Chaos, Fractals & Dynamical Systems
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Lecture No. # 14
Introduction to Fractals

Now this course deals this with fractals geometry, a discipline of mathematics that has developed over the last say above 20 years and it has found a lot of application in various disciplines of science and engineering and that's why it has attracted much attention and also imagination of adventurous to this. This course will try to cover in a span of a few lectures. The basic idea is that of geometry. Now what does geometry deal with? Geometry deals with objects and spaces. This is something that is often missed in regular geometry lectures that geometry deals with objects and spaces. Points, lines, spears, rectangular parallelepipeds these things are objects and they live in spaces. For example a one dimensional or a zero dimensional point can lie on a one dimensional line or a two dimensional plane.

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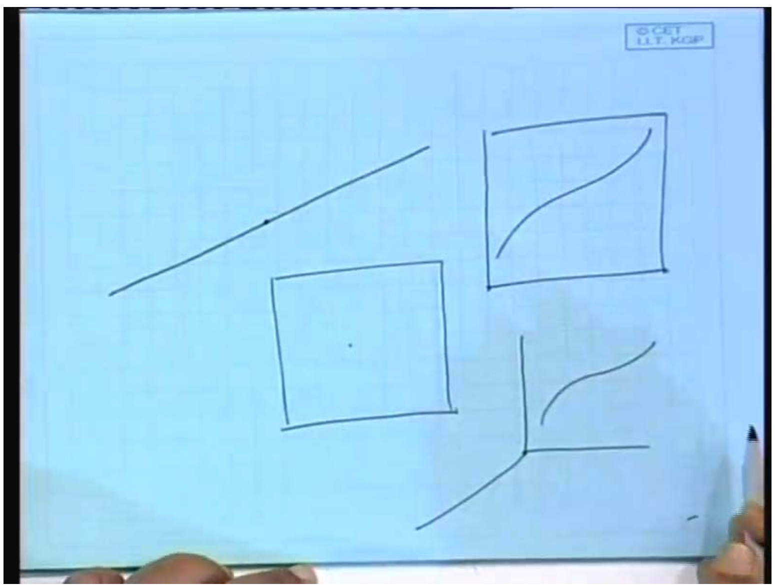

So a point which is zero dimensional object can lie on say a line in which case we'll say that zero dimensional object is lying on a one dimensional space or it can lie on a two dimensional space. Likewise a line can live in a two dimensional space or a curve can live in a three dimensional space. In other words what I am trying to convey is that there are two types of objects that are of interest in geometry, the objects and spaces. The objects live in spaces and when we talk about objects living in spaces, we always find that they live in spaces of dimension either greater than or equal to dimension of the objects. Now for much of the human history, people have studied only regular objects, idealized objects like a sphere, triangle, rectangular parallelepipeds. Things that we really never come across in our daily life spheres, triangles.

Obviously these are idealization from what we actually have in daily life and this is sort of a legacy of Pythagoras and Euclid. They device the concept of idealizing what we see in actual reality in nature, studying the property of those idealized objects and thereby we have a whole lot of Euclidean geometry it deals with such idealized objects. These are objects in the matter spaces, also we were talking about either a flat sheet of a paper for a two d space or a three d space with a flat characteristics which means mathematically that if we have two points their minimum distance is drawn by the Pythagoras law. So you have the idealized objects and the idealized space. Now by the end of the nineteen century there was a revolution in mathematics, especially in geometry through which to the work of Riemann, Gauss, Lobachevski people like that we sort of learnt, there could be other types curved spaces, positively curved space, negatively curved space and those things and Einstein found a very profound application of that in the theory of gravitation but objects still remained the ideal ones. We still talked about triangles or one of the classical things by which we illustrate the difference between a curved space and a flat space is the angle between a triangle.

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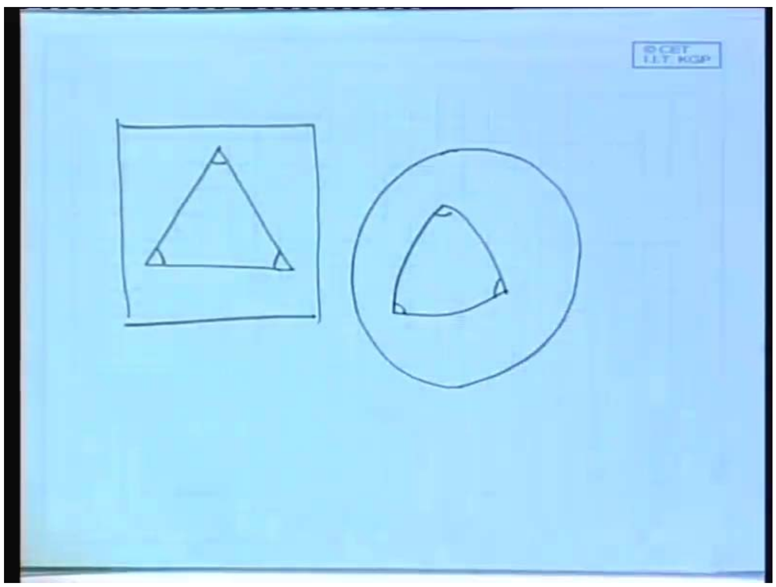

So if you have a triangle drawn in a flat piece of paper, the angles contained will be 108 degrees. While if you have a football or the surface on the earth and if you try to draw a triangle over that the angles contained will be larger than 180 degrees. So that was a difference between the 2 types of geometries, the flat geometries, flat space and the curve space but the object still remain a triangle, an idealized objects and only over the last say twenty years or so we are breaking out of the narrow confines of the idealized Euclidean objects. Now science is beginning to recognize that the things that we really see in nature, things that are far from idealized, things that are crooked, curved, bend and they do not look like the things that we study in school geometry text books. Those things are actually valid subject of study in geometry and fractal geometry is essentially concerning that. What we really see around us, the things around us, their geometry, their specific properties that is what fractal geometries all about.

So having laid this ground let me come to the point that what exactly is the difference between the things that we see in nature and the things that we studied in our school text books, the objects. One of the aspects that often if you think of it which will immediately catch your attention is that the non-idealize things or if you have an ideal curve given by some kind of a function, y is equal to f of x this is an idealized object given by some kind of function.

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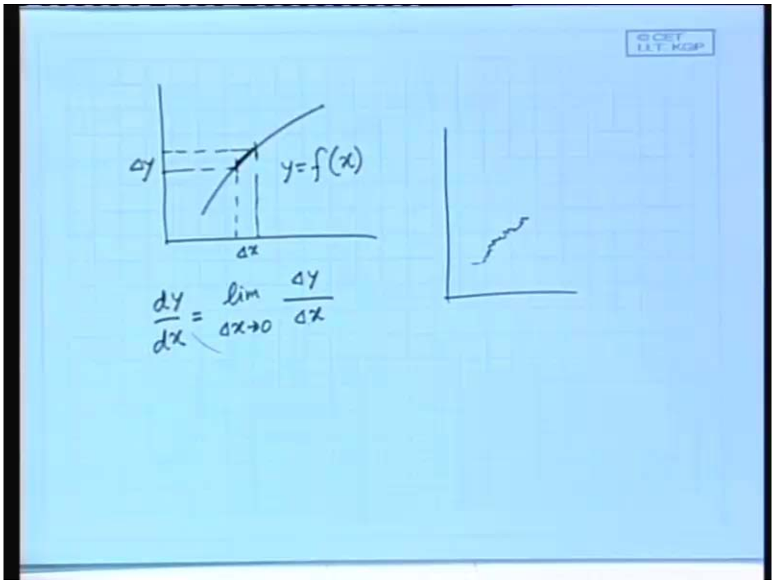

Now if I ask you what is the difference between this object and the kind of curves that you see in nature? One thing immediately will come to your attention that if you now look at a very small range of this then it almost resembles a straight line. Take any idealize curve like the circle, the parabola like whatever curve you can think of. If you look at smaller and smaller portion of that curve, it will always flatten into a straight line and in fact that is how the derivative is defined because the derivative is nothing but dy by dx is nothing but limit of delta x tending to zero delta y delta x. So what we do? We have this term, this part as delta x and this term as delta y. so we vary x by a (Refer Slide Time: 00:09:41) bit and we observe the resulting variation in the variable y. We have this delta y by delta x and we take a limit 10 into 0 and that is what we call dy dx derivative of y with a respect to x at this point.

If these be the definition, obviously it rests on the assumption that as you bring this one closer as this one comes closer, it actually approximates a straight line and this straight lines slope is this fellow. So there has to be the straight line existing but there lies the main distinction between the idealized curve and the actual curve that we see in nature. The main difference is that the actual curve that you see in nature are not differentiable anywhere. Meaning that you come closer this slop continuously keeps changing, it’s not differentiable. It’s this particular thing does not exist as a unique number. So that is the main distinction between an idealized curve and the curve that we come across in nature. The ones that we come across in nature are continuous but non-differentiable geometrical objects. Let’s give some examples for such curves in nature.

For example for the electrical engineers it will be easy to visualize, if you continuously keep a record of the load on a power station then you will get a some kind of a graph relatively but there will be variations all the time because somewhere somebody is always switching on and off some power, it's varying and that variation if you zoom on in a very small scale, you will find that this terms does not exist. Meaning that this is a non-differential curve. Similar is the situation with the recordings of the oscillations during earthquakes. If you have that as the curve, that as the geometrical object and if you zoom closer you will find that it is does not smoothen into straight line and thereby this term does not exist. So you see that there are existing curves in nature which does not follow this property that the derivative will exist. In fact they are curves in fact most curves are such that they are continuous but there is no derivative.

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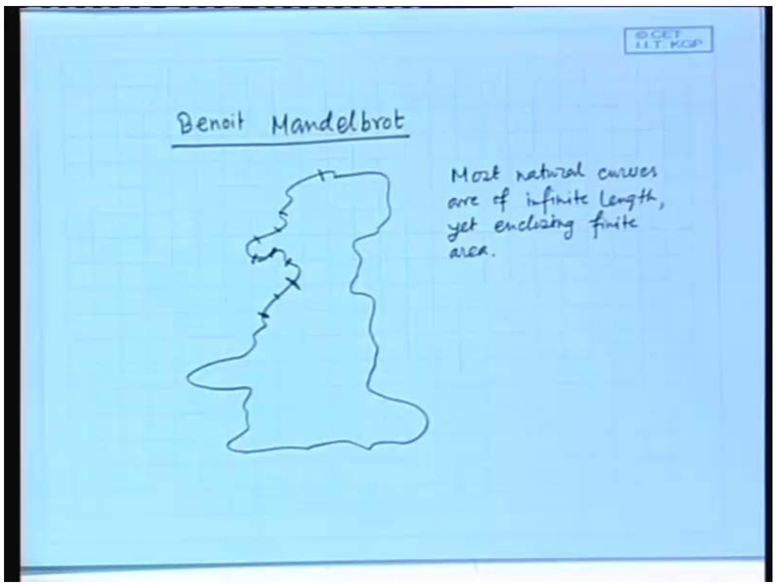

The next major distinction can be illustrated through the person who invented this fractal geometry. His name is Benoit Mandelbrot, this man asked a famous question. He said can we measure the length of the coast line of England, how long is a coast line of England? If you think it appears that okay, we can always measure, we will go around and we'll measure but the moment you start to ponder on this question, you will realize the amount that will measure depends on the yardstick of measurement. Imagine the course line may be something like this. The highland of England very badly drawn but nevertheless it is. So if you have the yardstick of measurement say this long then you will start from here, you will make one section here, then a section there, then a section there, then a section there. As a result you will miss whatever happened in between. You might say let's reduce it but so long as you had this yardstick of measurement, if you go on cutting this boundary and you simply add them up, you will get some number. You might argue to reduce it, so you come to this part, this part and so on and so forth.

As a result you still miss some creaks and crevices. As you go on further reducing it, you see more and more creaks and bends and finally when you make it one centimeter long, still you see bends around pebbles in the seashore and if you continue this argument and at every stage you find that the length that you are measuring go on increasing.

The more you are decreasing the yardstick length, the more the total measured perimeter of the Island is going up and as a result by continue this logic it is not difficult to realize that as the length yardstick of measurement that goes to zero, the measured length of the coast line goes to infinity, a startling result. This implies that the coast line of England or any Island for the matter is a geometrical object of infinite length, yet enclosing a finite earlier that’s important. So the conclusion is the most natural curves are of infinite length, yet enclosing. So that is the property of curves. What is the similar property of say surfaces? Take a natural surface for example our lungs. What is a purpose of lungs? The purpose is to absorb as much oxygen possible from the air, as a result the requirement is that it should have the maximum surface area exposed to air.

Yet it must have a finite volume, a small possible volume because it has to be accumulated within the ribcage and that immediately least of the conclusion that the surface of the lung must be of the same character that it is a surface of an infinite area yet enclosing finite volume. So you see these are the aspects in which the natural geometrical objects differ from the idealize ones, they are continuous but non deferential but this kind of objects then obviously need a different kind of characterization. The way we learn to characterize, the way we learn to understand geometrical objects, the same type of understanding will not work because we have to deal with non idealized objects as Benoit Mandelbrot once said that mountains are not cones, clouds are not spheres, lightening’s are not straight lines.

Obviously the way we understood those idealized object like cones, triangles, straight lines those things will not work when we try to actually understand the nature of geometrical object that we find in nature. So we need a different format, we need a different technique and for that comes the equation of dimension.

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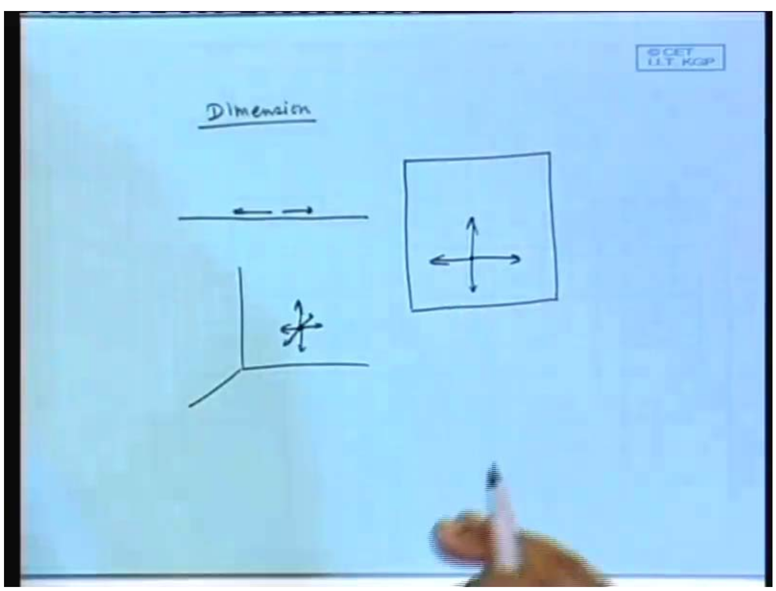

What is dimension? It is rather strange that we never come across the actual sense of the term dimension in school or college text books. Our idea of dimension is often based on common sense. I mean we sort of intuitively understand what dimension is like a point is a zero dimensional object, a line is a one dimensional object, a surface is a two dimension object this kind of sort of intuitive understanding we have and naturally our understanding is that the dimension is an integer. We have considerable difficulty in imagining any otherwise. Let me tell you that the ancient people had the same kind of difficulty when they had to imagine that there could be numbers, that will not natural numbers. the natural numbers can naturally do them like 2 men, 25 bananas, 6 children these are natural numbers but when man face the problem of measuring say length, area, time it was necessary to conceive fractions that just what they did and that is how the narrow confines of the natural numbers was broken.

Men broke free sort of limitation in concept. It seems that now it is time to ask to break free of the concept of the integer dimensions. Why? It must first be understood that the concept of dimension of an object and the dimension of the embedding space at two different things. This is something we often don’t understand and as a result we mess up the concept of the dimension of an object and the dimension of an embedding space. What is a dimension of embedding space? That is essentially given by the degrees of freedom. If you have say one dimensional space this means that you have freedom in moving left and right. That is the concept of the one dimensional space. If you have a two dimensional space, it means that you have freedom of moving left right as well as front back. If you have a three dimensional space you can move left right, you can move front back as well as up down, so you have three degrees of freedom.

So the dimension of the embedding space is related to the degrees of freedom. Naturally since degrees of freedom are integers therefore the dimension of the embedding space must be an integer but that does not mean the dimension of the object also must be integers because the object lives in the space and its dimension must be understood by the way it fills space and that is not a prior given that would be the same the dimension of the embedding space. Let’s illustrate how. What is the actual concept of the dimension of an object? The dimension of an object is as I told you essentially depend on how it fills space? So in order to illustrate what it means, how it feels space or what actually we understand by the dimension of an object let’s take a square as an example.

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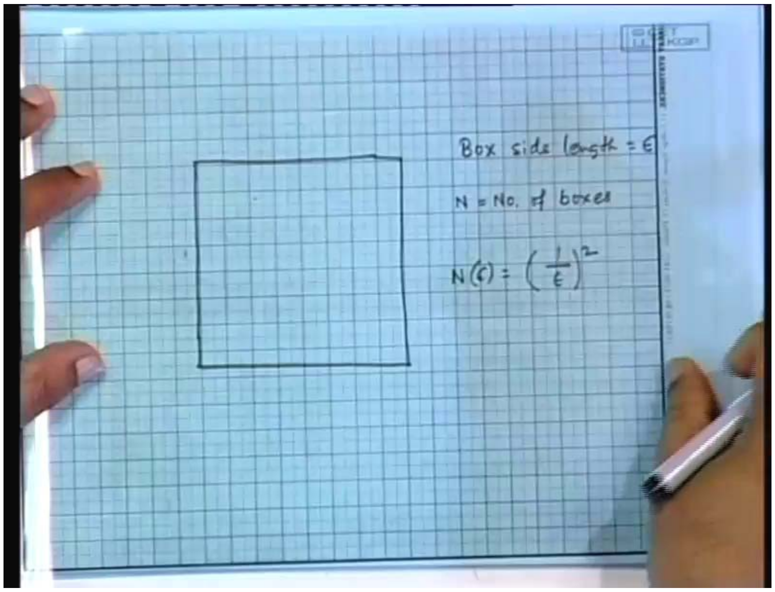

So this square as an object is living in the two dimensional space and we are trying to probe how does it fill the space. So what will have to do is to divide the embedding space, the space on which it lives into sort of boxes. We will have to divide the space into boxes and that we can easily do by placing a graph paper on it. So we simply place a graph paper on it, as a result of which we have essentially divided into boxes. Now we have to find out, these are the boxes so we have divided the embedding space into the boxes and we have to ask the question how does this fellow fills space essentially that means how many of this boxes are necessary in order to cover this subject.

So we can see that in this particular case we need 1, 2, 3, 4, 5, 6, 7, 8; 1, 2, 3, 4, 5, 6, 7, 8 and thereby we have 64 of these boxes. Now if I subdivide the space by a smaller box size for example let this be a boxes, smaller boxes. So each of the boxes that we consider earlier now let us divide into 4 boxes, probably you can see. This is one box, that is one box, that is one box, that is one box and so on and so forth. So how many boxes will be necessary now? In this direction earlier there were 8 boxes, now there will be twice the number of that which means 16 boxes, in this side also it will be 16 boxes so you have 16 square.

So in the next stage we can further bring down the size of the boxes, it will naturally go up, the number of boxes that will be necessary to cover it. How it will go up is that the box side length if that is epsilon and n number of boxes then we have the relation n which is a function of epsilon is one by epsilon square. That’s what we saw. When it was this big it requires 64, when it’s each side was half the size then we required 16 times 16 and even a smaller it will go up simply this way. So it will go up as one by epsilon square. So this is where the number two appears, you might have asked that you have taken a very simple object, what if you take not this object but something slightly different.

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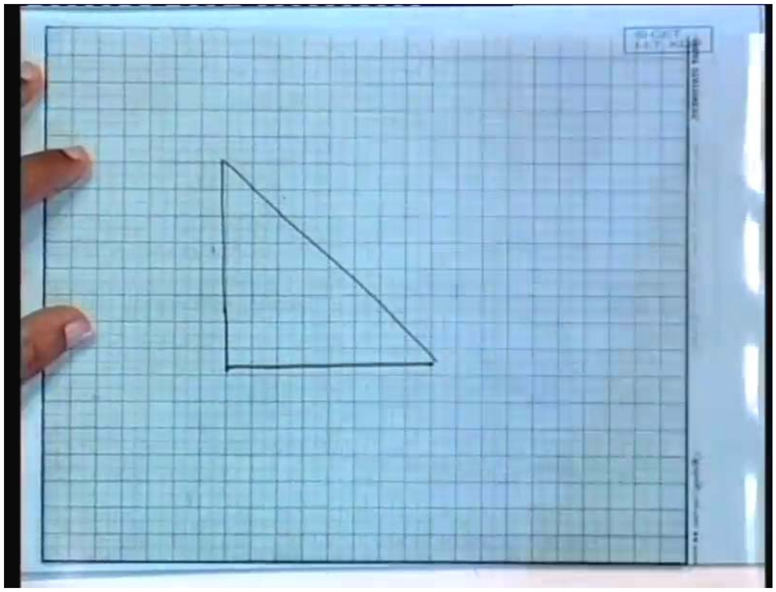

Say for example a triangle. So we take a triangle, we join them. Now this triangle if you place, if you take this triangle as the object that we have to investigate and we place the same graph paper on that then what would be the number of boxes that you need to cover it? You notice that initially you need larger number, you need this box because a part of the box is covered and is necessary in order to cover this object. Therefore these boxes fully has to be counted so 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21 and all that. So you can easily count but since it is not very well drawn triangle, I needed to have a scale but you can easily understand that you'll get a number. That is more than half the number that you needed earlier. Now if you further subdivide the boxes given then you would see, it will slowly approach whatever was necessary earlier, it will approach half that number but it will approach not that it will immediately be that number.

So initially you would need all these full boxes later you need only this box and this box and this box will go out, this box and this box and this box will go out and so on and so forth. So as you keep on subdividing, you will find that you are closely approximating this hypotenuse and as a result your number N. So this was for this square, for the triangle what happens N of epsilon will tend to I cannot say that it is exactly that because we have seen that it will tend to half one by epsilon square.

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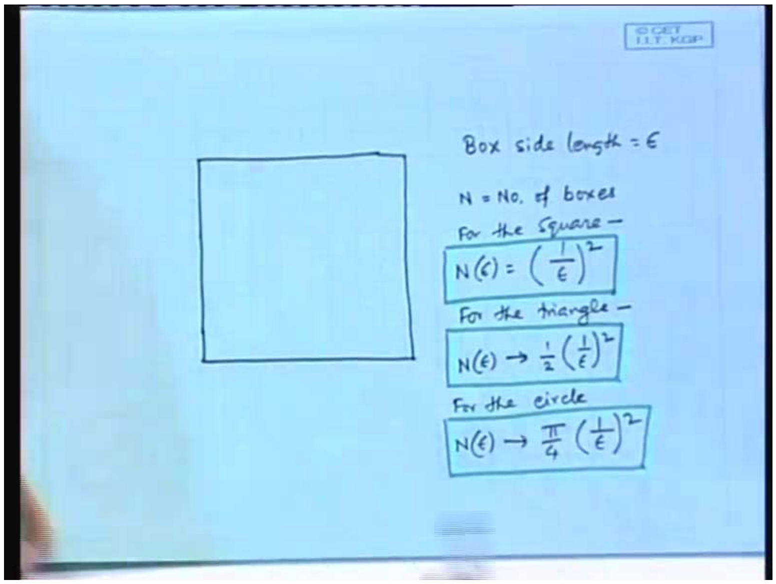

You might ask what if we take a circle, I am drawing by hand but it could have drawn by some more finer means and if we cover it will converge to this number. As you keep on reducing each box size then it will slowly converges, for the circle N epsilon will converge on to pi by 4, 1 by epsilon square. So notice these things this was for the square, this was for the triangle and this was for the circle and we know all these are idealized symmetrical objects. So as yet we have not gone in to the territory of non-idealized objects but we trying to extract what we understood by the dimension of the object.

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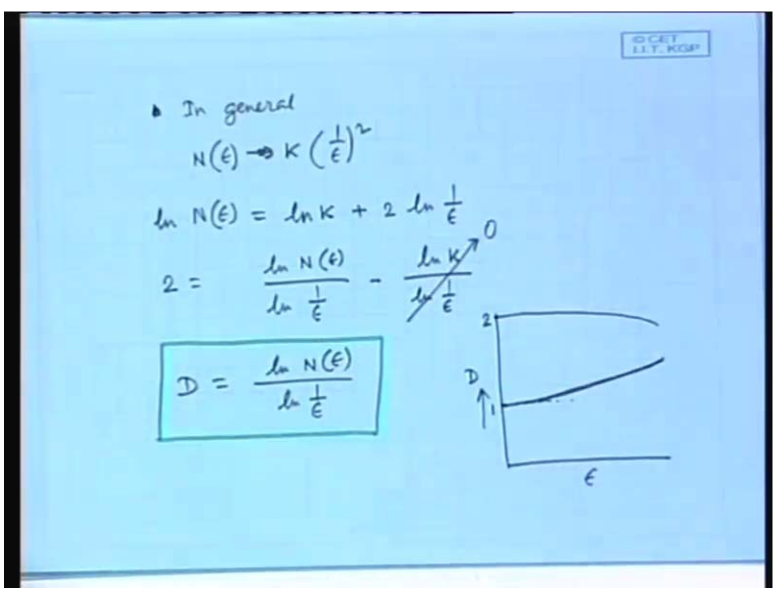

So once we have this, we can sort of in a general way we can write, we will keep this one for a future request but in general we can write some k into 1 by epsilon square. So it’s not equal to, it’s converges to. So far we had here k = 1, here k =1/2, here k = pi/4. In general for all these objects we can write k and then we ask the question how can we extract the number two from this. We can go the following way, simple school day algebra. We take the logarithm of both sides ln N epsilon is equal to, I am considering the situation after it has already tended to that is why I am using the notation of equal to. So this is ln k + 2 will come forward, ln 1 by epsilon. Now we are trying to extract this number 2 so we write it as, we take it in the left hand side we say the 2 equal to we have ln N of epsilon divided by ln 1 by epsilon minus ln k divided by ln 1 by epsilon, so this is the number 2.

Now we were seeking this number two because we knew that it represents the dimension of those objects. We had considered this square, we had consider the triangle, this circle we know that these are two dimensional objects so we trying to extract the number two. So as epsilon tends to zero what will happen to this one? This denominator, the second term will vanish because this term will go up so this second term will go to zero. We cannot say the same thing about this term, so we essentially conclude that the dimension of any object, in this case it was two but in general the dimension D is nothing but ln N of epsilon divided by ln 1 by epsilon but this is important. We have the rule that the dimension is actually understood by means of this.

So what we’ll do actually? We took any object, we sub divided the embedding space into equal number of covering boxes, something that will completely cover the embedding space and then we count the number of boxes that are necessary to cover that object. After we have counted, we reduce the box size epsilon and thereby as we go towards epsilon tending to zero, if we keep on measuring this then this should give me the dimension of the object. So for these objects initially if you start measuring, it will not be two but it will converge on to the number two but nevertheless it will converge. If you really do it, you will find that you will get something like this that if you vary the epsilon and if you plot the dimension measured this way then for any object say for a parabola, if you take a parabola and if you do the same procedure it will start from another value and it will converge on to the number one. If you take a triangular sphere then it will start for some value and it will converge on the number 2.

So ultimately as you reduce the epsilon what matters is where does this number converge? So that is our concept of dimension and finally if you apply this method to any geometrical object, the object that are found in nature if you apply this method wherever it convergent that will be the dimension. If this is the concept of dimension, this is the idea of dimension that we have to find out how many boxes of the embedding space are necessary to cover it and we reduce the box size to zero progressively and find out this number and wherever that number converges that is the concept of dimension then obviously for any non idealized objects we have to fallow the same procedure.

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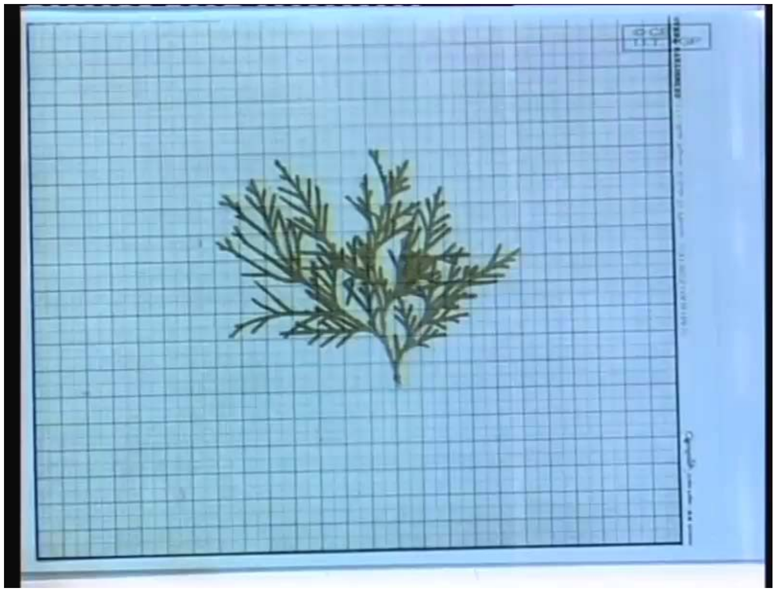

Let's illustrate how to do that. This is a non-ideal geometrical object. Just before coming to the class I had picked it up from one of the lands in our garden and so this is a natural object which everyone of you have seen. In order to measure the dimension, we covered embedding space by means of equal number of boxes and we start counting. So how will we count? The number of boxes that as necessary in order to cover this object. So 1, 2, 3, 4, 5, 6, 7, 8 but not this, these are not necessary 9, 10, 11 so on and so forth. If we count and if we then put it here, take the logarithm and for every stage of counting, its stage of counting means each box size that you have taken, in this case it is one centimeter box size then if you take that in place of the epsilon then you will get a number. Reduce the box size to half a centimeter and then you again do the counting and then you again put it in these two places, the half a centimeter here and the number of boxes that was covered here.

So what we generally do is to find out how many of these are necessary that means we have a total size and then we say that my box size is one tenth that size, one twentieth of that size, one hundredth of that size. So that the epsilon is also non dimensional number, we don't call it in terms of centimeters. So epsilon is non-dimensional distance that is in terms of the size of the total and if you go on doing this exercise for every iteration, for every stage of this counting that means in the every stage we have set a particular side length, in the initial you might say I have divided the whole into 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, so that that kind of twelve division, twelve by twelve square. In the next stage we make it half of that and so on and so forth. If you do it that way then you will find that always converges on to a number that is not integer between one and two something that has a fractional value.

So all geometrical object that are obtained from nature, yield this kind of the similarity of all geometrical object is that if you repeat this procedure, if you carry this procedure for any geometrical objects you always find a number that is non-integer. Now it is time to clearly state what is a fractals. Fractals are geometrical objects with fractional dimensions. It's important so let me write the definition of a fractal.

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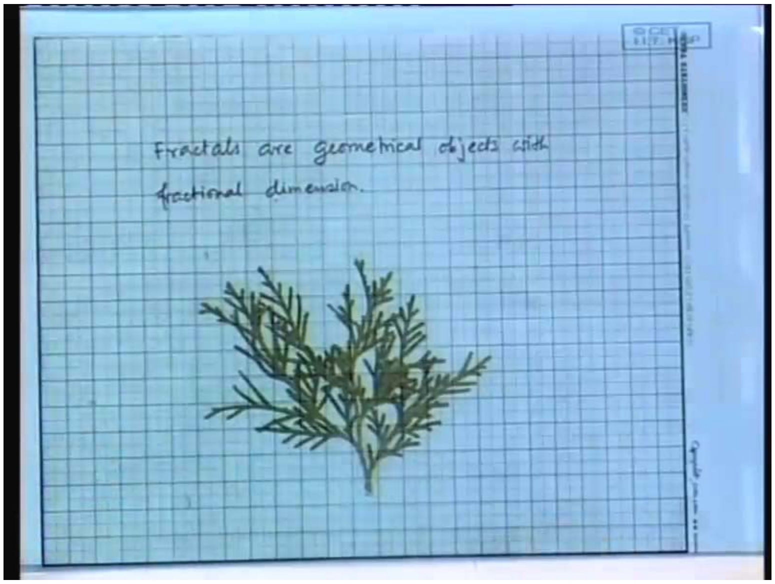

If it is made up of points spread over a line, a point is of dimension zero, a line is of dimension one. So if I have a set of points spread over the line then if I do this procedure, its dimension will come between 0 and 1. If I have an emending space of a sheet of paper and we have got some kind of a curved line and we measure its dimension this way then its dimension will come somewhere between 1 and 2. If you have a surface rather crooked surface embedded in a three D space then if you measure its dimension, it will come between two three and so on and so forth.
So the actual dimension of the object will be less than the dimension of the embedding space, it will always happen. Only in idealized objects say a cube embedded in a 3 D space, a cube means if it is idealized cube not made by carpenter, if it is made by carpenter the surface has always some non-idealities.

I am talking about an idealized cube then the objects dimension will be equal to the embedding spaces dimension. Hence it will always be less and unless it is an idealized say circle embedded in a 3 D space in which case the circles dimension as I already shown, it will convert on to the number 2. that is why its dimension will be say to be true but if that object is a non idealized object that particular curve will always converge on to a number that is not true. So this is the concept of fractal geometry.

The way to obtain that number fractal dimension, all that we do is if we have any object we simply put that object, we embed it by the space and we count in a number accordingly we write this equation. So we put it in this equation and will get the value D. Nowadays with the advent of the computers, this procedure has become significantly simple because all we need to do is we need to take a photograph of that, make it into a file that is a binary file. Binary file means these points where the pixels that contain this objects will be one and the rest of the space will be zero and then simply by means of a computer, we can do this procedure by embedding by means of different size boxes and counting how many of these boxes are necessary to cover this object. Ultimately the maximum extent to which will be able to reduce epsilon is the pixel size.

So we cannot really even in idea, in actual practice even if you do it by computer we cannot really bring it down to zero, we can bring it down to the maximum extent of the pixel size but when we do so when we draw the curve actually it would come to a point and then we have to stop and then we have extrapolate it across that so where it crosses that is the number.

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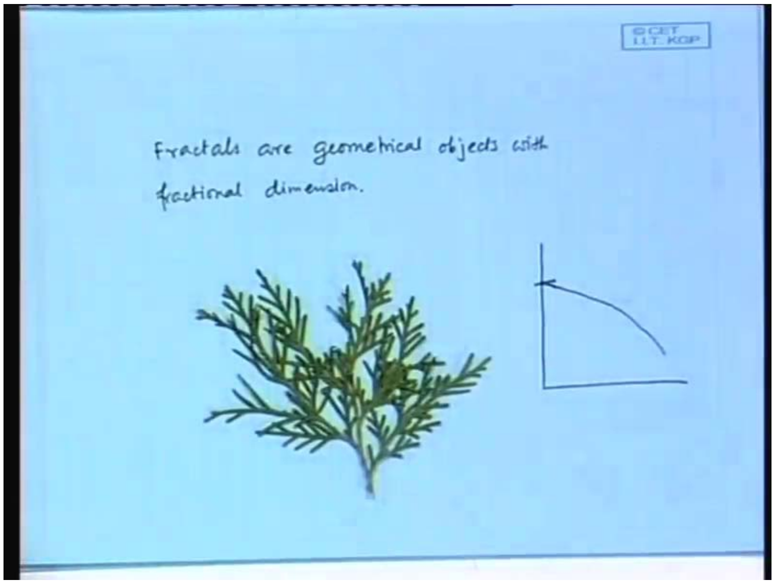

This is a procedure then you should yourself will be able to do. The hundreds of images available on the net, you can also take pictures. Many of you have a digital cameras, simply you have to make it as a binary image zero and one and then you have to count. What is the use of this number? After all this procedures you have got the number fine but what use is this number that is the important aspect. After all this number signifies the way it differs from the idealized once and the way it differs from the idealized once would be the way it is crooked, curved, twisted. So the ones that we generally ignore when we idealize things, the things that we had ignored the dimension or the amount by which the dimension differs from a integer dimension that will indicate that character how crooked it is, how bent it is, how twisted it is.

So that is what it will measure. Looking at an object we do have an intuitive feeling, how crooked it is, how bent it is, how twisted it is, how non-ideal it is. We have a feeling like that but this allows us to get a quantitative estimate from that. So far so good we have obtained the quantitative estimate, what do we do with it? Just think of the situations where that character matters. For example in the rivers silk particles flow. Why does it flow? Because it floats in the water and then it get carried by the flow of the stream. Which particles will flow and which particles will settle in the bottom as silt particles? That obviously depends on the specific gravity of the individual particles, the size of the individual particles and stuff like that but given the same value of these, the specific gravity being the same, the size is being the same, what will determine whether a particular particle will be carried by the stream or it will settle? It will depend on the surface properties, the more crooked the surface, the more probability it has to be carried with the stream. So here the surface property is the matter.

In automobile exhausts, you have fumes. You can see in the crowded Indian cities, you see the fumes as black suite coming out of the exhaust pipe. I think you have heard that causes harm to our breathing ducts. Out of those there are many other various particles, particles of various size and shape and the crookedness of the surface and obviously the more crooked the surface, the more probability it has to attached to the breathing ducts and cause harm. So here also you see that the character of the surface matters. When there is a normal electrical switch, you press the switch and two metal may contact. Do every point of the metal pieces may contact? Obviously not, because if you look at the surface by means of an electron microscopic, you see that there are so much imperfection that they look like mountains and there are two things which both have this kind of imperfection are obviously the ones that may contact simply the places where they attack each other. Obviously the character of the contact depend on how crooked are the surface is.

There again the surface character matters. The situation with say solid electrolytes where when we say using some kind of electrolyte and electrodes there is a electrolyte which say dilute solution of hydrochloric acid and you have put some electrodes to generate hydrogen and oxygen. Really the reaction takes place on the surface of the electrodes. So how crooked are the surface, what is the surface property, the generation of hydrogen oxygen will depend on that.

Similarly when you talk about the solid catalyst for example there is a reaction in which gases are taking part in the reaction and there is some solid catalyst. How efficient will be the catalyst not only depends on his its chemical property and stuff like that but also it depends on the amount of surface that is exposed to the reactors. So all these cases it is easy to see that the fractal dimension will matter. in geology there is a large body of knowledge in which you see when the rocks were formed they have formed in some process and there are big rocks, small rocks, pebbles sand grains and the geologists often cut the rocks to see what is inside. You see structures within structures and all these structures now geologies are trying to relate to the way they are formed.

So the geometry of those very irregularly shaped structures actually have some meaning. So you see in various disciplines, we have specific use of this concept of the fractal dimension. The economies data set for the exchange rates, the one that you find in news paper everyday that today the dollar rupee exchange was so much and tomorrow it is so much. If you go to any of those banks, you will find it is changing hour by hour. There is a computer which gives the hourly record of that. Now if you go to the places where actual exchanges are taking place that you will find changes minute by minute. So you see that is again a data set that has complexity within complexity.

Similarly the situation of the Sensex, the any economic index these things are all curves that are fractal in nature in the sense that you cannot differentiate them. If you want to measure the fractal dimension, you will get a number that is between 1 and 2. So today we will end with idea that fractals geometry is a new type of geometry that tries to represent actual natural objects. The main distinctions are one that the main natural objects are geometrical objects that are continuous but not differentiable anywhere. Secondly they are infinite surfaces yet enclosing finite volumes, thirdly most importantly they have a fractional dimension. So we will end with this today and continue in next class. Thank you.