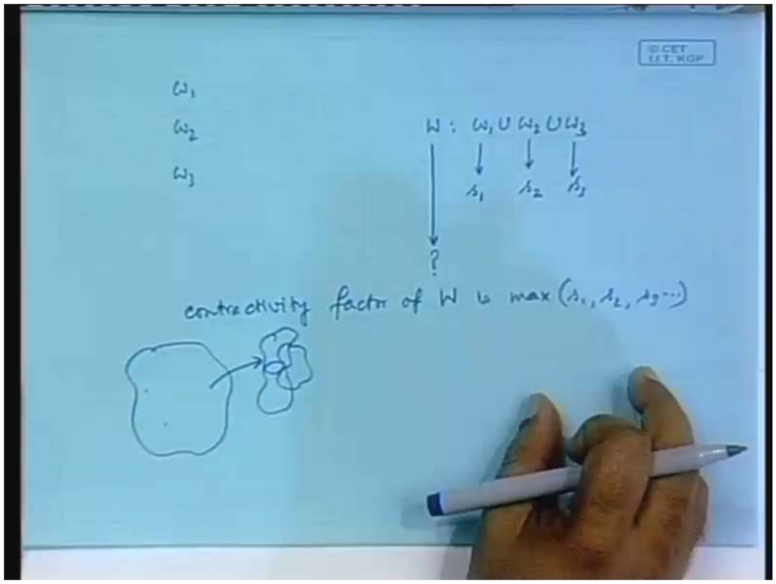

Now suppose I have defined a w_1, w_2, w_3 and ultimately I am saying that my w is w_1 union, w_2 union W_3 and W_1 has a contractivity factor s_1, w_2 has contractivity factor s_2 and W_3 has a contractivity factor s_3. Then what is the contractivity factor of s? Again that follows from commonsense and therefore there is no point is going to a detailed proof, it is a maximum. So the contractivity factor of capital W is… see I can go into the proof but all the proof will get a little crumply. You can see that ultimately you have one set, with the help of one of these you get a smaller one because it is contractive. With the help of the other one you get a another one with third one you get a another one. Contractivity from here to here is given by the distance between the points. It’s not that how much that does the total set shrink. No, that’s not a contractivity factor.

The contractivity factor is given by the distance between points in that set and that obviously in some cases it will have the contractivity given by s one, in some cases by s_2 but ultimately if I say what is the maximum contraction between points that will be the maximum of all this. A hand waving argument is like that but if you want to have a detail, just look at the proof of this. So if I choose a few maps with contractivity factor less than one and only make sure that all of them are less than one. So we have chosen the individual transformation w_1, w_2, w_3, w_4 and all that each we have to make sure that they have contractivity factor less than one that’s enough because then ultimately what we have must have contractivity factor less than one and therefore it must ultimately converge on to something.

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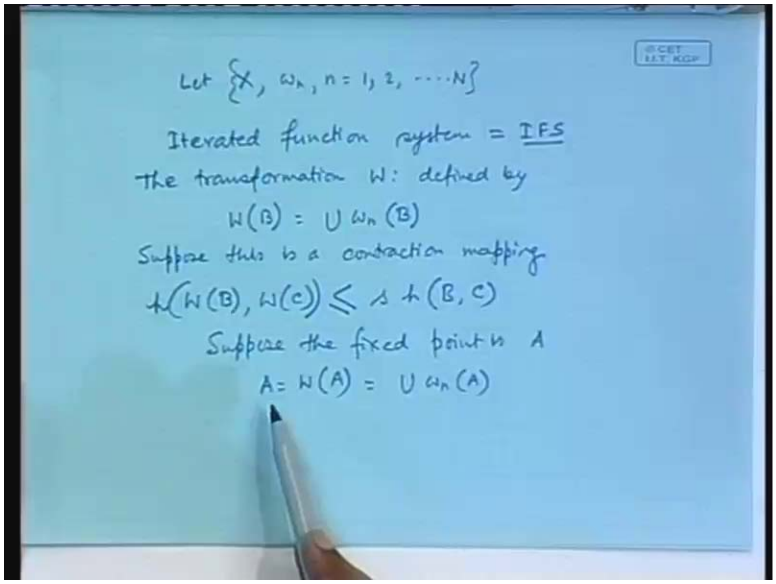

Now let us go to next theorem. Again X, X are the set and w_n are the transformations defined with various transformation w_1, w_2, w_3, w_4 and all that these are the transformations. These taken together would be called iterated function system. function system means this w's are the function taken together it is a function system and you can iterate it that means you can repeatedly apply this transformation, so it is called a iterated function system. Remember this particular word IFS will appear repeatedly in our discussion. So remember IFS, iterated function system. We generate all factors using iterated function system that is why this is very important. Now look at the theorem. It says if you have this an iterated function defined, the transformation capital W defined by W of set B is the union of w_n of B. Do you understand what I said? This is my set B, w_1 yields this one, w_2 yields this one, w_3 yields this one so ultimately the union is the result (Refer Slide Time: 43:55). So from B to capital W of B is the transformation. The transformation is not by the individual ones but union of the individual transformation is the ultimate capital W, what we are talking about. This is the iterated function system, capital W is the function system, go on iterating it that means you apply once, you get this, you apply again you get something else. Ultimately it will give a sequence of images and that is what we are talking about.

Now this is the mapping and suppose this is a contraction mapping which means the Housdroff distance between, now I will write h because we have already defined, a Housdroff distance housed of distance between W of B and W of C should be less than equal to some s times h of B, C. Then if this is a contraction mapping then the Banach contraction mapping theorem will say that this has a unique fixed point and the iterated steps will lead to that and that will be written as ultimately, suppose the fixed point is A, so A is equal to W (A). It is nothing but union of W_A(A). What does it mean? That means that A is unique for every such transformation we have defined, we can define A is the fixed point is unique it is an image and starting from any other image say B, we can repeatedly apply this transformation and ultimately converge on to A. So do you see the logic, slowly imagine for any given image i will somehow define this W and then starting from any given image i can repeatedly apply that transformation and ultimately will converge on to the final set, mathematics says so. Now it might get very counter intuitive because actually I wanted to show some of these things but nevertheless we can do that, I will just flash some of the images here so you can see.

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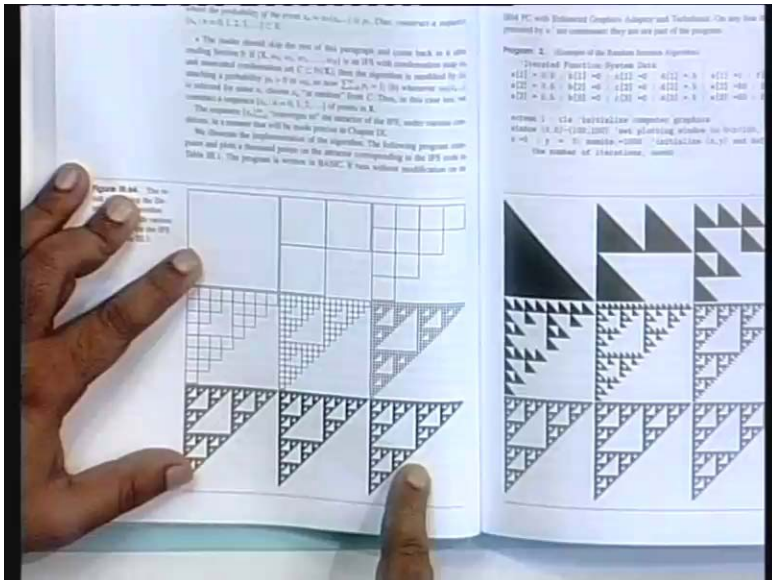

We started from a black triangle and we have a set of w’s given which give these three, each smaller thereby these are all individually contraction mappings keep on applying that ultimately it converges on to this here (Not Audible)) (00:48:12 min). It might be a difference starting point for example, you might start from a square, apply the same transformation we ultimately get same stuff. This is the attractor, can you see that properly? Yes. This is the attractor, for this attractor somehow we need to define the state of affined transformations and then I don’t care what my initial condition his. Wherever you start if i have this numbers ultimately I will converge on to this. This is the confidence given by the Banach’s contraction mappings here. Now let us proceed. Now what I’ll do is I will tell you how to do this. For example for the Sierpinski triangle that you just saw the abcdef, you will need a few of them you can easily see that unit three of them.

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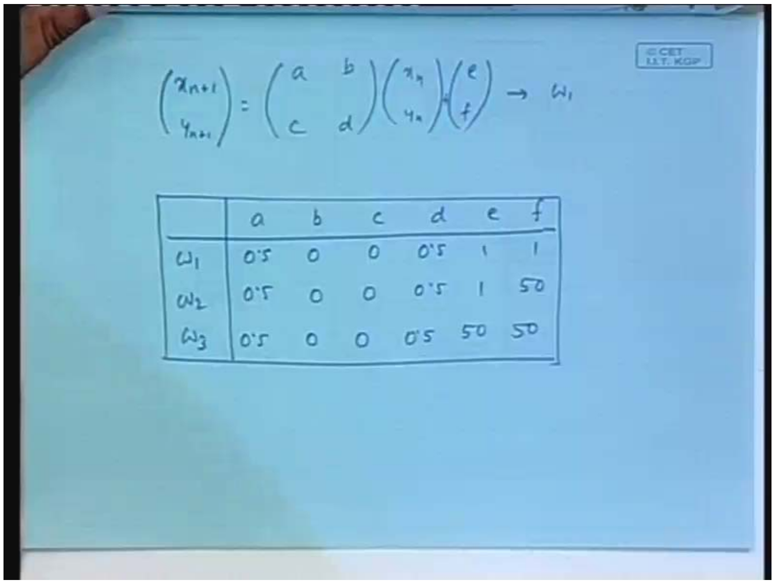

So you will need, I'm defining it as x_{n+1}, y_{n+1} abcd x_n y_n plus ef and you will need this is say w_1, you will need three of them. So you will need three values of abcd, three values of ef. I will just put that in a table. W_1, w_2, w_3 you will need three of them abcdef 0.5 0 0 0.5 1 1, 0.5 0 0 0.5 1 50, 0.5 0 0 0.5 50 50. How to generates this numbers I will teach you, don't worry but presently assume that god gift, somebody gave you and now today start is to do what I just show you that means starting from any given image to ultimately converge on to that. How to do that? what we will do is first start from any possible image say s square, say sr_2, say a arbitrarily shaped triangle whatever it is which will be nothing but set there will be points in that set.

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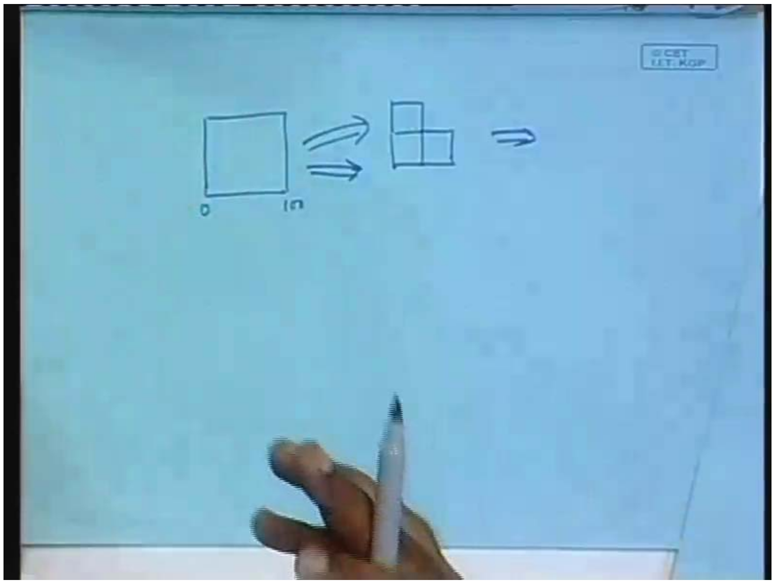

So suppose you have started with a square there will be points in that set and you may say that you are talking in terms of pixels, it will be easier for you to work in pixels. So say this is 0 to 100 pixels say, you might as well scale depending on the resolution of a square. Then take the first one w_1 which means a has 0.5, b has 0 so on and so forth, apply this to all the points of this square. All the points means you immediately run into the question, how many points? As many as the number of pixels in this square, so it will go somewhere. Now take the next one, do the same process there it will go somewhere else, take the third one it will go somewhere else then you started with this pixels filled, After the first iterate of w you end of with this pixels filled. In the second iterate you would start with these pixels and apply the same procedure, it will ultimately converge onto something. You will see before your eyes how an image as you apply the contraction mappings change in shape and ultimately it converges on to something very interesting. In the class you have also seen the image of what did I show? Probably the fern I showed and the snow flake. Let's talk about the fern just interesting. Write down the iterated function system of the fern and you write the program to generate the fern. You have understood how to write the program now, actually the program is very simple.

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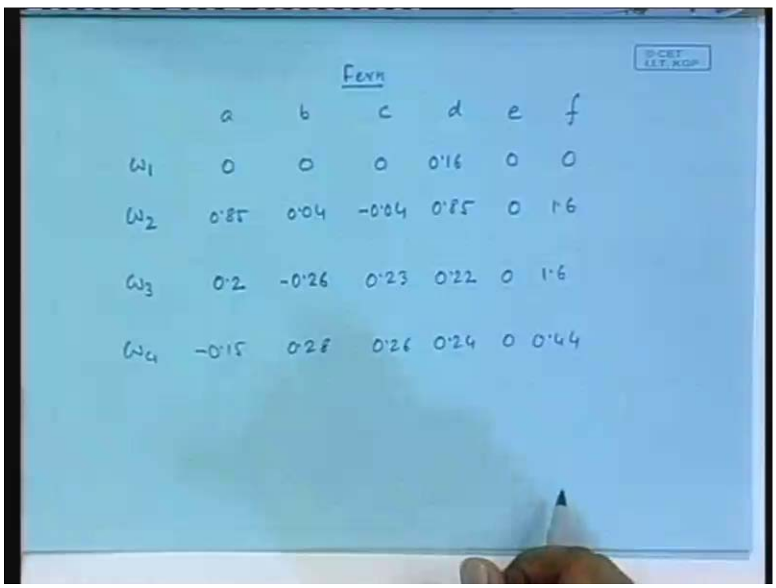

For the fern again you will need four of them w_1, w_2, W_3, w_4 abcdef 0 0 0 0.16 0 0, 0.85 0.04 -0.04 0.85 0 1.6, 0.2 -0.26. You might wonder who gave these numbers. Well at the end of the day you will be able to generate this numbers. 0.23 0.22 0 1.6, -0.15 0.28 0.26 0.24 0 0.44, given this numbers you will also be able to generate this numbers. So you have understood how to write the program to generate the fern. If there is a questions ask but try to do this before you come to the next class. This will not take more than an hour I can tell you, program. So that's end of the today's class.