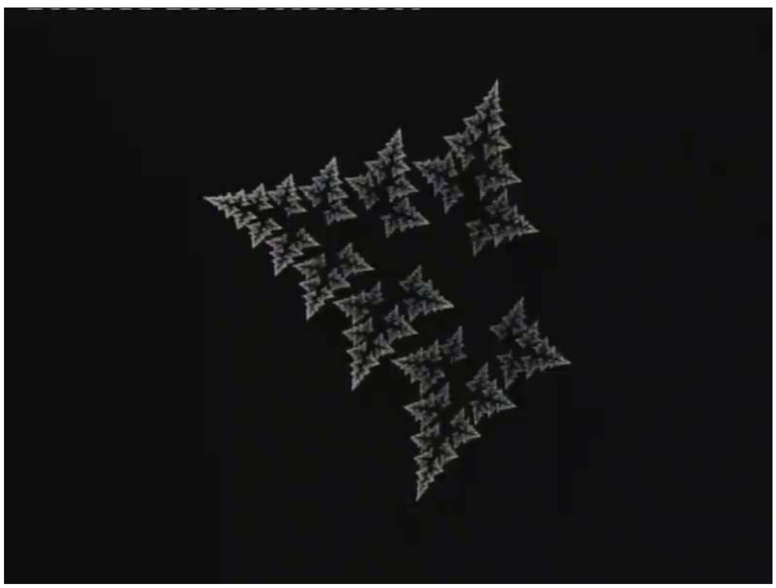
Chaos Fractals and Dynamical System
Prof. S. Banerjee
Department of Electrical Engineering
Indian Institute of Technology, Kharagpur
Lecture No. # 16
The Space Where Fractals Live

So far we have learned about how to measure fractal dimensions and we have also learned a bit of how to generate fractals by some kind of an iterative procedure. You have learned how to do the middle one third removal kind of algorithm to obtain a set and we have also seen that a set would be having a fractional dimension meaning that will be fractal but now let us check a holistic approach to the whole situation. What are we looking at? We are looking at pictures, images and if you think of images then in the 2 D space they are nothing but some collection of points. In order that they are nothing but a set. Especially if you think of binary images for example there is image on the computer screen right now.

(Refer Slide Time: 00:01:50 min)

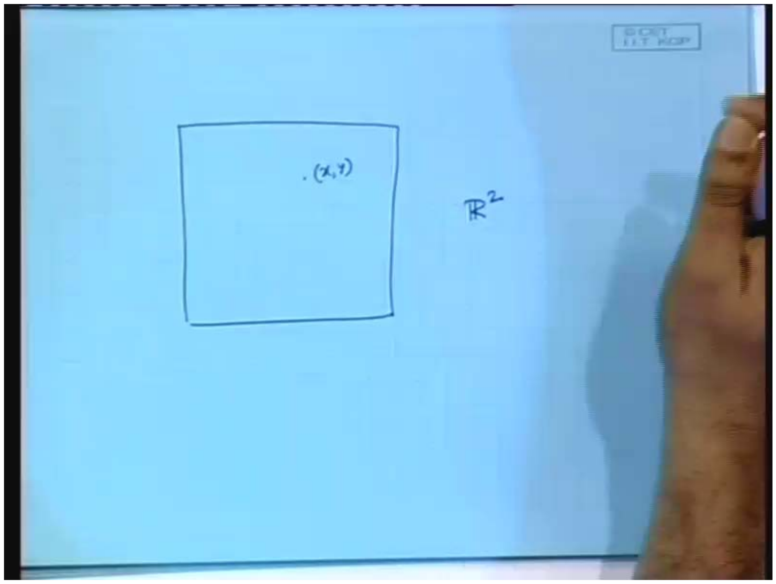


If you look at the image, this is the image of a snow flake and that is nothing but a collection of white dots in a black background. So it is essentially a set. To generalize that idea, it's not difficult to visualize that any picture is nothing but a set. Any image that is drawn in black and white is nothing but a set. We can later generalize to include the concept of colours and grey scale and stuff but presently let us confine our self to the idea of binary images, zeros and once which means that certain parts are dark certain parts are white.

So the point is that any image is nothing but a set. Now when mathematicians deal with such problem where they want to work with a certain type of entity, the kind of logic that they proceed with is that they essentially define a space where each element will be what we want to study.

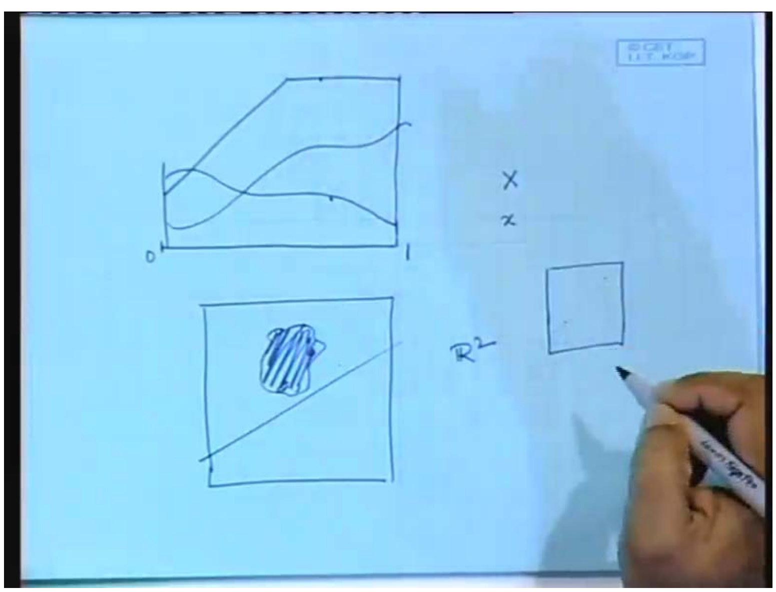
For example in mathematics we have the real number space where each real number is nothing but a point element of that set.

(Refer Slide Time: 00:03:23 min)



You have what is known as the R_2 space which is the space of 2 real numbers which means any point is a combination of x and y. So when a mathematician wants to study the property of such elements then they would say that I have defined a particular space, R_2 space and I am studying the property of the space. So whenever one tries to understand the properties of something, one defines a space.

(Refer Slide Time: 00:04:55 min)



For example when a mathematician wants to study functions, say functions between the point 0 and 1, you might define many functions between that. So for example say this is a function, this is also a function, this is a function so suppose we are trying to study the properties of such functions then a mathematician would say that let's define a space in which the elements are functions. So he would say the space of all possible functions between 0 and 1 that is my space in which I'll play where my elements will be nothing but the functions between 0 and 1. Then if you can define certain properties of that space, certain mathematically obtainable theorems in that space then all that can be used very fruitfully in understanding the kind of the things that we are trying to study, in this particular case function between 0 and 1.

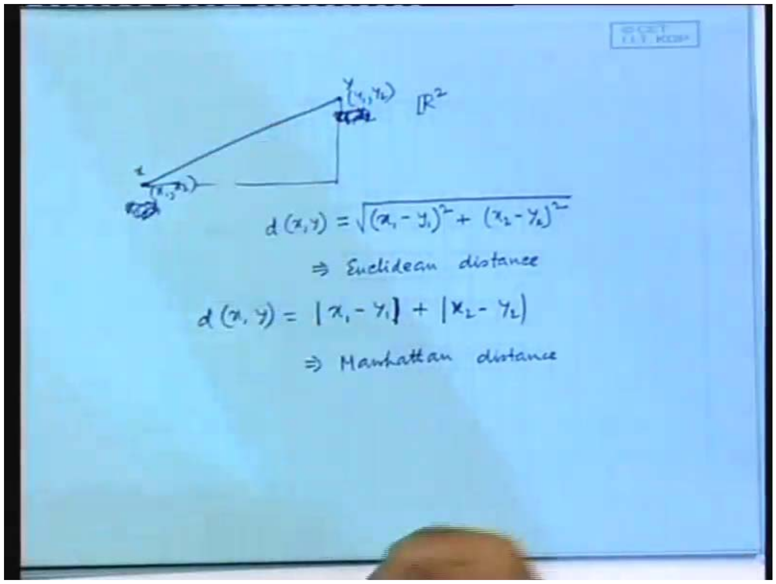
So in our case what are we studying? The kind of things that we are trying to study are essentially images. Images as I told you, if you talk again about binary images they are nothing but a collection of points in the 2 D space. The 2 D space is a space of two real numbers. So what we are talking about is, if say this is the space of two real numbers, any point is defined by two real numbers along the x coordinate and y coordinate. So the underlined space is R but what we are trying to study is not really the R_2 but you are trying to study like images or so. If you have images like so, you have essentially a subset of the R_2 space. So how would we define our space? Our space then would be, where the elements are nothing but all possible subsets of the R_2 space.

Notice what we are talking about. Now we are talking about a new space where the elements would be all possible subsets of the R_2 space which means that all possible images drawn, undrawn, thinkable, unthinkable and anything that can be a possible image would be member of this space. So it is this space we will work from now onwards. We will give it a name, a little later but then when a mathematician has to deal with such things, essentially he defines a space X where the elements are small x. so in our case the X capital is the space of all possible subsets of the real value R_2 space and the small x which are the elements these are nothing but the subsets of the R_2 space.

It will not suffice to say only subsets because this line extended to infinity is also subset. Obviously we are not talking about such things, we are excluding the possibility of parts of that image going to infinity which means they are compact, they are bounded, so all these properties will have to impose and for that you will need something additional, something more. What more do we need?

For example when a mathematician works in the space of all possible functions or simply the R_2 space then you can define two points. In the space of all possible functions this is a point, this is an element and this is another element. Can we then define a distance between them? Notice the moment we talk about things being compact and other things, we need the concept of these things. So can we define a distance between two points? We can surely do.

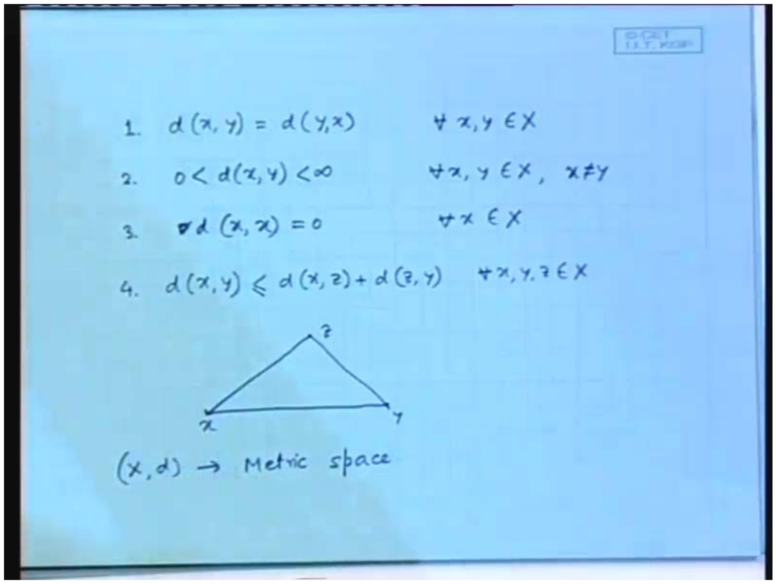
(Refer Slide Time: 00:09:54 min)



For example if we have two points in the R_2 space then how can you define the distance between them? Say this is x_1 y_1 and this is x_2 y_2 then we can say the distance between them is this (Refer Slide Time: 10:12). Can you put that in exact mathematical terms. Here you would say the distance between x and y, let us name them differently. This is point x, this is point y so this point x is x_1 x_2 and this point y is y_1 y_2 then this would be in this way $x_1 – y_1$ and this is called the Euclidean distance but you see this place could be a city like Calcutta or a city like New York in which it is not possible to go like that. You can only go along the horizontal direction and vertical direction, you can cross the road and then climb up the stairs to upwards. So you could go only like this, in which case the distance between the two points would be mod of $x_1 – y_1 + x_2 – y_2$.

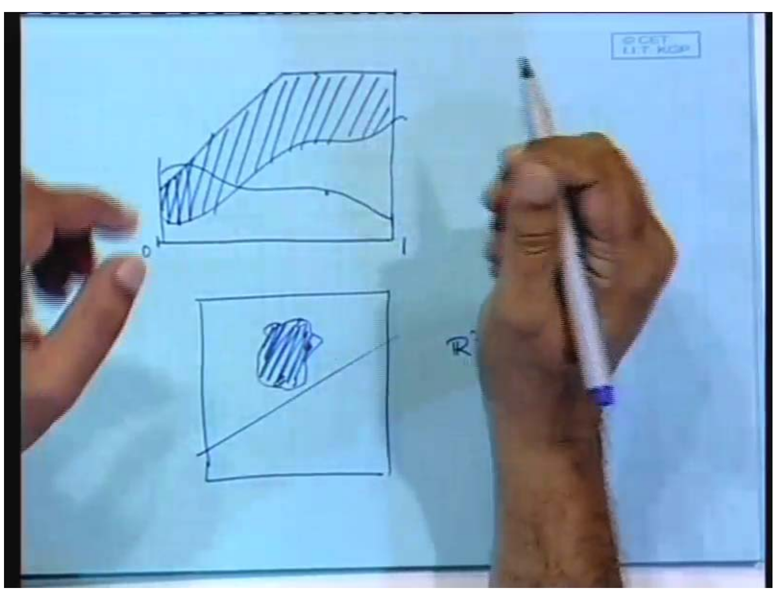
This is also a measure of the distance between the two points and this is called the Manhattan distance. In New York, in the island of Manhattan this is the only way you can go from a place to place, if you are not the Batman. So obviously you need to define this kind of distance. So the point is that in a space you can define a distance this way and it is rather simple to define such distances in the R_1 space R_2 space but can we define the distance in a space like the space of all possible functions between the number 0 and 1, non-trivial but possible. At this stage we will need to understand some properties of what we are calling distance.

(Refer Slide Time: 00:13:02 min)



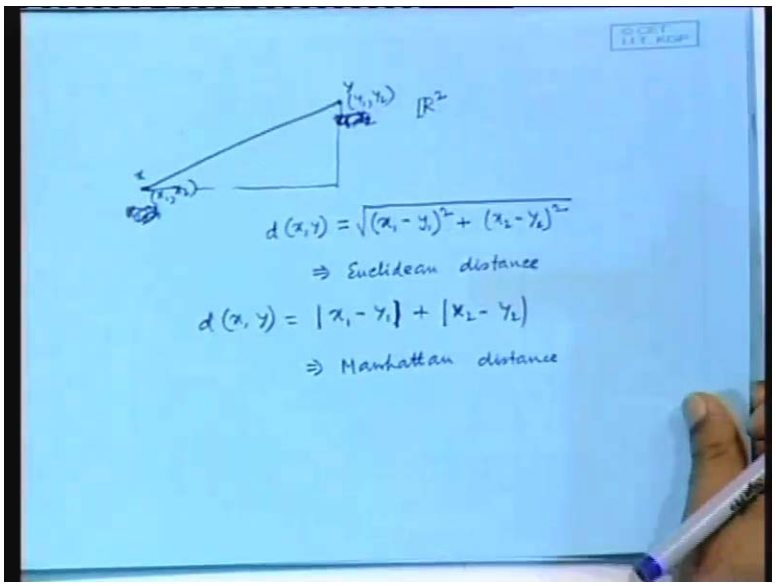
The properties are number one that the distance between x and y should be equal to the distance between y and x, for every xy belonging to the space x. The second property that we need is the distance always remains between zero and infinity, again for every xy belonging to x and x is not equal to y. We will also need that distance between x and x is zero for every x belonging to this space x and fourthly you will need that the distance between two points x and y must be less than the distance between x and a third point z plus the distance between z to y. It's called triangle inequality for example here is a point x, here is a point y and here your point z then distance between xy should be less than this distance plus this distance (Refer Slide Time: 14:50). So that is called the triangle inequality so this is for every xyz in x.

(Refer Slide Time: 00:15:38 min)



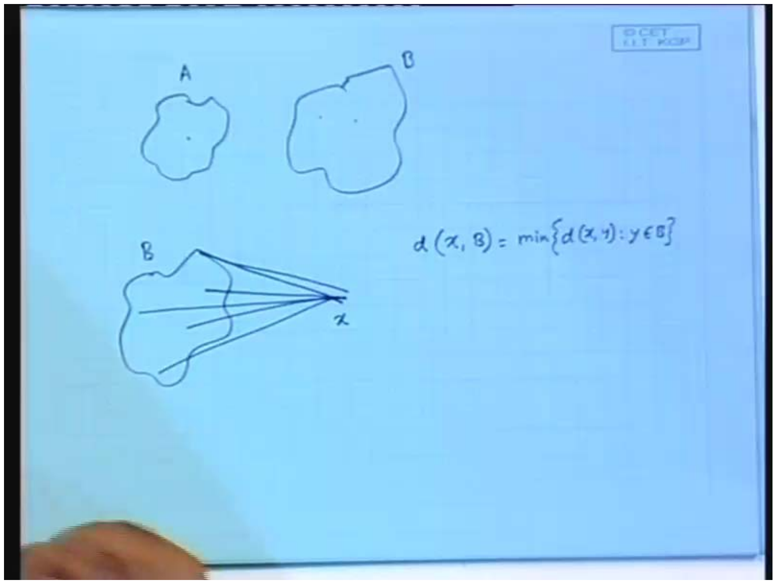
So these are the properties of distance that we need and indeed it's not difficult to check that the unfamiliar measures of distance like the Manhattan distance thus satisfy this rules. In the space of all possible functions, can you imagine what kind of distance would define? For example you might say that the distance between this and that will simply subtract. So this to this subtraction, this to this subtraction, this to this. Will that work? It will not work because the distance between x to y will not be equal to distance between y to x but if you define it as the area between the two curves, it might work. So my point is that it is possible to define distances, satisfying these criteria in all kinds of spaces not all kinds but wherever possible we will try to define this spaces. Now whenever we are able to define not only the elements of the space but also distances between them. Then that space is called a metric space. So will say not only x, not only the elements of space but also d together is a metric space. So the space and the distance together will be called a metric space.

(Refer Slide Time: 00:17:07 min)



So from now onwards we will not call it a distance, we will call it a metric. So this will be called a Euclidean metric, this will be called the Manhattan metric and so on and so forth. So you can understand that we need to define for the particular space that we are talking about. What space? That is where the elements are the compact subsets of the R_2 space, in that space we need to define a distance in order to talk in terms of the properties of the metrics spaces. So we need to define these distances. We will do that surely.

(Refer Slide Time: 00:18:03 min)

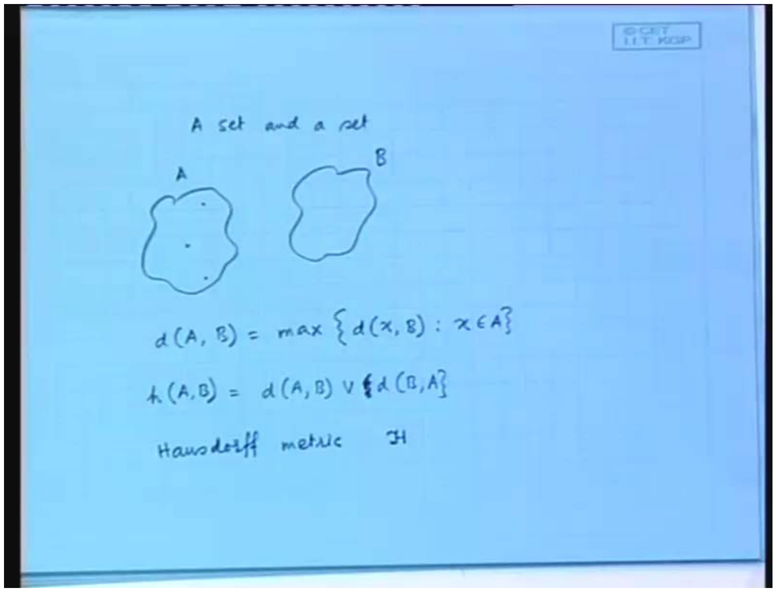


So we have an element and say another element, this whole element is a subset, an image and this whole element is an image. So this is an element of this space I'm talking about and this another element of space I'm talking about and this consist of invalid number of point that also consist of infinity number of points. How to define the distance between these two? That is a problem we are talking about. Now obviously these contains a large number of points, this contains a large number of points and therefore we need to talk about between these two points and that could be defined in all possible ways the Euclidean way, the Manhattan way and all possible ways. So let us chose one, as the distance between two points in R_2 space. So let us chose simply the Euclidean distance as the definition of distance between two points in the R_2 space. I have already defined Euclidean distance so it is like so. Thus two points in the R_2 space will be defined like this.

Now we need to define the distance between a set and a point. So first let us tackle the question of defining the distance between a set and a point. Obviously this set has an infinite number of points and here is a point, so you can define many distances like this. Out of all these which one would you take as the distance between the point and a set? Average, maximum, minimum something that satisfy all those properties. So in this case we will take the minimum distance between two. So we will define the distance between, here is a set A and here is a set B. so we are talking about the distance between a particular point say x and the set B.

Then the distance between the x and B will be defined as the minimum of all the distances between x and y where y belongs to B. so we are in a sense measuring all the distances and taking the minimum of that. Once we have done so, once we have taken the minimum then at least we have defined the distance between a set and a point. We have defined a distance between a point and a point, set and a point. The next step is to define a different between set and a set. So if we are trying to define the distance between a set and another set.

(Refer Slide Time: 00:21:47 min)



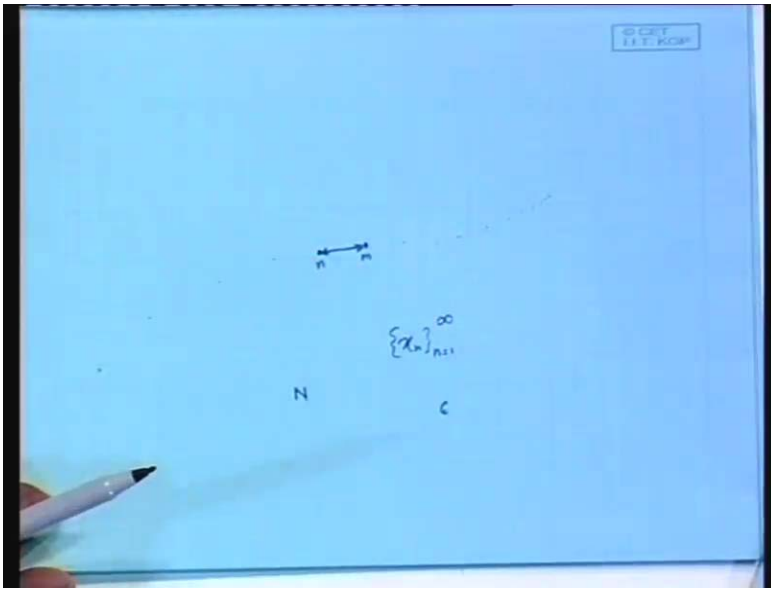
So a set and a set then all we need to do is here is a set and here is another set. Here is a set A and here is set B and we have defined the distance between the point x belonging to A and B. Now we can say that here is a point to which the distance to B was measured, here is another point to which I can measure it, here is another point to which i can measure it, like so we can measure the distance between each and every point a set A to the set B. now should we take the maximum and minimum? Again notice that we need to satisfy the properties that we talked about here, we will come back to that but the distance between set A and the set B, we take the max of all the distances between the set elements and the set B where x belonging to the set A.

So point to point is Euclidean distance, a point to a set is the minimum of all the Euclidean distances and set to set is max of all the distances between all the points of a set and another set. Now consider have you satisfied all these properties? Obviously third is satisfied, second is satisfied, first not yet, no it is not satisfied and so we need to do something more. These two we cannot yet call it a matrix because it does not satisfy the first property that is the distance between A to B should be equal to the distance between B to A. As here what we have done does not ensure that. So finally we'll define a distance h between A to B which is the distance between A to B as defined here and the distance between B to A the bigger of the two, we can also take that and that should be unique.

So finally if you take the bigger of the two that means we measure the distance a between set A to B and then the set B to A and finally we take the bigger of the 2. Then we have defined a concept of distance that satisfies all the requirements of being called a metric. This is called the Hausdorff metric and the resulting space where each element is an image, each element is a compact subset of the R_2 space, along with this definition of the distance will be a metric space. So that is called the Hausdorff space. What we have done is essentially, we have defined certain properties, certain things that we need in order to explore in the playground. So what we have done is we are essentially defined the play ground in which we will play.

What is a playground? We defined the space, not only that we have also defined the distance between two points in that space. Essentially something, somewhat counted into this, somewhat non-trivial, we have define the distance between 2 images. The image of Monalisa and my image obviously it's difficult to imagine a distance between them but there is obviously you can define a distance between the two images. Once we have defined this, we have obtained a metric space. The moment we have done all the result that are available on the metric space would be applicable. One of the important things that is applicable that is regularly applied on any metric space is the idea of sequences. What is the sequence?

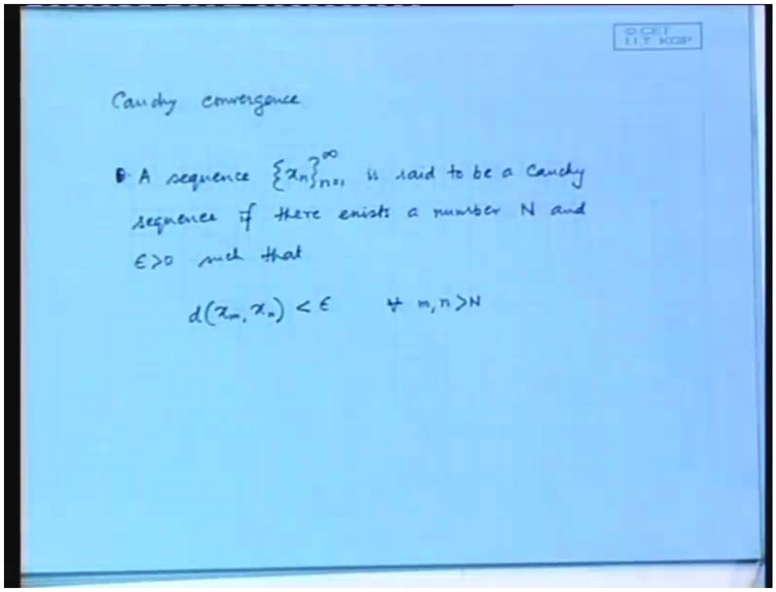
(Refer Slide Time: 00:27:36 min)



If you have R_2 space, you can define a sequence of point like this and infinite number of points. In order to define the infinite number of points, you have to say how do we go from this point to this point and this point to this point and so on and so forth. We will have to define what is known as mapping. How do we map from a point to point but once we are done so, we define a sequence and we will write the sequence as x_n, n is equal to 1 to infinity. So there is the sequence and such sequences may converge on to a certain point under certain conditions. When it will converge? The condition of convergence is something like this, it is called Cauchy convergence. The idea is that if you have a sequence like this, if it is convergent then you can see that the distances between them go down as you go further. As you proceed the distances become shorter and shorter.

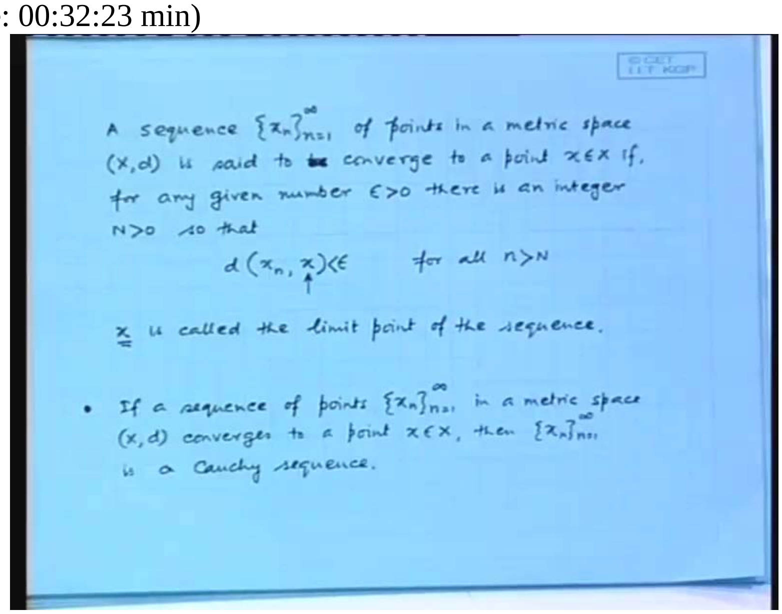
So in order to define such a sequence something that converges, you need the definition of distance otherwise you cannot really have convergence of sequences that is why we need to define distance. So you need to do that but then suppose we have the nth point and the mth point and the next point and the next point, so if you have such point then you can define the distance between them and as you go on this distance will shrink. So you can say that there will exist some number capital N so that if this n and m are greater than capital N then the distance will be smaller than some number c epsilon.

(Refer Slide Time: 00:30:04 min)



So how we will say that? We will say that a sequence x_n n is equal to 1 to infinity is said to be a Cauchy sequence, if there exists a number N and epsilon greater than zero, small number such that the distance between x_m and x_n will be less than epsilon for every m, n greater than N. what have we said? We have said that there is a sequence in which if I choose the n and m after this number N then I can make it the distance as small as a one. So in order to express clearly in mathematical term, we have said that we will define a small number epsilon such that the distance between the two points will become smaller than epsilon. There will always exist some capital N so that the distance will be smaller than this epsilon. So such sequences are also called Cauchy sequences and Cauchy sequences are convergent sequences and they always converge on to some particular point. Now where do they converge? For that I simply display what I have just written.

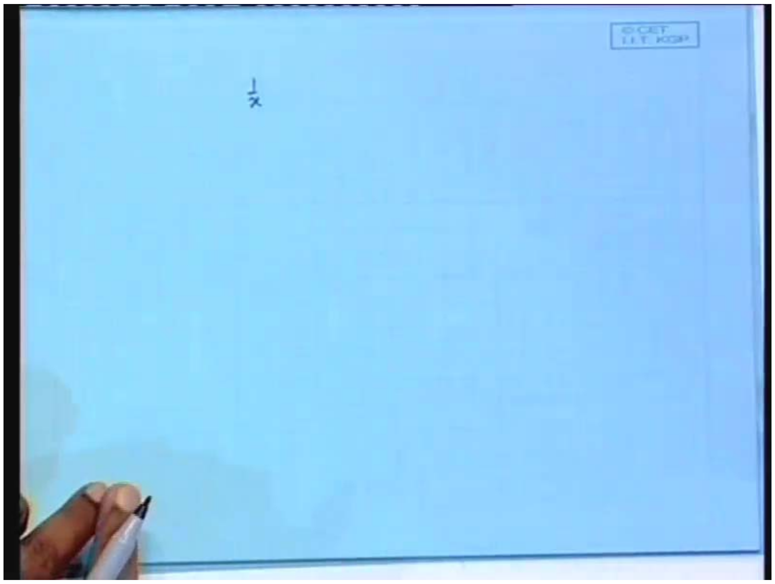
(Refer Slide Time: 00:32:23 min)



Suppose they converge on to the point or element x then following the same property of the Cauchy sequence you can say, a sequence this of points in a metric space x, d said to be convergent on the point small x which is the member of the set x. If for any given number epsilon greater than zero there exists an integer N greater than zero so that the distance between this is… (Refer Slide Time: 32:58). So earlier what we have said to be x_n now becomes the limit point x and we can say that there always will be some N so that for small n greater than N it will become smaller than a number. Why do you need all these? We need all these because actually it reaches the limit point after infinite number of steps and we cannot really count infinite number of steps. So we need to do something in order to still to be able to say that it is convergent. It is convergent on to this number or element x. So here looking at this, it is easy to see that it is convergent. On a real line if you have a number of points, it is not difficult to see that it will be convergent for example the sequence half, one fourth, one eighth, one sixteenth so on and so forth. We know that it is convergent.

So the moment we have defined this sequence, we can identify that it will converge and such sequences you can easily apply this property of Cauchy convergence and check that it does converge. So if it does converge, hence x is called a limit point. So we have defined the Cauchy sequence, we have defined a limit point, now the main point. Here we have defined the space and we have defined how to make steps, go from one point to the other, define a map and while taking the steps we will keep track of the distances between two consecutive points and if it satisfies the Cauchy convergence property we know that it will converge on to some point but where we converge does that point exist? That we will need to make sure.

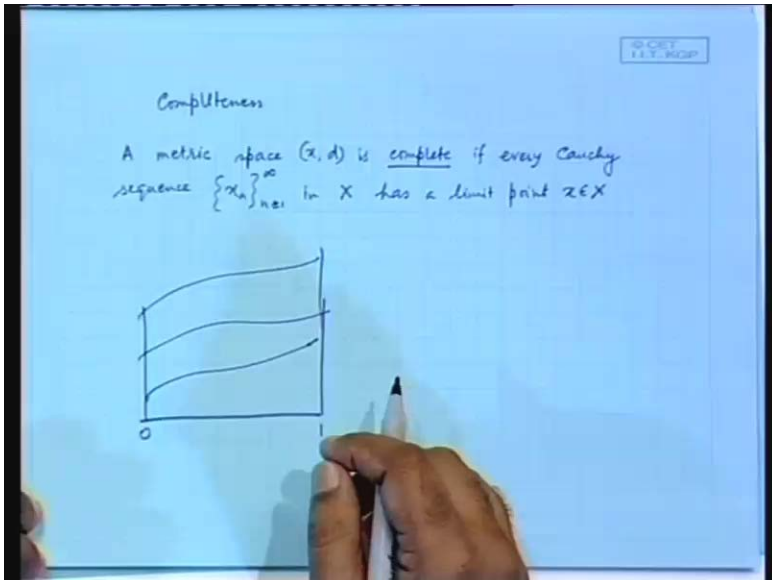
(Refer Slide Time: 00:35:18 min)



Why? Because you might say that 1 by x is a sequence, if so one by x is sequence as x goes on increasing, the ultimate point actually does not exist that's a undefined point. So we should not define space like that, so we need to ensure that while we take the steps where we land finally, our limit point is a member of the space.

Without having ensured that what will happen is that I am playing in a play ground but there are holes in it and while I am taking steps I may drop in one of the holes. Obviously I cannot do that so we need to define what is known as the property of completeness.

(Refer Slide Time: 00:35:59 min)

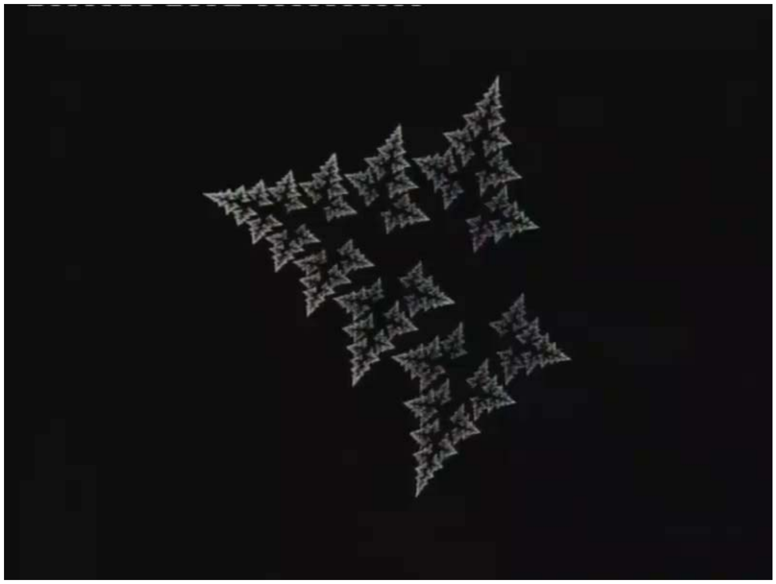


What is a property of completeness? A metric space is complete, if every Cauchy sequence; you might ask can there be an incomplete space? Imagine, suppose you are talking about functions between zero and one and suppose you have a space, you are defining as the space of all polynomials. So you are defining as a space of all possible polynomials in this. You would be able to define a sequence of such polynomials. You can show that will converge on to a function but that is not a polynomial. So such sequences exist and the point is that in order to play comfortably in a ground, we need to make sure that the ground is complete in every respect that means every point exists. While we take the steps, we know that where we are falling, our state is falling those points exists.

What we will essentially do? Can you see what are we driving at? We are driving at this that we have defined the space of all possible compact subsets of the R_2 space that means we have defined the space of all possible images. We have defined the distance between the two images but now we require these properties in order to make sure that the space is complete. Now what we will do? We will take a step that means we start from an image and then will take a step to obtain another image and if we keep on applying this property, this particular mapping again and again we will obtain a sequence which will converge some time onto something. What will that something be? An image, so if you want an image and if you can define a suitable way to take the steps, we can obtain any image as the limit point. So this is how we are proceeding. In order to proceed that way it is necessary to make sure that we define a space that is complete and in order to define the completeness, we need the property of distance. We need to worry too much about the distance and all that because this was necessary in order to ensure that we are playing in a complete space. Later for everything we will really not need to obtain the distance of space.

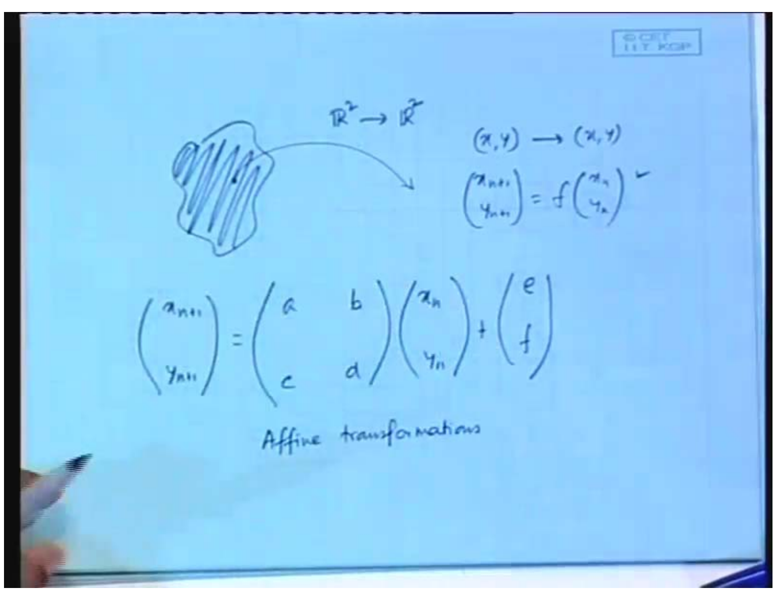
We have defined the distance in a particular way the minimum, the maximum, the greater of the two, it is a necessary but it is not necessary to do all that all the time. It is not necessary to actually obtain the distance between two sets all the time. Why, I will come to that later.

(Refer Slide Time: 00:39:58 min)



So we have defined a distance. We can see that here there was an image on the screen. This screen, this image is nothing but an element of this Housdorff space. If you deform it then that will be another image. If you plot any other image that will be a different image and we can then define the distance between this image and that. The question now is can we take a step, can you deform an image into another image by means of some mathematical function, can we?

(Refer Slide Time: 00:40:38 min)

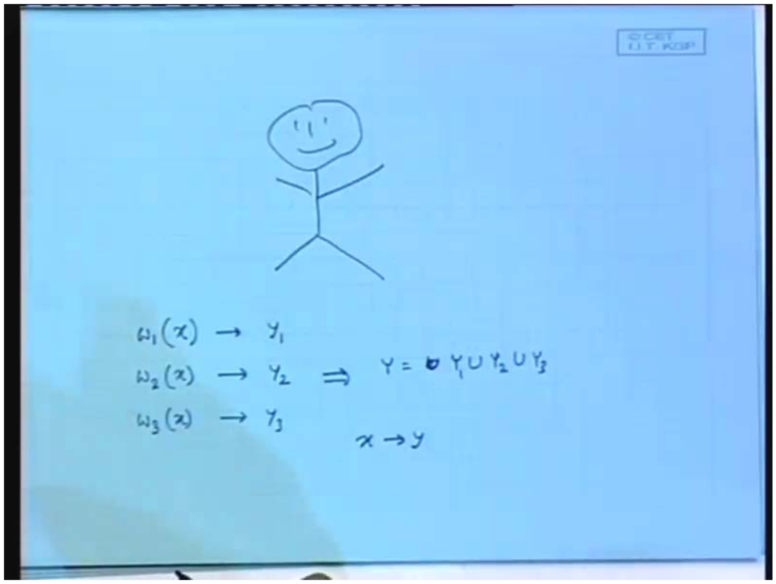


It is not difficult at all because the point is here any element of my space is nothing but a set. A set is a collection of points. If you can define a map from R_2 space to the R_2 space which means at any point xy will map to another point xy. If you can define that way then we have defined a mapping from x, y to x, y or x_{n+1} y_{n+1} as a function of x_n y_n. If you can define that, a point can be taken to another point and if this function is applied on the whole subset of the collection of points, what we will get? We will get another image. So from an image we can get another image which is nothing but taking step in that Housdorff space and that is why we needed to ensure that while we take the steps, where our steps fall, those points are there. These are members of the space. So we need to do something, We need to define something like this. How do we most conveniently define a step? Any point will be transformed to another point in the two dimensional space. What is a most simple way to define this? A linear mapping.

A linear mapping means something like this x_{n+1} y_{n+1} will be some a b c d x_n y_n plus like so. This is the simplest possible. Can you think of anything simpler? Obviously not, they can't be anything simpler; this is the simplest possible way to obtain the transformation. We will use that, we will not go further. We will use only these and these transformations are called Affine transformations. So essentially if you can define a b c d e f then we have defined the transformation, we can then transform an image into another image and by doing that we can ultimately converge on to something. It is possible but that will require certain property, we will come to that slowly. So what we have said is that using a mapping like that I can transform a point into a point and if this mapping is applied to all the points of the set, I obtain a set from a set and thereby essentially I take a step from a particular element of the Housdorff space to another element of the Housdorff space, an image to an image.

We might also make our life a little bit more interesting by saying that if one of the things I obtain after the transformation is say (Refer Slide Time: 44:50) this and I obtain another thing like this and I will have another thing like this then the transformed image will be union of all these, you can say that in order to make our life more interesting. That is from a particular image if we do some transformation and obtain a part from same image if we do some transformation and obtain another part and a third part and a fourth part and so on and so forth then we can say that our final image which we obtain after one iteration is nothing but the union of all of them.

(Refer Slide Time: 00:44:48 min)



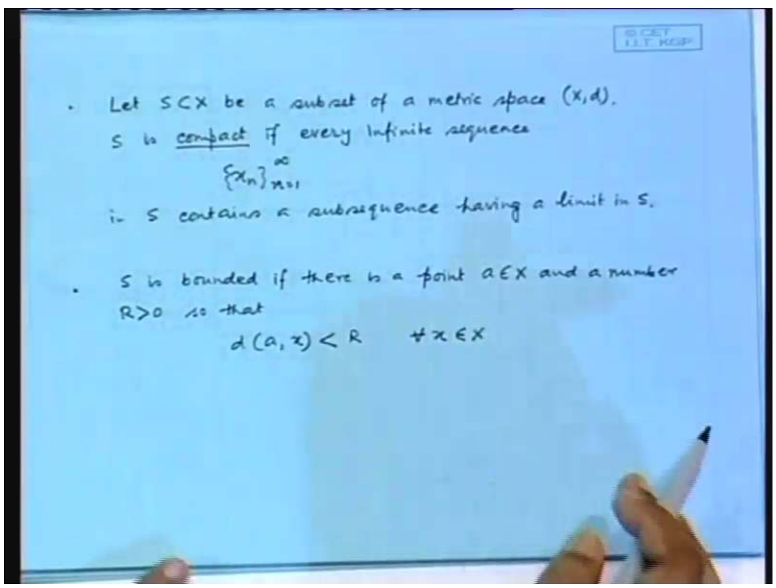
There by you can compose an image, we can make an image. What it will ultimately lead to is an image something like what is shown in this screen. This is actually union, can you see the union. If you look at the image on the screen, can you see that it is a union of three things, one part here, one part in the lower side, another part in the left hand side, another part in the upper side and the whole thing with some transformation yields this lower part, yields the left part, yields the right hand part and we have the whole image as a union of the that thing.

(Refer Slide Time: 00:46:37 min)

So let us try another. So here is an image which is pretty common image. This is the image of the fun. This is also obtained by the same method and here also this is obtained as a union of a few parts. You can see there are many parts put together gives this image. So the point that I am making is that in order to obtain such interesting images, it will be necessary not to have just one transformation but a collection of transformation making a full transformation. So we will say that from the set say x, if we apply say w_1 on x we get say y_1. If you apply w_2 on x we get y_2, if you apply w_3 on x we get y_3 and so on and so forth and then our final image will be the y is the union of y_1 then we achieved the transformation from x to y. So a single transformation can be given this way.

So we have defined the property of completeness, we have defined the compact subsets. Have we defined compactness? No. We sort of intuitively said that our elements are compact subsets. What is compactness? Compactness is also a property that is obtained from the idea of distance but that idea is a bit mathematical. Let us go by the common-sense idea. The word compact would mean that it is bounded and close so the word compact is actually a representative of the idea of boundedness and closeness. So what are these?

(Refer Slide Time: 00:49:23 min)



A set S is bounded if there is a point a and a number R so that distance between a and the elements of the set x is less than R. So this R should not go to infinity that's all. So this will be the concept of boundedness and the actual idea of compactness is this that if S is an element or subset of the space x then this would be said to be compact if every infinite sequence x_n like this in S contains a sub sequence having a limit in S. So you have the whole space in which a subset is our S then within this you can define a sequence. So that contains a subsequence whose limit point is also included in this set x. that is what I said this definition is mathematically necessary but for our purpose we can very well do with the commonsense idea that we are dealing with essentially elements, subsets of the R_2 space that do not go to infinity. They are bounded, only that much property we will really use.

(Refer Slide Time: 00:51:02 min)

Look at the set on the computer screen, this is after all bounded. So we are talking about only the properties of boundedness, the rest are mathematically necessary but for our purpose to understand the basic idea that will not be necessary. So essentially what we have done is, we have defined a space in which the elements are compact subsets of the R_2 space. we are defined a concept of distance between a two elements then said that if we can properly take the steps then we can go from a point to another point in that space which means that I move from image to another image. We can define sequences of images and if that follows the Cauchy sequence property then we can say that will always on converge on to a particular image. Every sequence will converge on to a particular image and since this space is complete, this proof is rather involved so I will not go in to that. That is how that space is complete that means all possible Cauchy sequences do converge on to point that are members of the space.

Since that holds, we know that if we can define any Cauchy sequence that will always converge on the image and that image will be a member of the space and therefore we can always play with that space. So far we have done this. In the next class what will we do is, we will play with the mapping. We have already said that we can define a mapping as an affine transformation and we can define the total transformation, total mapping as union of the things that have been obtained by individual affine transformation.

The question then is can that yield a Cauchy sequence? Will that always converge? If that converges what will it converge on? So we will tackle with these problems but I can tell you, if you look at this screen whatever we have shown so far they are all products of this kind of Cauchy sequences which is defined on the Haursdorff space. We will need to identify and understand how to choose this number a b c d e f of the affine transformation. So that we can generate something as beautiful as this and the end of this set of lectures, you will be able to set this numbers to generate any fractal of it on the computer screen, that ability you will get at the end of the day. So that is the basic idea that we trying to convey in the next few classes.
Thank you.