Chaos, Fractals & Dynamical Systems
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Lecture No. # 15
Mandelbort Sets and Julia Sets

In the last class we had talked about the definition of fractals, the concept of fractal dimension and we understood that everything in nature, all the object in nature that we see they are all fractals and that is to be understood in terms of fractal dimension. Now today we will handle the problem of generating factors. Can we generate fractals by some kind of iterative procedure? In order to tackle that question let us first consider fractal with embedding space of a line. that means we have already learnt that factors are objects that lived in embedding space, underline space and the dimension of embedding space is either equal to or greater than the dimension of the fractal object. So in this case we are considering the simplest possible situation where the underline embedding space is one dimensional which means we are considering the line as my embedding space, so it's like this. So this is my space on which we are considering the generation of a fractal.

Now if this is the embedding space then on that we will chose a particular portion on which we will try to generate the fractal which is the distance between 0 and 1, say these 2 points. So we are attempting to generate an object between the range 0 to 1 on the real line and let me describe an iterative procedure in which we will first say that this distance between 0 to 1, let me first mark it but this is the range you are considering. Now in the first step what we will do is, we will divide into 3 equal parts and then eliminate the middle one third, remove the middle one third. So that we will get after the first iterate something like this. Here is one chunk and here is another chunk. So from this point to this point and from this point to this point which means we have removed the middle one third. So this is a procedure of as you said that it is a procedure of removal of the middle one third. So this is coming from this full line 0 to 1 to this is one step, one iteration.

In the next iteration what we will do is we will continue this procedure which means at this stage we have left with two chunks, we will take each of them and remove this middle one third. So we will have an object like so, from here to here, from here to here, from here to here and this. So this is the second iteration of this iterating procedure and in next step we will have to further subdivide this 4 segments into even smaller ones and remove the middle one third, so we have this and so on and so forth. If you continue in this procedure at infinity term then what do you thing we will have at the end? At the end if you go on doing this procedure, you can see that it every step we are removing some part of it. So we started from the full real line but in every step we are removing some part of it, ultimately when we do this procedure infinite number of times but we are essentially left with a collection of dots on the real line. Each of the segments will shrink to zero size or will be subdivided, subdivided and so on and so forth finally will be left with this sprinkle of dots on the real line, look at it in some place.

Now the question is finally what we have is a sprinkle of dots, a collection of points a set. A set is called cantor set. The set is obtained by the removal of middle one third procedure, finally we land up with a collection of dots. The question now is that a fractal, in order to address this question we will have to obtain the dimension of the subject.

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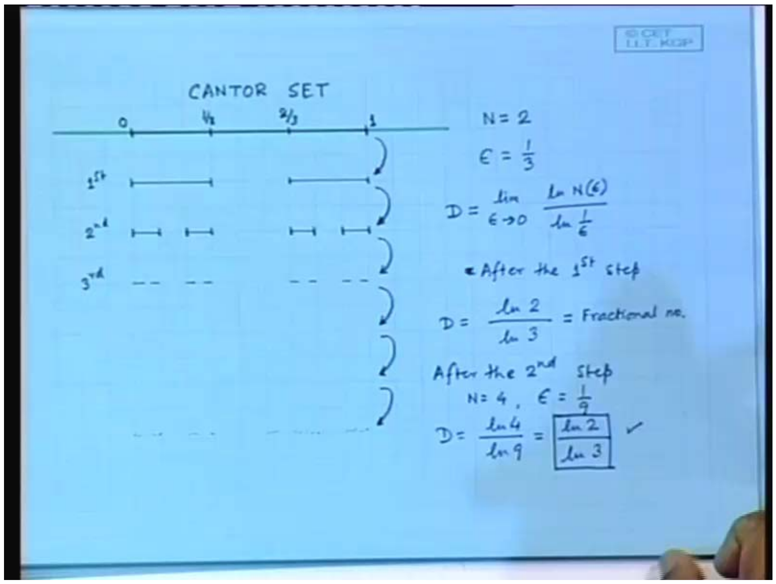

Now in order to obtain the dimension of the subject as we have learnt already, we have to divide the underlined space into boxes and count the number of boxes that are necessary to cover this object. So here we need only to create boxes within the range between 0 and 1 and we could say that we will subdivide this into ten number of boxes so 0 to 0.1, 0.1 to 0.2 so on and so forth and then count and further subdivide the boxes and go on counting but it is not to difficult to see that if we subdivide the zero to one range into boxes in a different way, into boxes of size one third, zero to one third, one third to two third and two third to one then we have some advantage. What advantage? Let see after the first step if you have divided the boxes into this 3 boxes, range between 0 to 1 into 3 boxes the first one between zero to one third, second one between one third to two third and third one between two third to one. Then how many boxes do you really need in order to cover this object? 2, so we have N =2 and what is our epsilon? Epsilon is one third of the whole range, the range is one and therefore one third of that so epsilon is one third.

So by the definition of dimension that we have already learnt about, if we look at the dimension then dimension is limit epsilon tending to 0, it is ln N of epsilon the functional of epsilon and by ln 1 by epsilon. That’s what we have learnt as the definition of dimension. So if you do that, if we consider the dimension that can be measured after the first iteration, first step then will say after the first step if you measure the dimension then D is equal to, then this point does not arise we are only trying to measure this quantity (Refer Slide Time: 09:15) this will be ln, N is 2 so 2 by ln, 1 by epsilon is 3, this is a fractional number lying between 0 and 1. Now you might say that this is not the final step, you obviously go on doing the procedure over and over again.

So let's go to the second step. At the second step if I consider after the second step we are here, so this is the first step, this is the second step, this is the third step so on and so forth. So we are here now, at this stage if you say that we will further subdivide the boxes into 9 boxes that means earlier it was from here to here, from here to here and from here to here. Now we will further subdivide the boxes into three inch that means this part is divided into 3 boxes, this part divided into 3 so on and so forth totally 9 boxes. Then how many of these boxes are now required to cover this object? At the stage N = 4 and epsilon is 1/9. So D is ln 4 by ln 9 which you can easily say this is equal to ln 2 by ln 3. So what we achieve? See by going from step 1 to step 2, we find that the dimension as measured like so remains the same.

As a result it's not difficult to conclude that if you go on doing this procedure at infinity term and at each step you also make the epsilon, the box length further smaller by the same degree that means each one is one third of the previous one. Then we end up after the infinite number of iteration with the same dimension therefore when we have arrived at the final thing, the collections sprinkle of dots the cantor set its dimension should also be the same which is ln 2 by ln 3, this is a fractional number. So we conclude that what do we have here is a fractal which is in the embedded space of one dimension and therefore that should have a fractional dimension between 0 and 1 which is true in this case. This cantor set is very important in the study of nonlinear dynamics and we will have to understand that in order to study of the nonlinear dynamics later. Now let us consider another situation where the embedding space is 2D.

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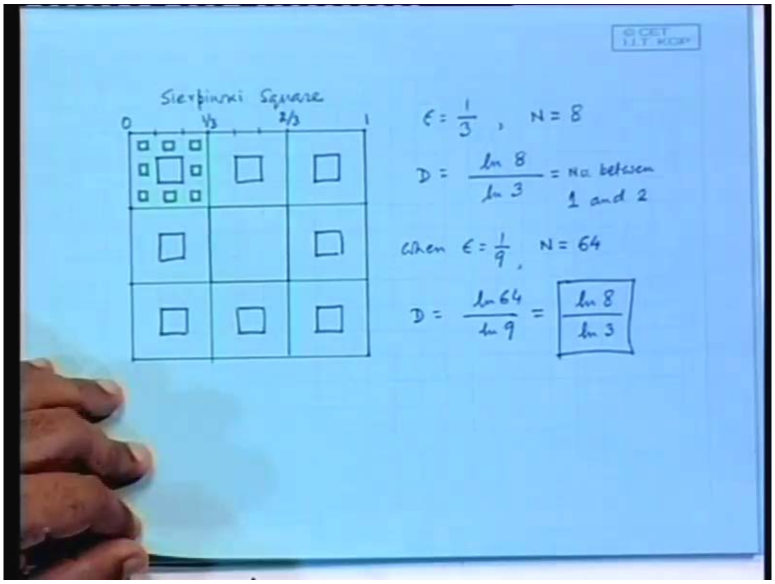

Now consider we have a square to start with. So even though our embedding space is 2 dimensional spread at infinity term in all directions of down, left, right but it will be sufficient for us to consider the range of space that is confined within this square. Now here also we will apply the middle one third removal procedure. What is it? We will remove the middle one third of the square. What is the middle one third of the square? It is essentially this square. If you remove that you see what we have done, we have divided the side into 3 equal parts and this side into 3 equal parts so out of that we have taken the middle one and we have removed that.

So that is the middle one third removal procedure as apply in 2 D space. So this is the first iterate. In the second retired what will we do? We are now left with 1, 2, 3, 4, 5, 6, 7, 8 of these smaller squares and we will remove further the middle one third. So what will we do? We will remove this, let me do it by hand because these are smaller. Here we will remove this, after this we will remove this and so on and so forth (Refer Slide Time: 14:55) so this is the second step.
In a third step whatever squares are remaining around it you can easily see that if we complete these then we are left with those squares, so here are those squares. Further if we take these then here is one square, another here, another here, so in these square there are how many left? 1, 2, 3, 4, 5, 6, 7, 8 and naturally in 1, 2, 3, 4, 5, 6, 7, 8 there would be total 64 squares left and in the next iterate you will further subdivide these into those smaller chunks and remove that and so on and so forth. This procedure will be repeated for each of these squares. Ultimately what will be left with? in this embedding space of two D we will be left with only some parts filled, rest of the parts remove empty and that will be a collection of points in the two D space another set. This set is called sierpinski square.

Now let us consider obtaining its dimension, this particular objects dimension. Now again we will have to cover this object, this particular embedding space with grid, with square because this object is a square object to start with so will subdivide but the question is how it will be most convenient to subdivide this particular box, initial starting point is the square box? How it will be most convenient to subdivide the square box in to the boxes? thereby the same logic it will be convenient to subdivide if this is between 0 and 1 then it will be convenient to subdivide into one third and two third sides. So that in the first step, you will meet these at the box, these as the boxes and then you will ask how many of this boxes are necessary to cover the resulting object after the first iterate.

So ultimately you have obtained a particular geometrical object, this sierpinski square and we are asking how many of this one third to two third to one, these square boxes are necessary for this object. Obviously when epsilon is one third which is a side of the box then at that stage, what is N? 1, 2, 3, 4, 5, 6, 7, 8. So N is 8. So at that stage that means when epsilon is one third what is the value of the dimension? Dimension is ln, N epsilon which is 8 by ln 1 by epsilon which is 3, so we have a number that is between 1 and 2. So we have this particular number as the representation of the dimension but when epsilon is one third. You might argue what happens epsilon is brought down further. for example it will be convenient to bring down epsilon to one ninth may be at the next stage so we have this chunks and so we are considering these boxes as a grid's and we are considering and we are asking how many of this grids are now necessary in order to cover this object.

At this stage so when epsilon is 1/9 then how many of these are necessary to cover this objects?
Notice in this particular box there are 1, 2, 3, 4, 5, 6, 7, 8 likewise there are 8 of this fill boxes and therefore we will need 64 of this in order to over this object. So at this stage D is ln 64 by ln 9 which is nothing but ln 8 by ln 3 so you see even if you have reduced the epsilon further, we have obtained the same value of the dimension so its stands to reason that if you keep on reducing the epsilon at infinity term, at every stage we will keep having the same value of the dimension and therefore the dimension is really invariant, while epsilon tends to zero. So this is the measure of the dimension of the object ln 8 by ln 3. So we have understood the sierpinski square.

There is another variant of the sierpinski construction which is the sierpinski triangular which is more popular than the sierpinski square.

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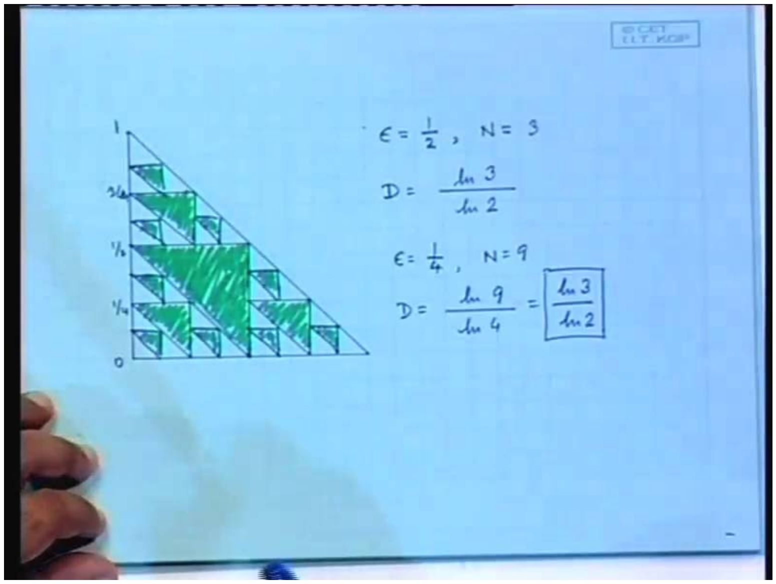

In this case we construct a triangle as a starting point and from there we apply the same middle one third removal scheme. So here is the starting piece and then we apply the middle removal scheme. so now here how many of the boxes 1, 2, 3, 4, 5, 6, 7, 8, 9 so half of that is 1, 2, 3, 4 and 1/2., 1, 2, 3, 4 and ½. So if we now construct it this way then the triangle is divided into 4 equal pieces and imagine that we are removing the middle portion of that. Essentially we are cutting this part out, to be left with a triangle shape. So here we have three triangles left, one triangle taken out so that is the first iteration in this iterative procedure.

In the second iteration what we will do? We will take each of these chunk and we will remove further and we will do like so. Here this is removed, here this is removed and here this fellow is removed (Refer Slide Time: 24:10). So this is the second iterate of that procedure where we are removing this part, this part and this part. In the third step we will remove even further. We remove this part, this part, this part and so on and so forth. So in every iterate we are keeping on removing parts and thereby taking out some chunk from the initial set the triangle and so we have removed this, this, this and so on and so forth. After the whole thing is completed, this is not complete of course there are triangles remaining. In the further iterates we will take again one third out of it.

So ultimately what remains is again a collection of a set, a collection of dots and we are now considering what will be its dimension. When we do that again, we have to subdivide the embedding space into the grid. The grid has the property that something completely covers the embedding space equally divides the embedding space. Now how we construct the grid is very much in our hands. we could say that we would do it the same way as we did for the square that means we take the whole box and then subdivide into squares and then count the number of square but it’s not difficult to see that a further refined procedure would be while we say that our grid squares or the embedding boxes these are nothing but the triangles. That means this is the whole triangle which is necessary to cover the whole object but then in the next step, we subdivide into this triangles that completely cover the space. In the next step we subdivide into the smaller triangles and then we count the number of triangles that are necessary to cover this object and carry on.

So initially the step was, we will subdivide this into 1, 2, 3, 4 this many triangles so that epsilon is the side length of each triangle. That means if the whole thing is zero to one then my side length here is half and now at this stage how many of these are necessary to cover this object? N is 1, 2 and 3 so 3. So at this stage when we have subdivided the embedding space into these bigger triangles then your D is ln 3 by ln 2. This is the fractional number, you might say that the definition has a limit epsilon tends to zero as the definition. So we will have to further subdivide epsilon so let's further subdivide into further half. So in the next step epsilon is 1/4 which means 0 to 1, here is one fourth, this is half, this is three fourth and this is one. So these are the triangles we are considering. How many of these triangles are necessary in order to cover the object? At this stage notice that this has been removed, this has been removed, this has been removed (Refer Slide Time: 28:34) so we need 1, 2, 3, 4, 5, 6, 7, 8, 9 so N is 9. So at this stage dimension is ln 9 by ln 4 is nothing but ln 3 by ln 2.

So you see even though we are subdividing the embedding space making epsilon smaller, we are getting the same value of the dimension which means again by the same logic, if we go on doing this iterating procedure at infinity term, we still end up with the same value which is this ln 3 by ln 2. Notice that this is the number again between 1 and 2 so in the underlined two dimensional space, we have created an object which is fractals. These are procedures to create fractals, very simple procedures and the determination of the dimension is also simple if you cleverly choose the embedding boxes but this may not be true with all fractals as we have learned. If you take a fractal coming from nature then we will really have to do it in the grid counting way. That means box counting, we subdivide the space by placing a graph paper on that and then counting the number of boxes necessary to cover that object but in such special cases where we are creating an object not taking in for nature but doing some artificial procedure to create a fractal then that dimension can often be obtained in such simple way. You might ask that we have also talked about special types of curves like the show line, the course line of England. So can we artificially create an object that is a same kind of property that it has an infinite length but yet it contains a finite area?

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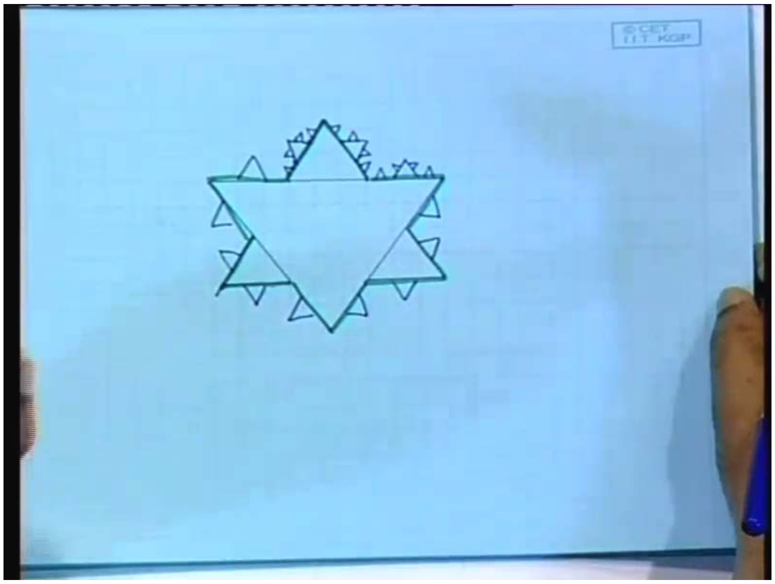

There is such a curve for example if you start with this triangle, now in the next iteration that is the next step we do not subtract something from it rather add something to it. So what we do is we divide each side into 3 parts and add a triangle to it. so we have this one here, some add the triangle to it so ultimately after this stage you have got this star like shape object remain. Now in the next step whatever sides are remaining, you again subdivide that into three parts and add a triangle to it. so you will have this one divided so added triangle to it, this one divided add a triangle to it, this one divided add a triangle to it and so on and so forth. In the next step take all the sides that are remaining and do the same procedure, so you will add here, you will add here, you will add here, you will add here and continue this process at infinity time.

Now in each step you are adding some link to it, naturally if you continue this process at infinity term, after infinite number of step you will have an area that is finite, you had the perimeter length would become infinite. So we can by some procedure create such objects that resemble in essential way the geometrical character of course line. Now we consider another way of creating fractals. This way comes from the study of dynamics. In the study of dynamics you have some kind of a dynamical system represented often by means of x_{n+1} is equal to some function of x_n which means that you start from the x_n, a particular position you go to x_{n+1} then that process can be continued. So that this defines a dynamical system where a point moves from place to place.

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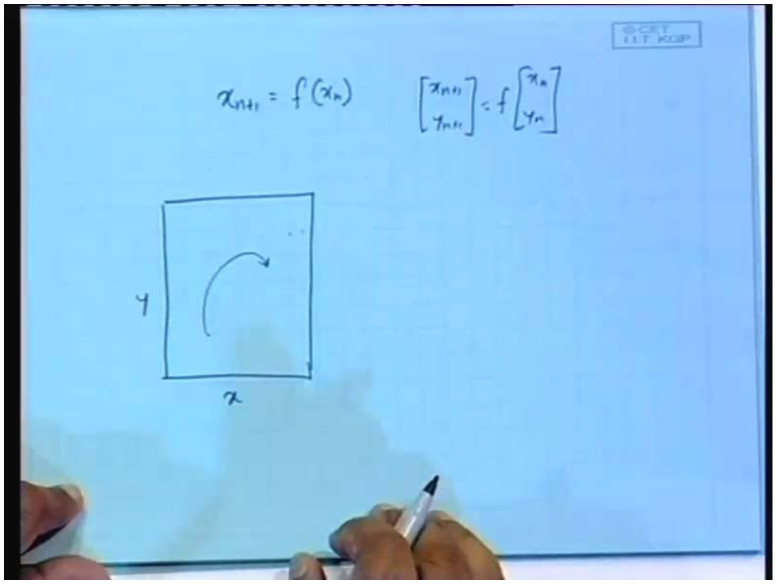

Now you can also say that these are vector that means x_n and y_n to x_{n+1} y_{n+1} so you have an x_{n+1} y_{n+1} which is given as a function of x_n y_n. So that what happens is if you take a point in that xy space, this point is a point by this function it maps to another point and so on and so forth, so that is a dynamical system. Now this way we are actually doing operations on vectors but it can be further simplified some time. If you define this nothing but as a single complex number which means it will have a real part and imaginary part. So that you will define it something like Z_{n+1} is a function of Z_n where the Z_n is nothing but Z real and Z imaginary.

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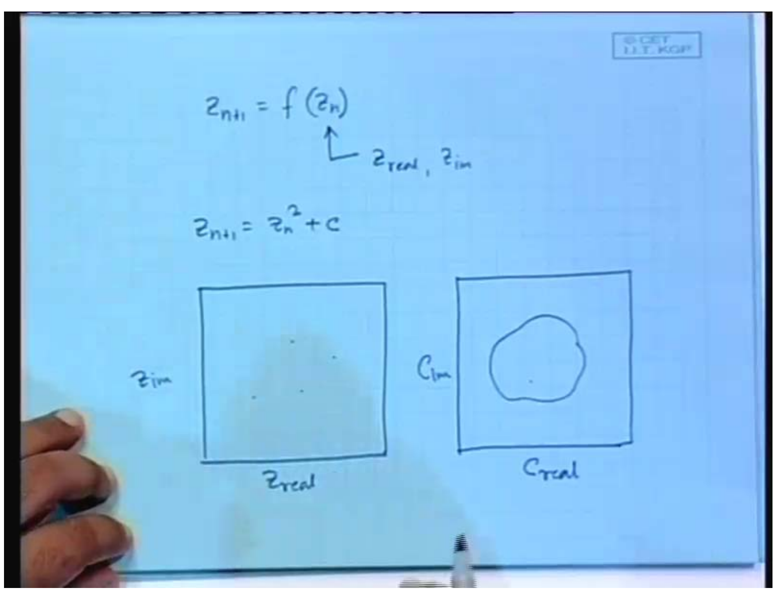

So there are 2 component of the Z and you define a function which takes that particular point in the complex plane to another point in the complex plane. In doing so let us define a particular very simple function like Z_{n+1} is equal to Z_n square plus c. They have seen that every dynamical system has the variables which as Z but they also have parameters, this is the c. So c is also a complex number, consist of the real part and the imaginary part. So Z is a complex number, c is a complex number and this is complex function that define the next iteration of that variable Z_{n+1}. So you might say that now I have a complex plane here, this is Z real and this is Z imaginary here and the starting point Z_n is say here. Then if you know the value of c then by computing this you can obtain this Z_{n+1}. If you know Z_{n+1}, if you know c, you can compute Z_{n+2}; Z_{n+2} gives Z_{n+3} and so and so forth, you have got a dynamical system.

Now the character of this dynamical system will depend on the character of the parameter, the value of c and c has two parts the real part and imaginary part. So we can also construct a parameter space where c real and c imaginary. So the value of c that we checked is nothing but the point in this. What happens is that for some values of c the iterations go to infinity. For some other values of c, if you go on iterating this remain bounded. So some values of c take the iterations to infinity, some values of c keep it bounded. So if you now say that I will construct the set that means a set of points in c, this parameter space c so such that if you take the c within the set then the system will remain bounded.

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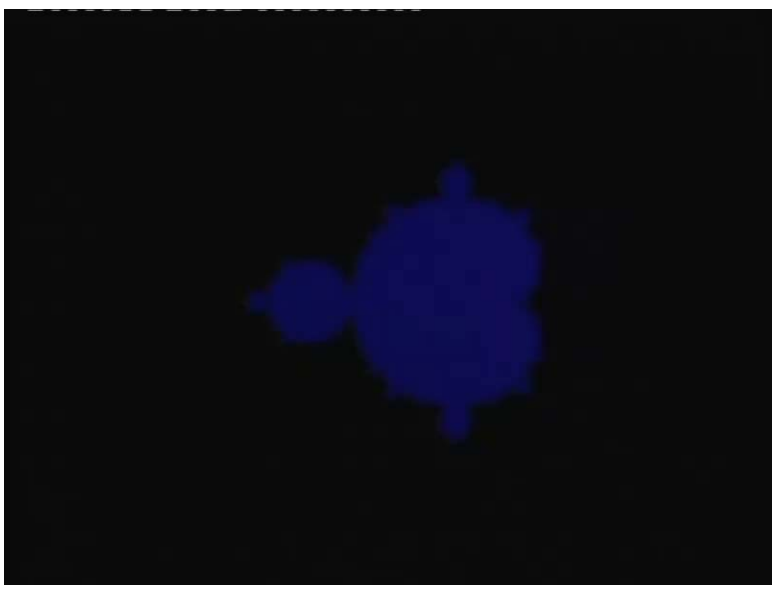

If you do that then we have what is now shown in the computer screen. just take a look at the computer screen, here is an object in the computer screen, the blue object is that set of c which keeps the orbit bounded and for each of these values of c, you might say that there should be some region in the state space which is the constituent of the real part of Z and the imaginary part of Z, some parts some set so that if the initial condition is here it will remain bounded, if the initial condition is out it will go out. So for every parameter c there will be a set in the state space for which if the initial condition is within that set it will remain bounded.

So here is the set in the parameter space and here is the set in the state space. What we are seeing on the computer screen now is this set in the state space, this blue set. Now the question is; is this a fractal? For each point of this, in the right hand side if you can see a particular set if we now blow it up, you see there is a set here. This is the set in the state space and the one earlier where the set in the parameter space.

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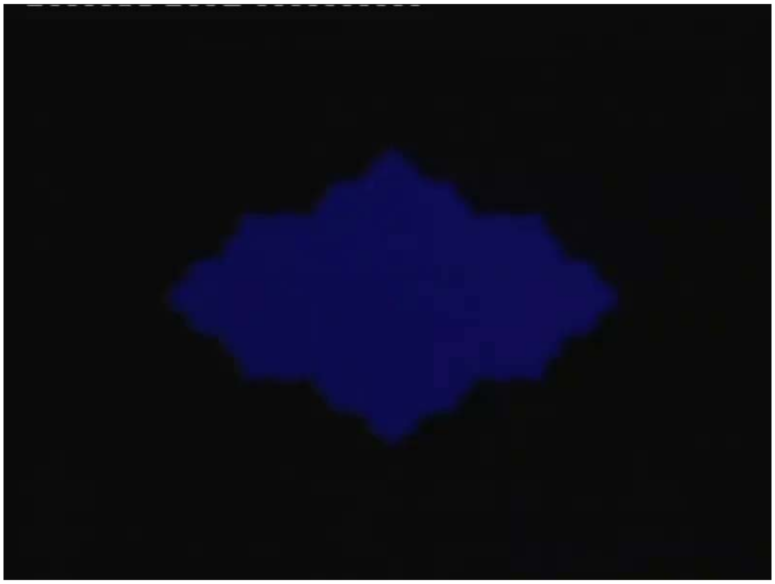

This set is called the mandelbrot set and this set is called as Julia set. So essentially this Julia set is the basin of attraction and the basin of attraction can be constructed for every value of the parameter so every point in the mandelbrot set corresponds to a particular point in the Julia set which you can see here.
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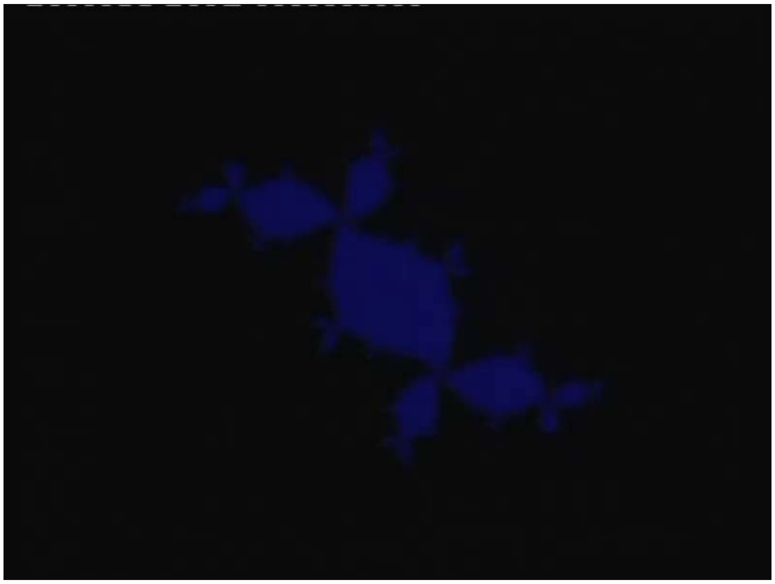

So for every point in this if I change the point, you see the Julia set is changing which means that we know look at this Julia set for that particular point, this is the Julia set which means that is the basin of attractor if the initial conditions are within this basin of attractions then it is attracted to some kind of fixed region. If it is out it goes to infinity. Are this beautiful? You can make them even more beautiful by saying the region that is showing in blue outside this region points goes to infinity but they go to infinity at different rates. So if you now say that I will color the outside region depending on how fast they go to infinity then we can do that by choosing proper option.

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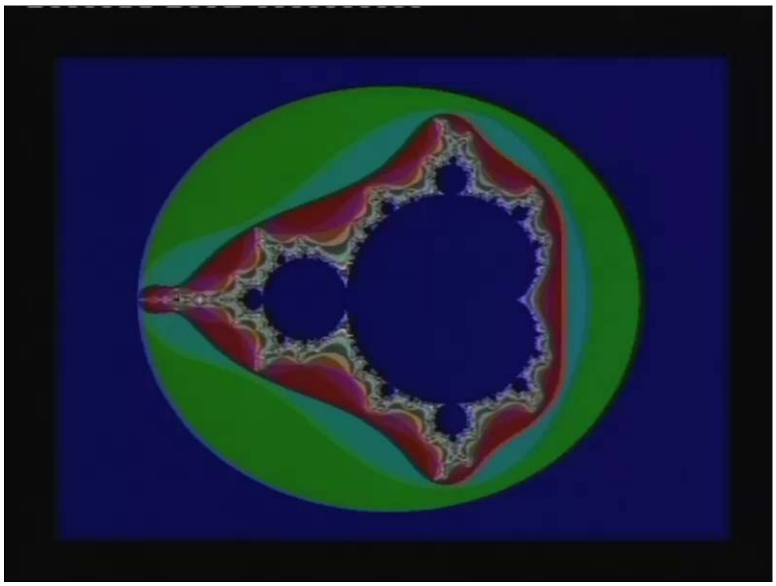

So here it looks beautiful much more appealing because the outside part we have colored depending on how fast they go to infinity and here also, if you choose the particular region, if you now move the cursor somewhere then you can see a particular picture. So this is the corresponding Julia set.

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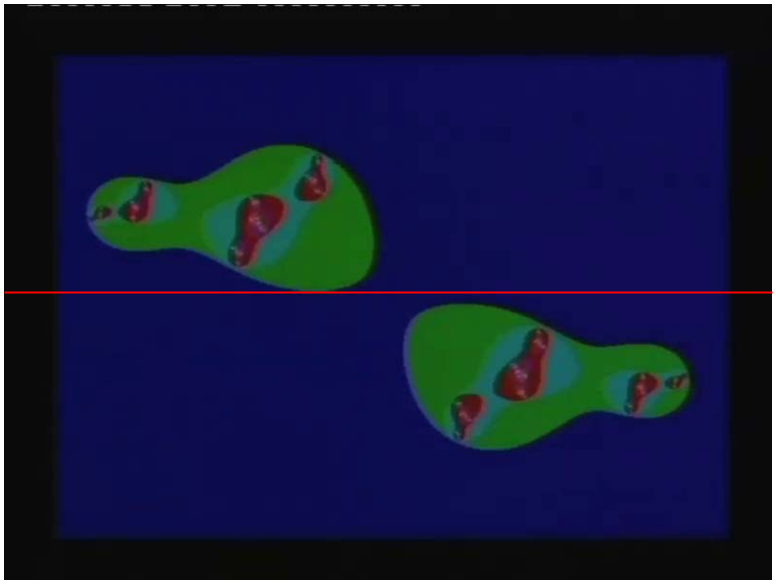

Suppose we probe this question. Is this a fractal? Here we are considering this perimeter. The perimeter of this middle blue region is that a fractal? What we will have to do? We will have to enlarge it so let us try to enlarge it by moving it in a proper place and then expand it here.

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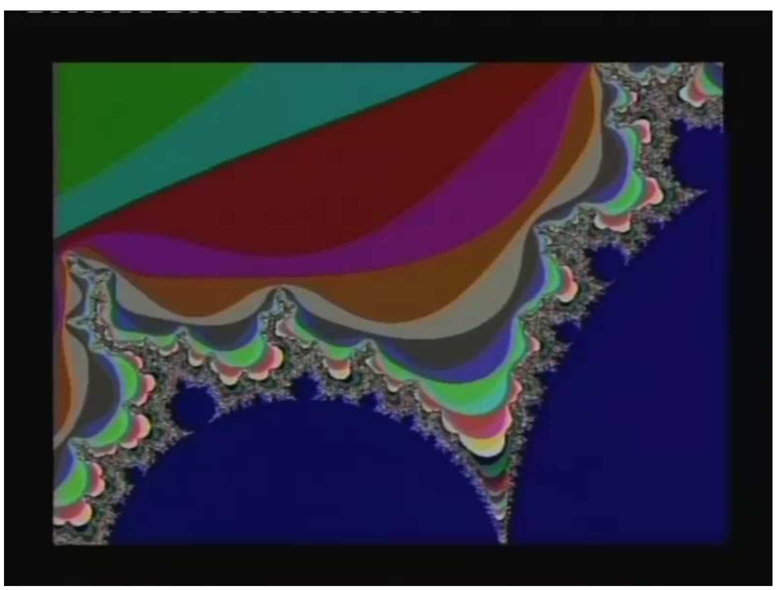

So let me expand it. You see more and more colors and more and more shapes and sizes. You might say let us further enlarge it so we will make a particular region and then we will enlarge only that part.

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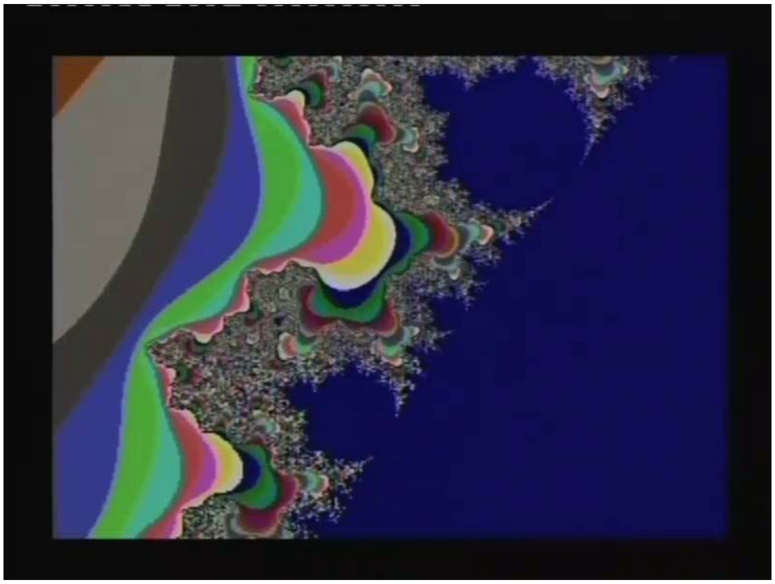

See more and more colorful beautiful shapes. Let us further enlarge it, so you see structures with in structures.

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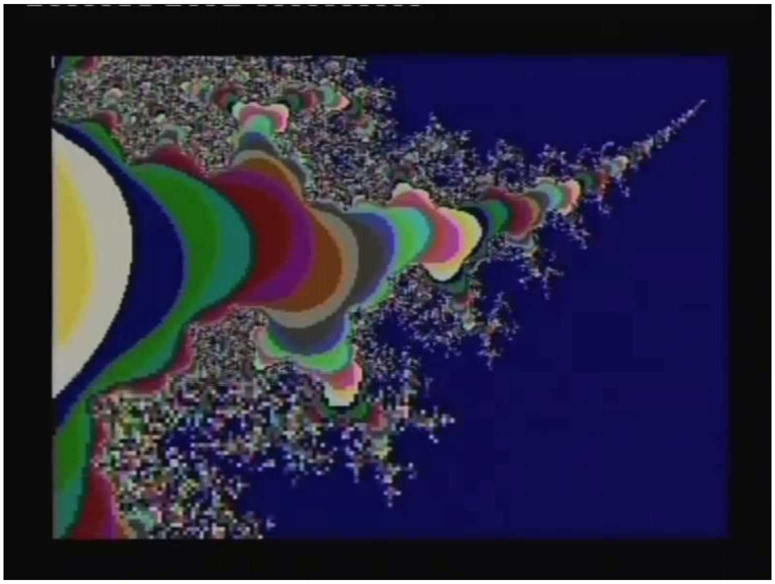

This is structures with in structures that were not visible, unless you go deeper into it and I can tell you that here is a shape, here is a structure. If you go even deeper you still see structures with in structure which means that this object is a fractal object. So let me go back to the original one, here was the original one. We said that if we choose a particular part, we see its corresponding Julia set. So we have corresponding Julia set so now let us consider what we are really looking at. If you concentrate on the image on the paper, we are considering the Mandelbrot set.

This is nothing but the parameters space of the system. You know that any system has some parameters, also for any system electrical system will have inductance, capacitance, input voltage and stuff like that as parameters. A mechanical system will have the forcing function, the value of the forcing function, the value of the mass, the stiffness of the spring and stuff like that as parameters and this tells us that if we now paint a picture of the parameter space for which a certain kind of the behavior will occur. In this particular case we are considering bounded behavior, bounded behavior will occur then we get an object on the computer screen which is a fractal object.

Again for each point in the mandelbrot set, if we consider the basin of attraction that means the set of initial condition for which the orbit remains bounded then we get another set which is the Julia set. So for every point of this mandlebrot set we are getting a Julia set. Now here it is not difficult to write a program to do this. It is rather simple. All you need to do is to consider the complex number and consider some kind of a range, if it goes outside that range you will paint it in one color, if it remains within that range after some number iteration say hundred then you paint it with another color and this coloring scheme is after how many iterations does it go outside that range. So here is a readymade program that is called fractint, I will quickly demonstrate it.

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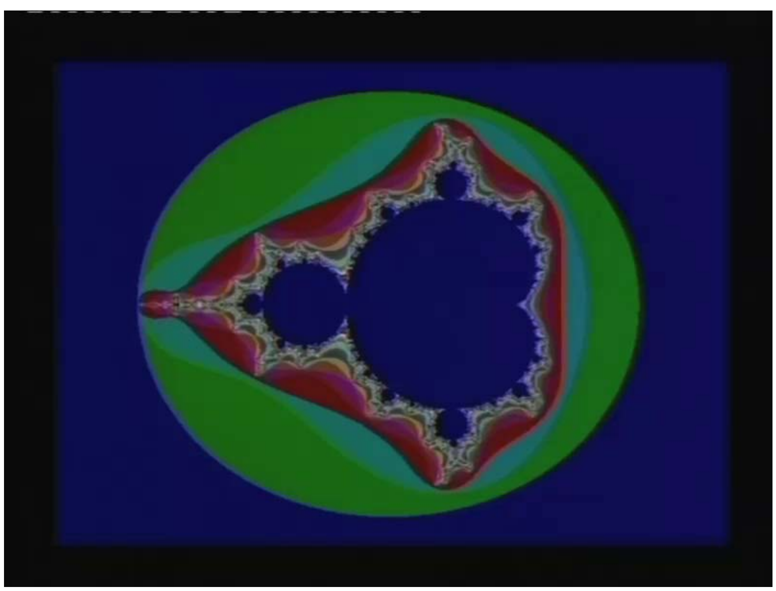

This program is called fractint. So first we have to select a video mode. We are selecting the video mode of this one. So the moment you choose, you have got it. So for every point in this, you get a plane. So now I am moving the cursor to a part that is closer to the edge.

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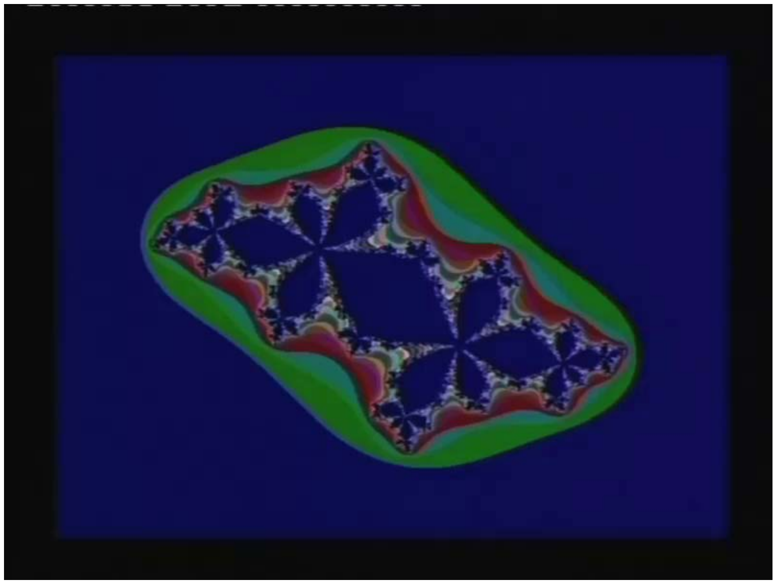

Now if I plot the Julia set, I get a beautiful structure. Now again I go back to the Mandelbrot set and I go to another point in the space say here, if i want to plot the Julia set, it's another color. So the point that I am making is that in this mandelbrot set, there are infinite number of points and for each point there is a Julia set which has a different shape, different structure. So this is an infinitely complicated geometrical object and infinitely beautiful also. The moment you start plotting it with this colors, you find that it has immense beautiful. So now let me again do that by taking the point somewhere there.

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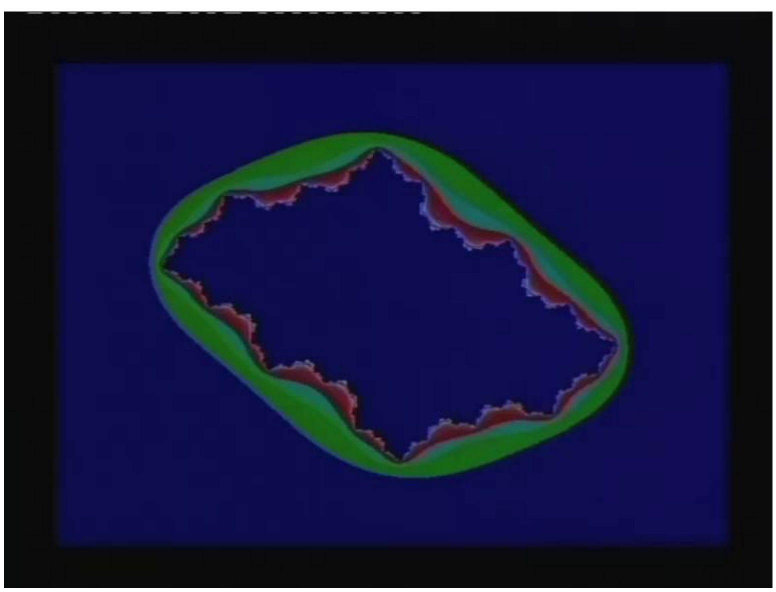

If you take a point inside you get a region filled with blue that means here if you move the cursor around it, you will find that this is the region, if you choose the initial condition beyond that it goes to infinity and coloring outside the region is by the same method that has just illustrated. That is if it is within the red region then it takes a certain number of iteration to go outside the area that we have said. That if it goes outside the area we will say that is going out, running to infinity. So how fast it is running to infinity that is designated by this colors. If we take the parameter inside the big circle, big area, big region then you get a relatively structure less object but if you go to the edge then you get very many structures.

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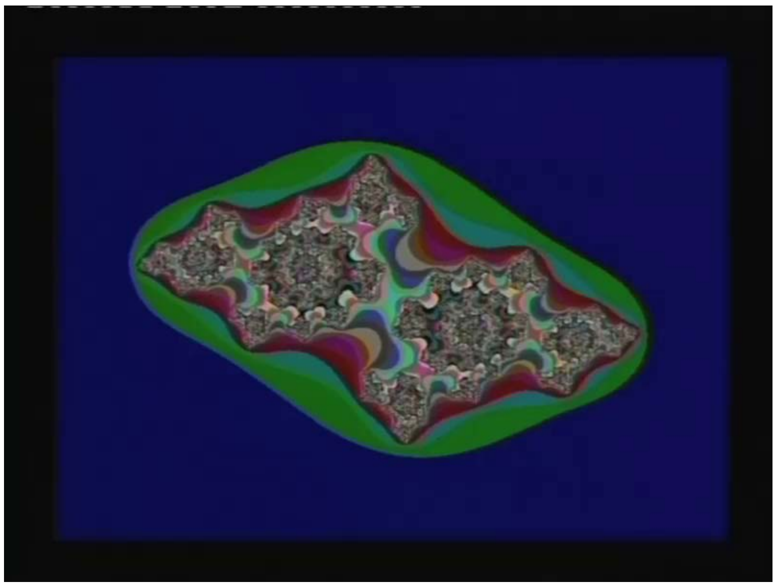

So let us illustrated it even further by showing a few points on this.
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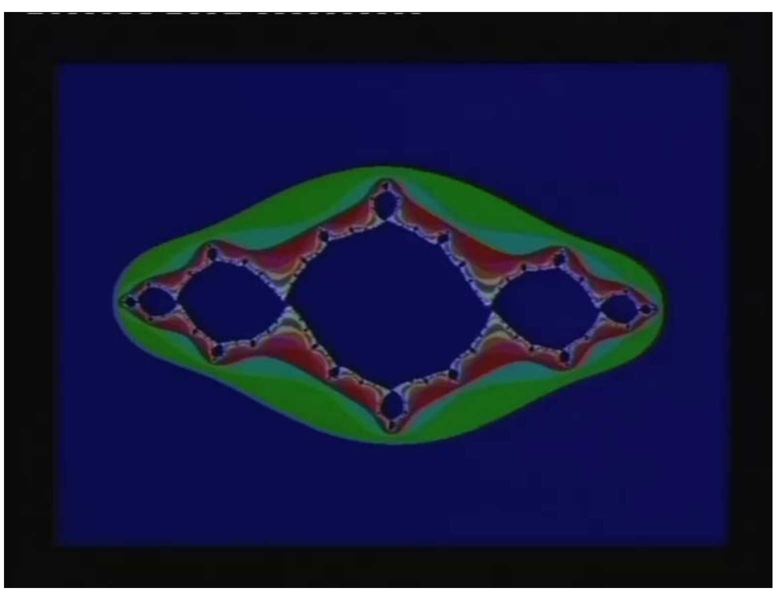

If you go to the relatively smaller region then you have got another shape of the corresponding Julia set. If you go further to this region, you get another shape of the Julia set. Quite beautiful.

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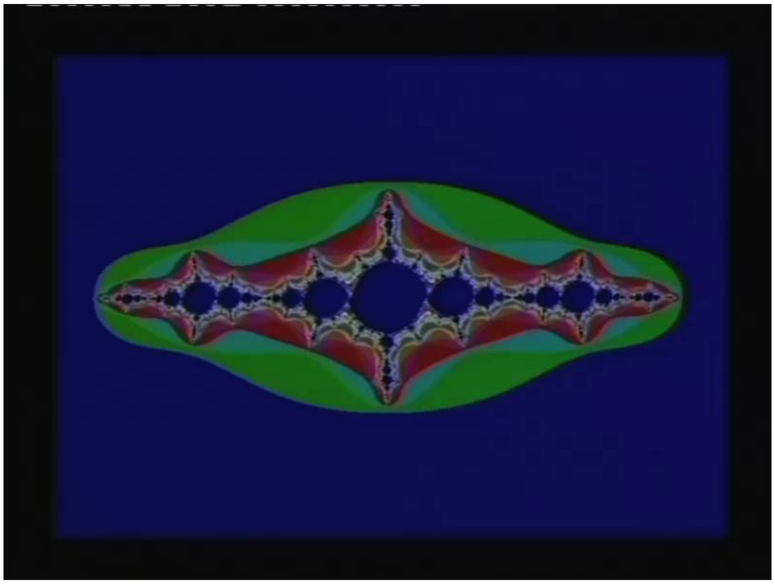

Now if you go further up, we get a wonderful Julia set. So the Julia set are things of very large complexity. Now if you want to probe the issue whether the Julia sets are fractals are not? Then we can enlarge that.

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We have got nice pictures, again we can enlarge some part of it.

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So we see where we initially believe that these are relatively simple regions, they are not simple enough.

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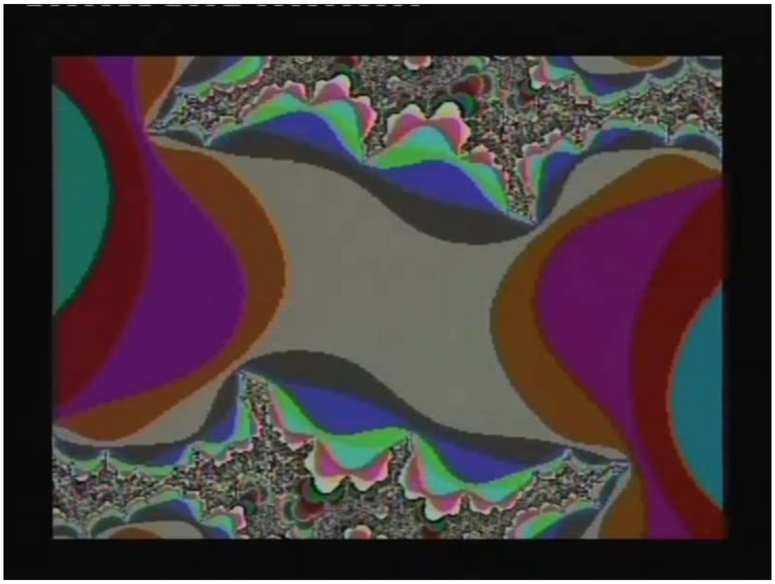

So here if we look at each small part of it, you will still see nice fractal structure and I am reducing and bringing down to an area which may be of interest.

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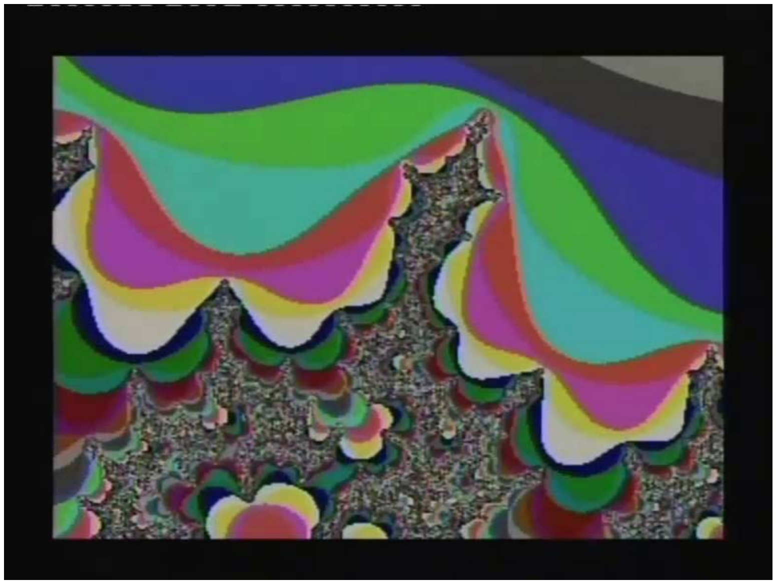

So you see there are further colorful structures. So the Julia set and the mandlebrot set have become the epidermis of fractals but one thing that is often missed in text books is that they really represent dynamical systems and the lesson to be learnt is that in every dynamical systems there are parameters and you can construct a parameter space. There are states and you can construct the state space. This is the model, a toy model of a system that is dynamical system and you can have both in the parameter space as well as in the state space you can have fractal objects. There are certain conditions under which the basin of attractor becomes fractal. There are certain conditions under with the set in the parameter space becomes fractal but in general you can expect fractals to exists in the parameter space of dynamical system as well as in the state space of dynamical system.

This is the lesson that you have learnt. It is possible to obtain the fractal dimension of mandelbrot set and the Julia set but the procedure is not as a straight forward as you have applied seen for the shape must be triangle or the shape for the square or the canker set so we will live that you can easily obtain it and apply the box counting procedure in order to obtain it. So there should be end for today but in the next day we will handle further procedure of generating beautiful fractal objects as a valid subject was studied in general.
Thank you.