Chaos Fractals and Dynamical Systems
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Lecture No # 13
Bifurcations in Two Dimensional Maps

In the last class when we are talking about the two dimensional maps we had seen that the essential idea is that we are placing a Poincare section and if the original system is 3 dimensional the Poincare plane is two dimensional, so we will see things on the Poincare plane. That means we will obtain two dimensional maps in this form x_{n+1} y_{n+1}.

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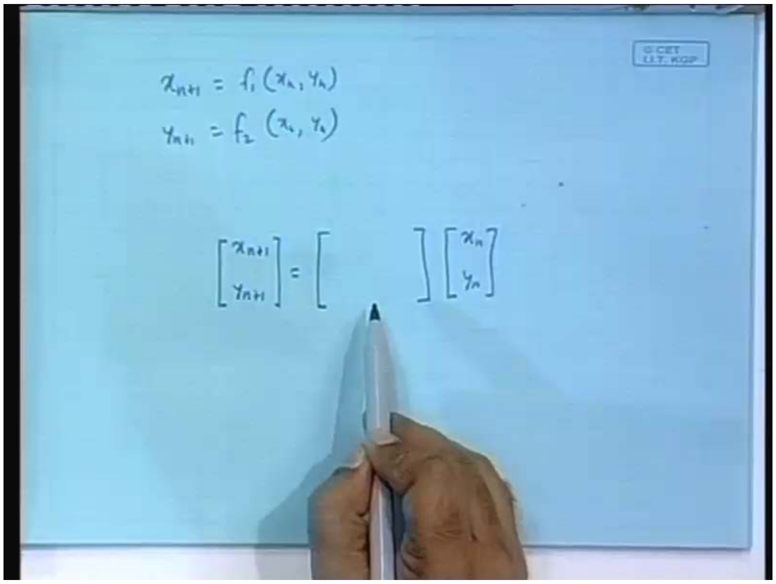

This is some function of x_n y_n and this is some other function of x_n y_n. then we said that having obtained this the next step would be obtain the fixed point by substituting x_n in the left hand side that means the fixed point is the one where the x_{n+1} is equal to x_n, y_{n+1} is equal to y_n. If you substitute we get a pair of equations two equations, two unknown. We can obtain the fixed points and the next step was to obtain the local linearization in the form of the Jacobian matrix. Having done that what we obtained is a linear equation x_{n+1} y_{n+1} is a matrix time's x_n y_n. This matrix is the Jacobian matrix. Then we said that the next step is to obtain the eigen values of this matrix and eigenvectors and depending on that if the eigen values are real than the eigenvector, there is no point in obtaining the eigenvectors. So when we do that and in a next stage we said that it is now possible to understand the dynamics in terms of the eigen values and eigenvectors, like we managed to give some names.

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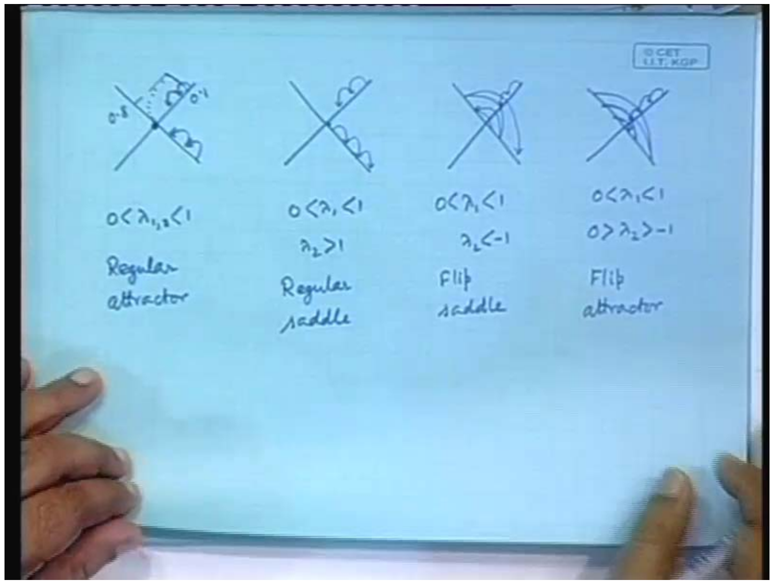

We had said that supposing here is a fixed point then here two eigenvectors, if both the eigen values are positive and real and between 0 and 1. So 0 less than lambda_1, 2 then the behavior would be like this, from here also it would be approaching. So this would be called the regular attractor. What would be the behavior if you start from somewhere else? How will it proceed? Obviously this will approach the fixed point but how? Again try to recall the logic that we had given earlier. This has to be broken up into two components, along the two eigen directions and then this distance will progressively reduce as multiplied by the eigen value along this direction and this one will reduce as multiplied by the eigen value at that direction.

If say here the eigen value is 0.1 and here it is 0.8, how do you think it will proceed? This one will be multiplied by 0.1 therefore it will shrink fast, while this one will be multiplied by 0.8 it will shrink low (Refer Slide Time: 04:30). As a result of this it will go like this. So this is a regular attractor and if one eigen value is between 0 and 1, the other eigen value is greater than one it is a regular saddle. So the behavior would be, if this is the lambda_1 direction it is contracting, while if this is the lambda_2 direction it will be like this. All the time try to figure out how the behavior will be if the initial condition is placed elsewhere, not on the eigenvector. You should be able to visualize that.

Next fixed saddle because this is less than minus one, it is outside its magnitude is greater than unity so this is saddle and this is negative therefore flipped. So how will the behavior would be? If this is the regular direction and if this is a flipped direction whose behavior would be like this. It will actually go from one side to the other and therefore it will be flipping. It is a flip saddle. This is what we have done. Now apart from that can you figure out what other possibilities are there? Take any flip attractor of course, flip attractor would be where the eigen directions are zero less than 1. So how will the behavior be? In one direction it would be like this, in other direction it would be nevertheless coming closer. These are the possibilities when the eigen values are real, this is flip attractor.

What happens in these cases? You normally say that it goes out to infinity but remember that actually does not happen. Why? Because this is only the local linearization. This is the only the local behavior so you might at most scientifically say that this particular set of iterates go away from this particular fixed point but you cannot really say that it goes to infinity. We will come to that later like what can happen to such systems but let us now tackle the case of complex eigen value. So lambda is equal to a ± jb. What will the behavior be?

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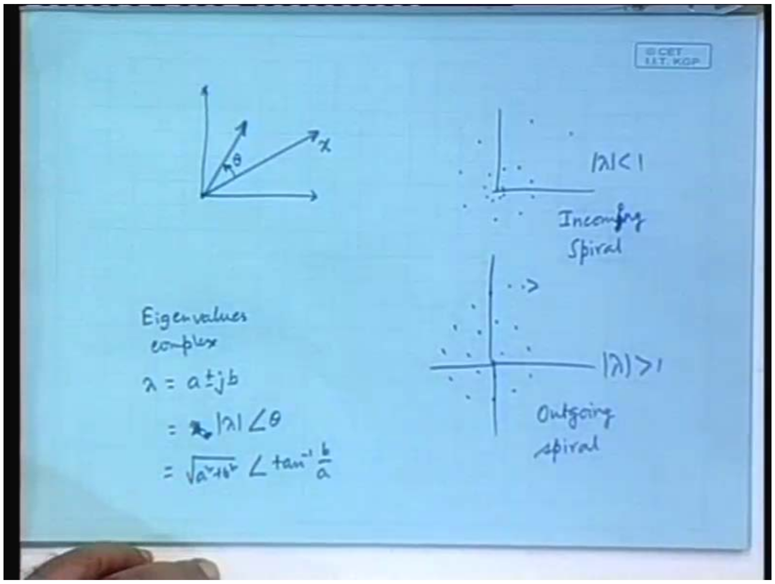

Say this is our axis and say this is our initial vector x. obviously as a result of the multiplication with that matrix, these vector will be operated on by a complex number and the complex number can also be written as say lambda real or magnitude angle some theta. So the moment you write it like this, you recall your ideas you learnt in the first year electrical circuit course, you understand that as a result of multiplication of with this what will happen is this vector will be multiplied by the magnitude and it will turn and it will rotate counter clockwise by the angle theta. So what is this lambda? Lambda is a square plus b square and the angle is tan inverse of b by a. So as a result of this what will happen? If this root over a square plus b square or the magnitude of the lambda is less than one then it will become a smaller vector and it will turn by an angle theta.

What would be the result of repeated application of this? That means if you are keeping on iterating this what will happen? in every iterate it will rotate by theta and it will shrink and as a result of which you will get a incoming spiral but remember it is not a continuous motion as you had encounter while dealing with the continuous time system, differential equations. This will be like a set of points that jumping like this but ultimately… (Refer Slide Time: 00:10:45). Is this point visible on screen? So this would be an incoming spiral behavior, likewise if you have the root over a square plus b square term, the magnitude of the eigen value greater than unity then it would be outgoing spiral. It is obvious.

So there two possibilities for this would be incoming spiral for lambda less than one and if it is lambda magnitude greater than one then it would be so on and so forth. So these are the two additional behaviors. This is the incoming spiral and this is the outgoing spiral. Now notice that these nomenclatures pertain to these fixed points. Fixed point has the nomenclature. In these cases the fixed points, the one that is around which you have this either incoming spiral or outgoing spiral behavior these are called 4 side. That means these each one is a focus. So this focus, this specific term is used where the eigen values are complex conjugate. It could either be a stable focus or an unstable focus. if one says it is a stable focus, you know that the behavior around is a incoming spiral and if one says that we have a unstable focus, you would be understanding immediately that one is talking about the outgoing spiral behavior. So we have now understood the types of fixed point that can be there.

Now we had said that we are ultimately trying to understand this stability of periodic orbits by placing the Poincare section and looking at this. As stability obviously now can be understood in concrete terms. What is the stability then? In this cases under what condition was the fixed point stable? You are talking in terms of real. If it is complex conjugate then a magnitude has to be one. So actually it will bald down on to whether or not the vector itself that means the complex number itself is inside or outside the unit circle.

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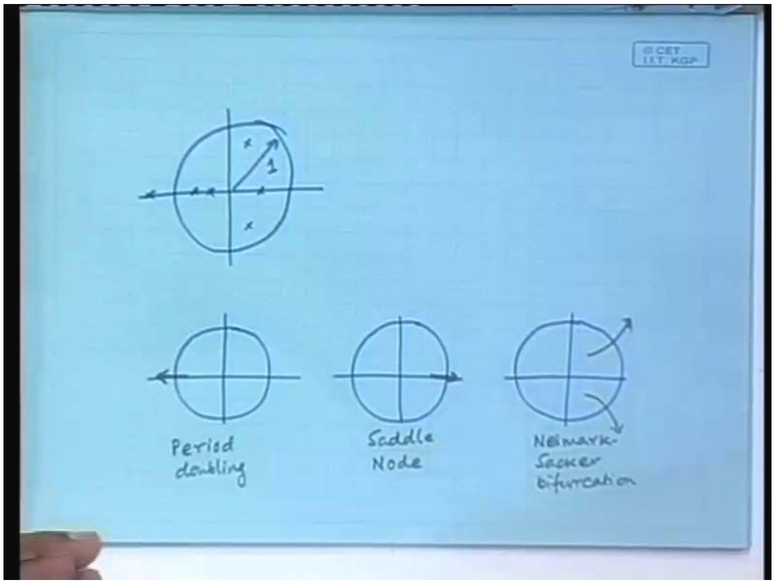

If it is outside the unit circle, it will be unstable but you can now relate the positions of the eigenvectors on the unit circle or (Not audible- Refer Slide Time: 14:19) two of them are here and here, flip attractor. What will be the behavior if they are here? It is a stable focus incoming spiral behavior. What will be if this and this? Saddle. Now we have to remember that this stability condition is different from what you had in case of continuous time dynamical system. In the earlier case you had the eigen values, the real part would have to be negative but here in this case the eigen value itself has to be inside the unit circle.

Now if I ask you, in how many possible ways can such a system lose stability? Means essentially that will bald down to, in how many possible ways can eigen value exit the unit circle. It's not difficult to see, there are exactly three possible ways. one say the unit circle is like this and it goes like this if possible and another possibility is where you have the unit circle like this and one eigen value goes like this and the third possibilities is a pair of eigen values leave the unit circle like this. These are the three possibilities, fundamentally different possibilities. Obviously it could go this way and could go this way also, there is no fundamental differences between them. That is why these are the fundamentally different ways in which a fixed point can lose stability. So let us try to understand what happens to each of them but from our earlier concept of one dimensional map can we relate. What is happening here and eigen value is becoming minus one. What is our anticipation? In case of the one dimensional maps what happens when this curve become minus one? Period doubling bifurcations.

All these are bifurcations because in all these cases the fixed point is losing stability, something else in gaining stability and so you have bifurcations but in this case you anticipate a period doubling. Why? Because of the similarity of the equation with the one dimensional map but whether or not that what happens we need to check. In this case eigen value becoming plus one what happen then? Tangent bifurcation are we called saddle node bifurcation and a third we did not come across because in one dimensional system you cannot have a complex conjugate pair of eigen values. So these two we recall what we learnt in 1 d and try to extrapolated in 2 d and try to understand what will happen.

Let me write down the name because here we understood that it should be the period doubling. We will prove but it is actually period doubling, this is saddle node and this is the special situation, I will treat this little later, it is called Neimark Sacker bifurcation. First let us handle these two and then we will come to this one. His question is we had seen that pitch fork bifurcation also occurs when the eigen value becomes plus one. Pitch work is you recall where the bifurcation diagram would look must the same like the period doubling bifurcation only thing is that in period doubling what is created a period two orbit while in this case what would be created is at two period one orbits both stable. I also said that they are somewhat real situations. So this similarity another thing as I told you that it comes sort of intuitive prediction, it might not be true. You might argue that no, it should be similar to the pitch fork bifurcation. That is why we need to examine each case separately and after having examined, if you find that it is similar to the tangent bifurcation case then we will be convinced before that don't be convinced. Let us explore by means of a specific map called the Henon map.

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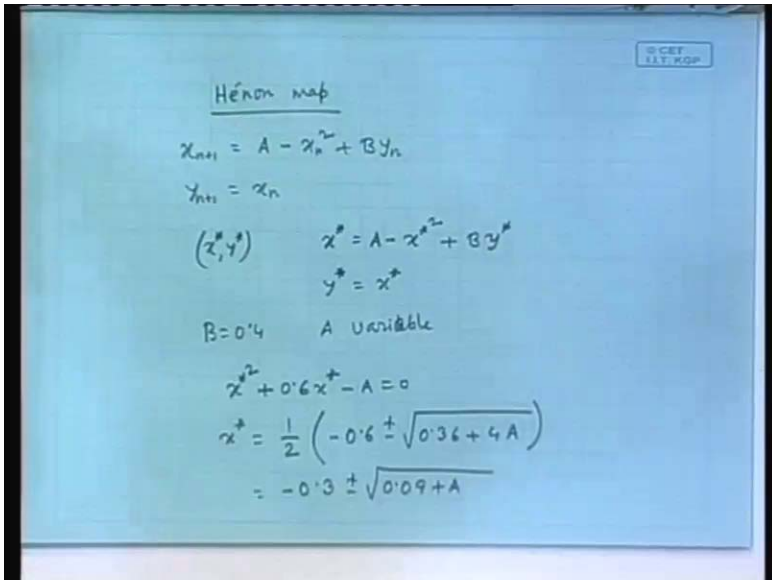

This map is x_{n+1} is equal to A – x_n square plus B and y_{n+1} is equal to x_n, 2 d system. So when you proceed what will be the first step to locate the equilibrium points? Do this, let this be an exercise we do today, so that will trace all the steps that we need to do in order to understand the behavior of this system and the first step is to locate the equilibrium points. In order to locate the equilibrium point we have to set the fixed point. Let the fixed point be x star so we would write x star is equal to A minus x star square plus by star, x star y star is the fixed point and y star is equal to substitute it here, you get a quadratic.

Let us proceed by choosing one of this parameters. There are 2 parameters A and B, let us choose one of this parameter say let's say B = 0.4 and A variable and we will be trying to understand what happens as A varies. So that this fellow is now 0.4 (Refer Slide Time: 22:00). If you substitute what is the equation you get? Let this goes to the left hand side, x star square, these two get subtracted so you have plus 0.6 x star minus A is equal to zero. So the x star is half minus 0.6 plus minus root over 0.36 + 4A. For the sake of simplicity let's take 4 out and cancel this. So this is equal to minus 0.3 plus minus root over 0.09 +A. Now this immediately tells you that so long as A is less than 0.09, the position of the fixed point is you get a complex number. Position of fixed point cannot be a complex number and therefore it doesn't exist. So the fixed point doesn't exist for A< -0.09.

Now suppose we are varying the parameter A and we are varying this way and here is – 0.09 before that there was no fixed point. Beyond that what will happen? There will be 2 fixed points and there locations are given by x star one and x star two by virtue of these (Refer Slide Time: 24:47) you would say this is also y_1 star is equal to -0.3 plus root over 0.09 +A and y_2 star is equal to -0.3 minus root over 0.09 +A.

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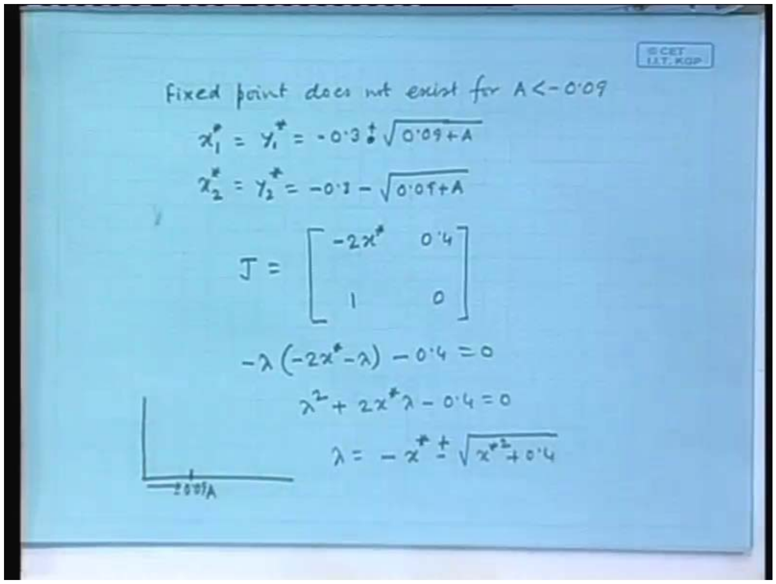

So these are the two fixed point which means that beyond that two separate fixed points are born. Is that point clear? Is not that one is born, two are borne so these two are born. Now we need to find out what they are, what kind of behavior do they have, how we will do that? We have to take the Jacobian and we have to study each one individually. So this equation you take the Jacobian, this is your f_1 x, y and this is your f_2 x, y. So what will be the first term? First term would be minus twice x_n. So the Jacobian is minus twice x star, second one is f_1 with respect to y which is B we have taken it a 0.4. The third one is f_2 which respect to x so 1 and this fellow is zero (Refer Slide Time: 26:27). In the next step we will try to investigate this and that separately, so first substitute this here. So what do you have? Its characteristic portion would be let's first determinant in terms of this, so it would be minus twice x star minus lambda, here is minus lambda so minus lambda minus twice x star minus lambda minus 0.4 equal to zero.

So you have lambda square plus twice x star lambda minus 0.4 is equal to zero. So you have lambda is equal to minus x star plus minus root over x star square plus 0.4. Fortunately here we have a square term, so we can substitute here and make it simpler. So just substitute this one here and see what is the result and what are the eigen value in this two cases. First let us assume A is equal to 0.09, if A is equal to 0.09 then this term vanishes. So you have 0.3, just put it here and see. What is lambda? It was minus 0.3 here, you are substituting this, so this fellow goes to zero, minus 0.3; if you put minus zero point three in front there is a minus, how can you get? [Conversation between Student and Professor – Not audible ((00:29:18 min))] Don't give me the wrong feed backs, it cannot be negative.

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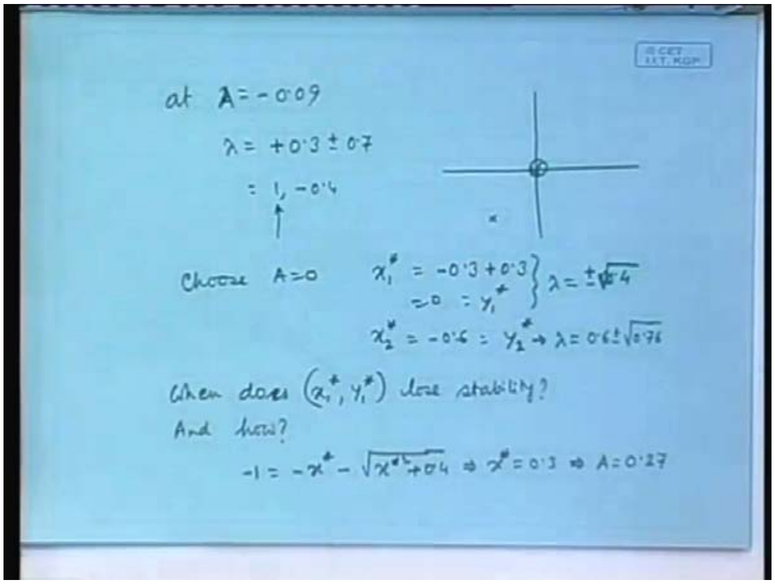

So if you substitute this, at A is equal to minus 0.09, lambda is equal to minus plus 0.3 plus minus, so one. So this is the point that I was trying to arrive at, you see at that value one of the eigen value is exactly one. So does that answer your question now Manjun? so here the eigen value was plus one but at the time, the phenomenon that we observed here was that at this parameter value two fixed points begin to exist, these two fixed point before that it was not there after that they were there (Refer Slide Time: 30:40). Now the next point is when they begin to exist that means when they are existing what are they? We have already categorized different types of fixed points, which category do they belong to?

In order to choose a value of A slightly bigger than this and a nice choice would be A is equal to zero slightly bigger than this. If you choose A is equal to zero, the advantage is that you can take a square root. Do it. It would be -0.3 + 0.3 =0, this is equal to y_1 star. So one fellow is situated at the origin, the other fellow is situated at… (Refer Slide Time: 32:00). So if you have this at the state space one is here another here. What are they? For that you will need to substitute them here, so when you substitute the position of the first point which is x_1 star is zero. If you substitute it here you have, for this one… [Conversation between Student and Professor – Not audible ((00:33:02 min))] root of 0.4, that's possible and this fellow is if you put 0.6 here it will be 0.36. So 0.6 plus minus root 0.76, is it?

The point I am driving at is this fellow is surely both are within the unit circle and one is outside the unit circle the other is inside the unit circle. So this fellow is an attractive node and this is saddle, so the two things that have been borne are a saddle and a node that is why it is called a saddle node bifurcation. So the meaning of the term saddle node is clear only when you look at the two dimensional system but the same phenomenon since that also happens in case of the one D system. It's also called a saddle node but there is no good explanation of the term saddle when you talk about 1 D, its either stable or unstable but in case of a 2 D system there are various types of unstable things it could be a outgoing spiral, it could be a saddle that's why when you talk about saddle it becomes more clearer.

This fellow is stable, this fellow is unstable (Refer Slide Time: 35:10). Which one is stable? This fellow is stable and this fellow is unstable and this is flip saddle. Now as you change the parameter further what do you expect to happen? there these two fellow will shift in position because the position that depended on A, as you change the A this fellow will shift in position but since all the time x and y's are the same therefore all the time their position would be on the 45 degree line but nevertheless their position will change and also their stability status will change which means this (Refer Slide Time: 36:02). Now can you find out here is there was a fixed point which was stable, as you change the parameter you do expect at some point of time it will lose stability. When will that lose stability, can you find out? Rather simple really because all we need to do suppose you want to find out whether that fixed point undergoes a period doubling. Period doubling means minus one so all we need to do is put -1 here and find out, when does that happen? Can you trace back and do that.

The next question we are asking is if we conjunct here that it loses stability by having one of the eigen values reaching minus one then all that we need to do is to find out under what value of the parameter A would that happen? So here you put -1, the negative thing will happen and it is logical to say that in that case you will have to take -1 here. There are 2 eigen value plus and minus, the one with the plus will be more to the positive side than the one to the minus therefore if one goes negative then only this fellow can. So you will have to substitute it like this -1 is equal to minus x star minus root over x star square. So what is the solution? First you need to find out x and from there you need to find A, so x is equal to x star is equal to 0.3 and that tells you A is equal to referring back here (Refer Slide Time: 38:33).

So you know that as you change the parameter further at 0.27, the other phenomenon will happen where one eigen value reaches minus one. What do you anticipate? It should be period doubling but make sure in order to convince yourself what will you need to do? You need to find out the second iterate of map and find out its eigen values. Finding the second iterate and finding the eigen values by now it should be accustomed because it came in the mid sem and I found that some of you could not eliminate those known fixed points that give me some trouble but this should not give you trouble. So all you need to do is to find out the second iterate of map which is to be obtained this way x_{n+2} is equal to A - x_{n+1} whole square plus 0.4 y_{n+1}. Then y_{n+2} is equal to x_{n+1}. In the next step substitute x_{n+1} y_{n+1} here, X_{n+2} is equal to A minus this fellow square, A – x_n square plus 0.4 y_n square plus 0.4 xn; y_{n+1} is equal to 0.4 x_n, y_{n+1} is equal to x_n.

Now y_{n+2} is equal to x_{n+1} which is A minus x_n square plus 0.4. So you have got the second iterate of the map. Do you? How to find out the fixed point of this? you would notice that it is getting a bit massive because this will give rise to fourth order term but nevertheless since you know that earlier two fixed point are still there, therefore the same step eliminate them, find out the fixed point, find out the Jacobian of this and find out whether they do exist or not and then if they do exist whether they are stable are not?

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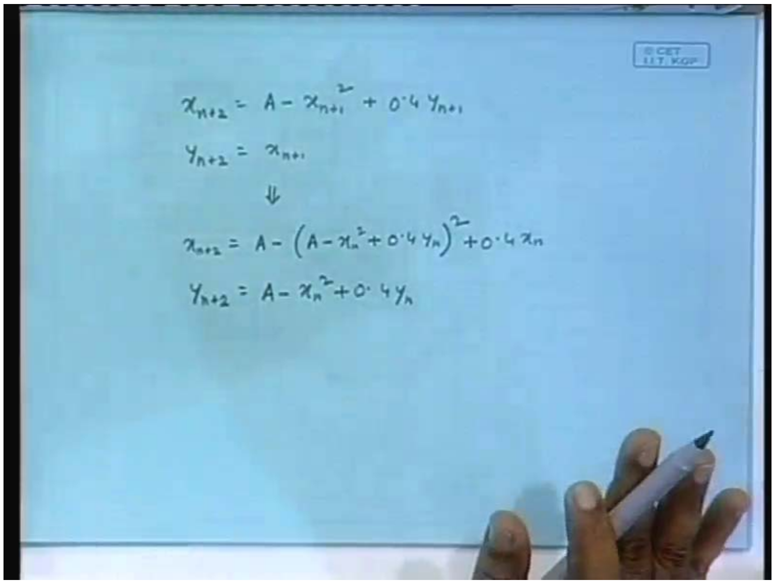

I have shown you how to do it but do it and convince yourself and let us not do it here because though this is tactable it will take some what longish time for me to do it. So I will leave you to do it. If it comes in the exam you should able to do it, that is a point. So we found a sequence of events that happens for this one and in that process we understood two phenomenon that when the one of the eigen values becomes exactly equal to one that is when a saddle node bifurcation takes place. A saddle node bifurcation means a saddle and a node are borne.

Now suppose you are at a parameter value at which you have a saddle and a node both existing and you change the parameter in the opposite direction. What do you observe? they will come closer to each other, as they do the eigen values also comes close to each other and finally at that particular bifurcation point of the parameter value what will happen is that they will collide and disappear and when they collide both the eigen values will become +1. So you don't really talk about the eigen value crossing +1.

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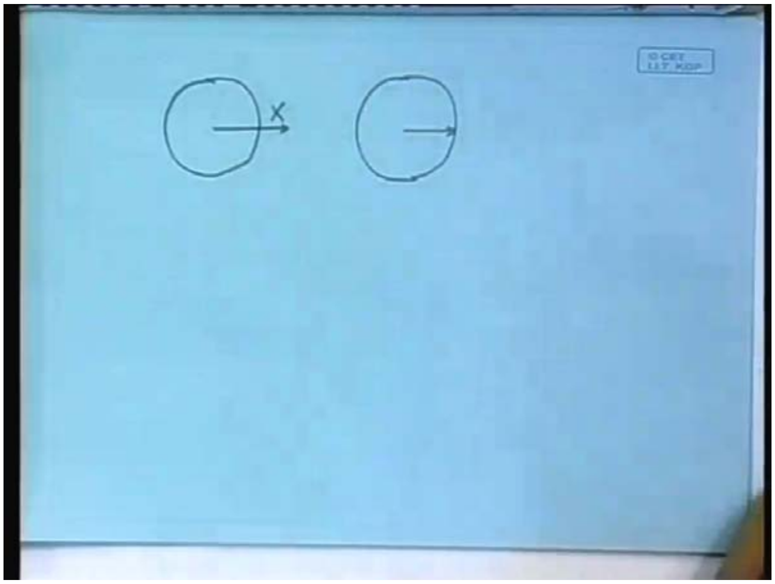

So this representation is somewhat erroneous, it doesn't really cross, it doesn't happen. What happens is that this is the unit circle, it goes hit and disappears, but it does cross to the negative one. So this event is the saddle node bifurcation, we have to understand that.

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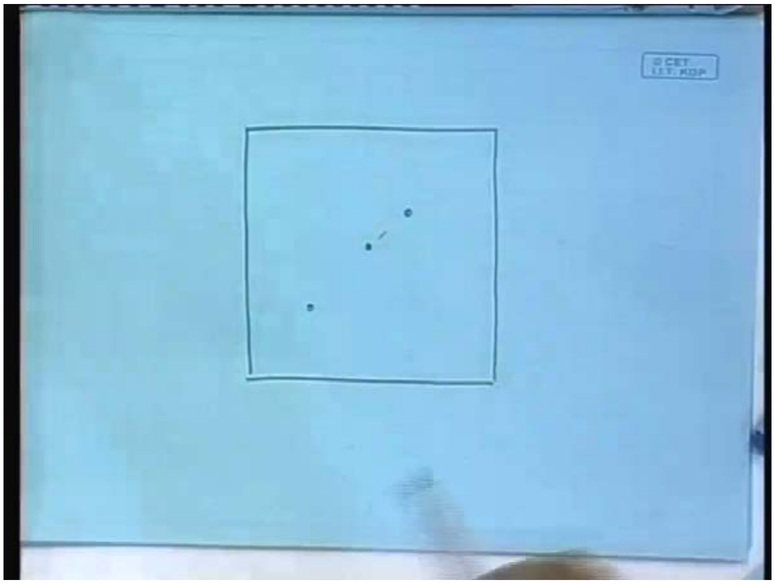

Next as we change the parameter further, there were two fixed points. Let's illustrate them in the state space, you have got the state space like so and here is a zero. We had a fixed point here and some where -0.6 there was another fixed point. I will change the further, they move and when they period doubling bifurcation happen what was the parameter value and what was the positions?

You have already calculated x star is equal to 0.3, so by then from zero it has moved to 0.3 and whereas the other one move? Never mind, it has moved and it has gone somewhere, did not stay there but its eigen value did not cross the unit circle and therefore nothing fundamental to this happens to this one. Fundamental thing happens to this one and it became unstable.

As a result of which two fixed points where now borne rather A period two fixed point was borne. Where would they be located? Now I told you that you can do that simply by solving the forth order equation and finding the location of the fixed points but logically also you can proceed. With logic you can predict where will it be. In order to understand that let us understand a few others issues.

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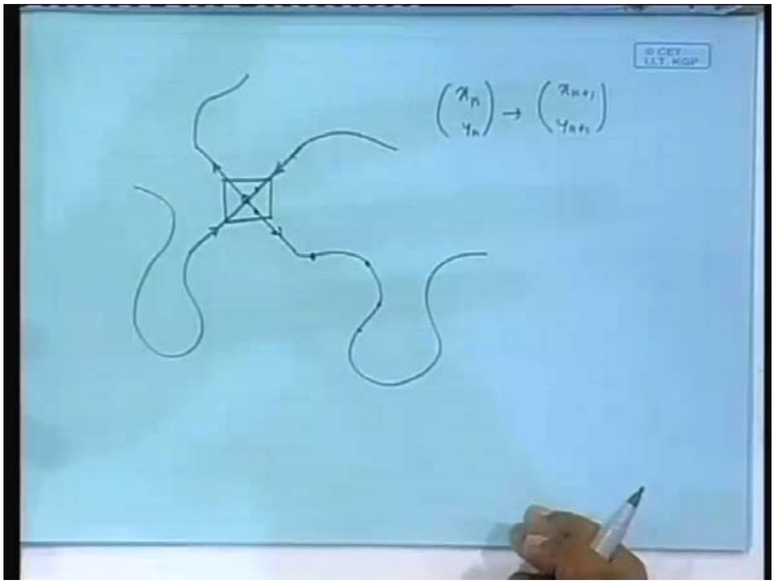

When we said that we have got a saddle fix point which means where the eigen values are such that you have got an outgoing direction, one incoming direction. Now this was the local linearization and the eigen vectors are defined by this concept that if you place initial condition on the eigen vector, it will forever remain on that eigen vector but that resulted found the local linearization. now if you go out of the local linearize zone that means where the local linearization is no longer valid then obviously these lines will no longer remains straight lines but still you can identifies the lines which have that property that if you start from the point on that line, it will forever remain on that line. We have done by that argument in the continuous time systems also. Same thing will be applicable here.

So in general expect that these fellows should bend and turn and twist and go anywhere they like. There is no reason to believe that they will forever remain in straight lines. While dealing with the continuous times systems, I gave some names to them. These were the stable manifold, unstable manifold. So this one would be called as the stable manifold and this one would be called the unstable manifold. In other words what is the definition of a stable manifold that if you take a point on a stable manifold and in further iterates it approaches the fixed point then you call it a stable manifold.

What is an unstable manifold? If in the reverse iterate it approaches the fixed point because if you are starting from here and if it is going away, how do you know that it started there? In order to really define it, you have to talk in terms of the reverse. How to actually draw them? It's not difficult to see that in the close neighborhood of the fixed point, the eigenvectors are tangent to the manifolds. So the way to draw the unstable manifold for example would be first to identify the eigenvector which you can do analytically. Suppose you have identified these are the eigen vector then take a point along that, reasonably close to that direction so we know that still linearization is valid and then keep on taking iterates. All the iterates must fall on the… but that will not allow it to draw it because we have noticed that I have put one point here, another point here. How do you know it bend this way?

So in order to do that all we need to do is take a last number of points between the first two points and see where they also go and accordingly you can trace the unstable manifold. You will be able to trace the unstable manifold to the right or to the left and you will have to do 2 different things in order to locate them, with one routines you will be able to locate one to the right another to the left but the same cannot be done for this stable manifold. It can be done only if because if you take a point along the stable manifold further iterates will go into that and therefore how do you know where it goes? This way, in order to do that you will need the inverse of the map.

Therefore immediate conclusion is that the stable manifold easily be drawn if the map is invertible else it cannot easily be drawn. there are routines, it will have little routines for that but nevertheless they are some are complicated tuffs I am not going into that but you will be able to write a program to draw the stable manifold and unstable manifold, if the map is invertible. What do you mean by invertible? What you mean is that x_n y_n this gives x_{n+1} y_{n+1} uniquely. Now if you know x_{n+1} y_{n+1} do you know x_n y_n uniquely, if you do then it's an invertible map, if you don't then it's not. Just refer to the situation in one dimension, you have the logistic map given as this. Is it invertible? No it's not, because for every value of x_n there is a unique value x_{n+1} but for every value x_{n+1} there are two values of x_n. So it is not invertible.

Similarly, is the Henon map invertible? Just try to work it out and tell me. In this case your problem would be, can you find out x_n y_n in terms of x_{n+1} y_{n+1}, can you write it down as an expression? If you can fine, you are through but if you lead to a plus minus term you are not true because it will lead to 2 possibilities. Now this stable and unstable manifolds play very important roles in determining the dynamics. Therefore this concept should be very clear otherwise you will not able to go-ahead. This will be very clear because many things in dynamics of such discreet time dynamical system depend on the structure of the stable and unstable manifolds.

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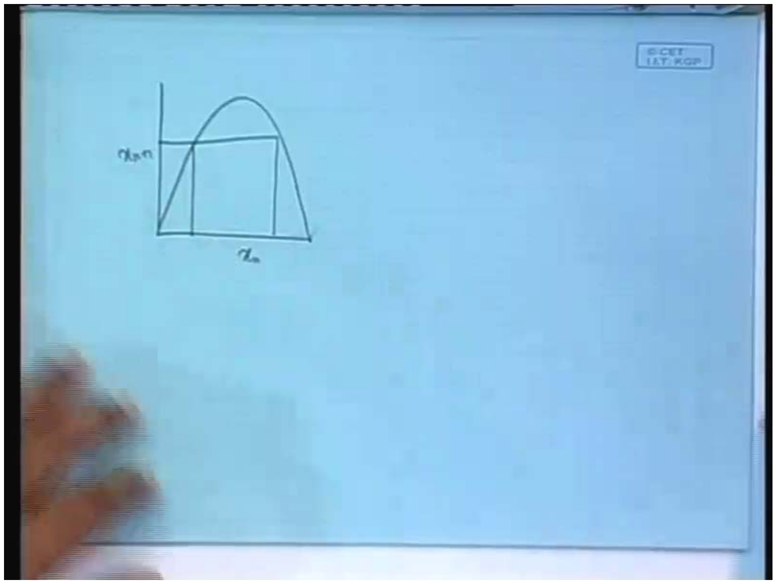

In case you have got a stable fixed point with real eigen values both would be stable manifolds. If we have a rippler which means both the eigen values, outside the unit circle but real in that case you have both the unstable manifolds. You will have stable and unstable manifolds in case you have saddle and that is why again you will find that much complicated and interesting dynamics happen in systems where there is a saddle fixed point. That is at the route of chaos I will come to that later. So in the next class we will further elaborates on this ideas of the stable and unstable manifolds.