Chaos Fractals and Dynamical Systems
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Lecture No. # 12
Two Dimensional Maps

There are a couple of things that I forgot to tell you yesterday that needs to be covered before we go into the two dimensional maps. The first thing is when we were talking about the phenomenon of chaos, we understood that there has to be sensitive dependence on initial condition and therefore when we talk about the one dimensional maps then also this phenomenon needs to be understood and then while trying to understand what happens in the state space, we realize that there has to be an expanding behavior. When talking about the continuous time dynamical systems we said that suppose you start with a collection of initial conditions enclosed in a ball then we have to observe how the whole ball evolves as the time progresses and then we concluded that if the ball as a whole shrinks then we have only a periodic orbit or we have a fixed stable equilibrium point but in order to have chaos in order to have sensitive dependence initial condition this must expand, stretch.

So we realize that there has to be stretching and folding because if it goes to stretching then it will run to infinity. So another condition was that it has to be bounded and so there has to be folding, so stretching and folding that was the mechanism of generation of chaos but that we understood in the context of differential equations or flows and that we need to understand in the context of the maps also. How does the stretching and folding happen in say the logistic map that we need to understand in order to actually comprehend how chaos occurs inside the system?

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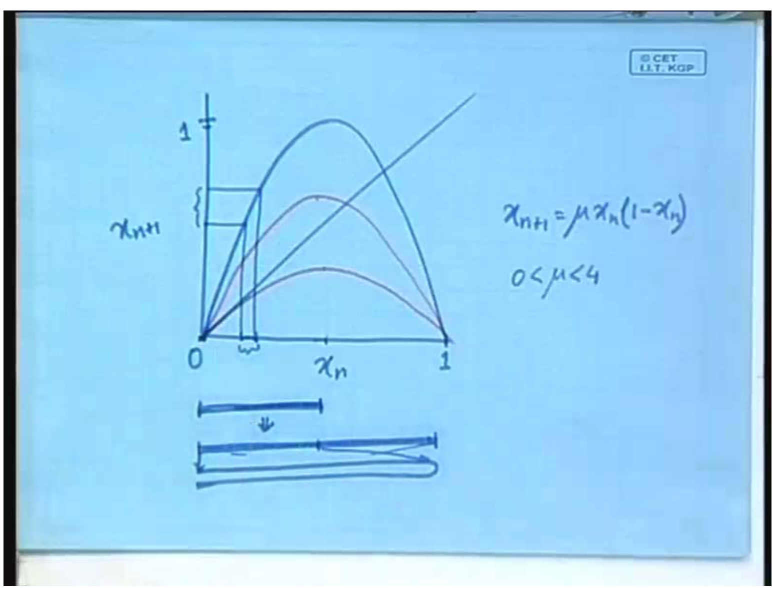

For example in the logistic map given by x_{n+1} is equal to mu x_n 1 minus x_n, the graph of the map we had drawn it from 0 to 1.

The height of the graph would depend on the value of mu and what is given by the peak position? It is symmetrical, so mu by 4. It will be a graph something like this and so on and so forth and this peak position is given by… So substitute half here what you get as xn plus one that is a position, so that you get as mu by 4. So if the range available is 0 to 1 then at what value of mu will the peak..? (Refer Slide Time: 04:00) This is the value of 1 at 4 that is why we often in this system take the mu range as 0. Remember the origin of this map, it was invented as a representation of a population scaled to have the maximum value of 1, so you you need to keep mu comparing to these values.

It will be good for us to go very between 0 to 1 on the x axis as well as 0 to 1 on the y axis which means that we are having mu is equal to 4. The graph would be something like this. What I will be talking about that will be applicable to other values of mu also for which the behavior is chaotic but for 4 it is definitely chaotic. You can see the chaotic behavior in the computer screen so we know it is chaotic.

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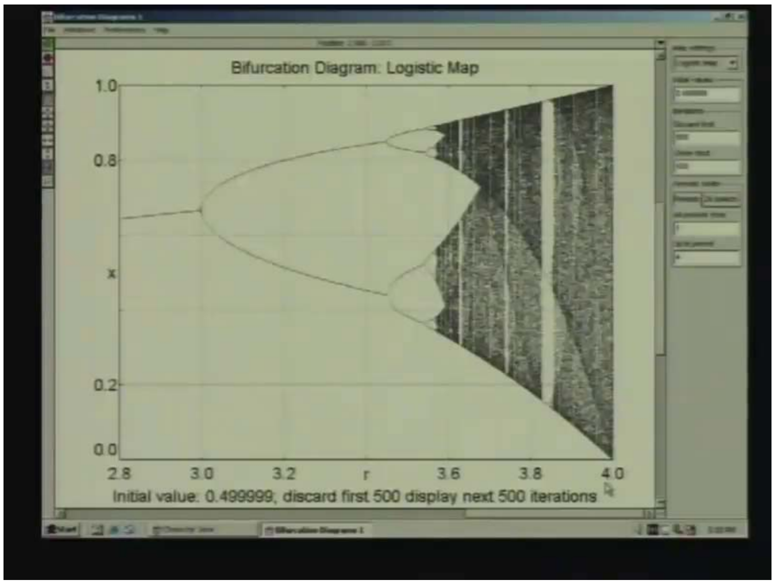

Now the question is how is the stretching and folding happening? Take some small length along the x_n direction and this length maps to this range (Refer Slide Time: 5:32). So here is a range where you start and here is x_n and here is x_{n+1}. So in the next iterate it expands to this length, so it stretches. If you take two very close initial conditions, the distance between them will increase as it goes there, so it stretches. Though you might argue that it doesn't stretch always because here the slope is less and therefore it would take a range here, it will shrink actually but it's not difficult to see that overall if you take the distance between zero to half that maps to the range between 0 to 1. So zero to half has stretched to 0 to 1, so you have a stretching.

Likewise you might say, let us let us depict it like this. If you have the length zero to half, in the next iterate that stretches to zero to one so there is stretching. Now if you start from the range zero to one next iterate where will it go? Again notice that zero to half will stretch to zero to one and half to one will stretch again back.

So what is happening is this one will stretch to… (Refer Slide Time: 07:31). 0 to 1 say, 0 will map to 0 but 1/2 will map to 1 and 1 will map to 0. So effectively what has happened is that this line has stretched and again folded and that is why you have sensitive dependence on initial condition. This process keeps on repeating. [Conversation between Student and Professor – Not audible ((00:08:10 min))] zero will map to zero because zero is a fixed point, zero will always map to zero. So this point always maps to the same point but the other things go and come back, stretched and get folded.

Now notice that we have talked about the Smale's horseshoe mapping, same thing is happening here. In the next iterate what will happen? This fellow will again be stretched and folded so that the whole thing remains between zero and one, boundedness as well as sensitive dependence on initial condition is ensured. So that is exactly why you have chaotic orbits in this map. That is how the sensitive dependence on the initial condition as well as boundedness is created in this one dimensional map. [Conversation between Student and Professor – Not audible ((00:09:16 min))] zero is a fixed point because if you draw a 45 degree line, it passes through zero. So zero is a fixed point, if you start exactly from zero it will always remain on zero but if you start from a neighborhood of zero it will not remain there.

Now you might be interested in knowing what happens beyond 0 to 1. I mean we have sort of define things in such a way that things remain bounded to 0 to 1 but at least mathematically there is no reason to think in terms of this only, you might say that let my starting point be say minus 0.2 or say 1.2 that's also possible. Just look at what happens, you try to work out logically yourself.

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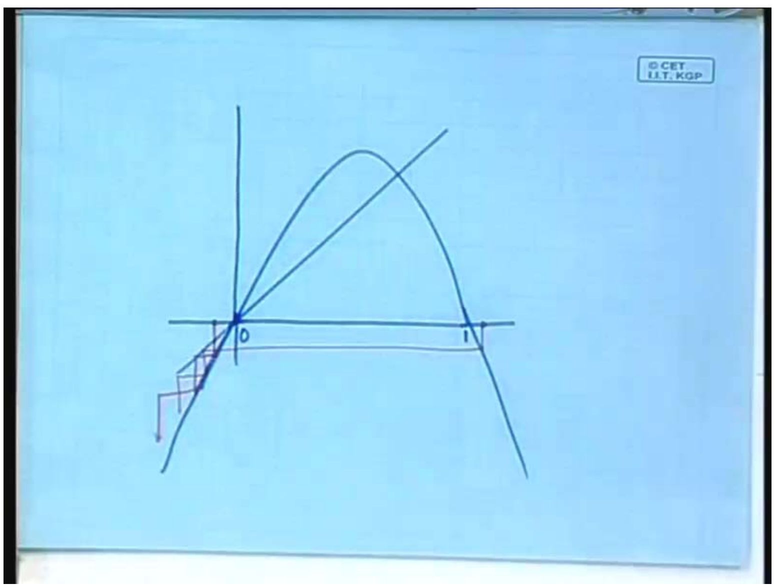

You have the 45 degree line here and you have the graph of the map going like this. So now you are talking about a starting point that is this side. So we will need to expand the 45 degree line and we will need to expand the graph of the map. We will have to extend it.

Now we start from a point that is in the negative side. So how we will iterate? Again we will not praise the calculator, we will do it graphically. so we will first go to the graph of the map, we will go to the 45 degree line, we will go to the graph of the map, we will go to the 45 degree line so it will go out to infinity. So any point outside the 0 to 1 range we will actually diverged to infinity. You might ask what happens here? Let's see if you start from a point that is say outside here, it will map to a point like this, again you have to go to the 45 degree lines and it will go on, same way it will go out. So anything outside this range will be divergent and anything within this range, zero to one range will converge.

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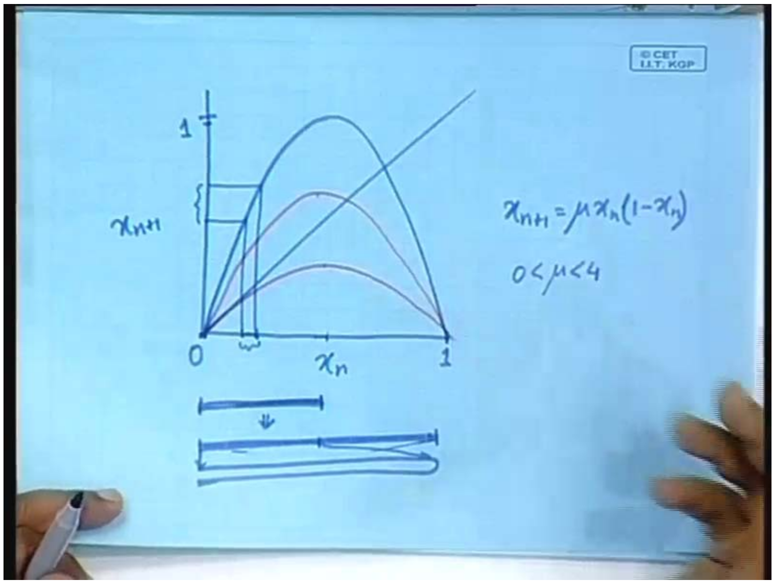

The other reason is see we have taken a range of parameter value zero to four. What happens if it goes beyond 4 say 4.1, 4.2. It is not difficult to see that then height of the map will be larger than one. Here this is your 1, 0 to 1, 0 to 1 and this fellow has gone to a value that is greater than one. As a result if you have a chaotic orbit, the chaotic orbit will go on moving and sometime it will come to this range that is above the one value. How will it make the further iterates suppose it has come here? It will come there and it will be mapping to somewhere outside, again it will go and it will go out which means that if the parameter value is beyond 4, the orbit cannot be stable it will go out.

The details of this, we will come to a little later when we understand what this actually means. The other thing that you need to know about this kind of a structure that you have seen on the computer screen, the period doubling cascade, we have noticed that there are such periodic windows that means a period three orbit comes into existence here, a period 5 orbit comes into existence here and so on and so forth.

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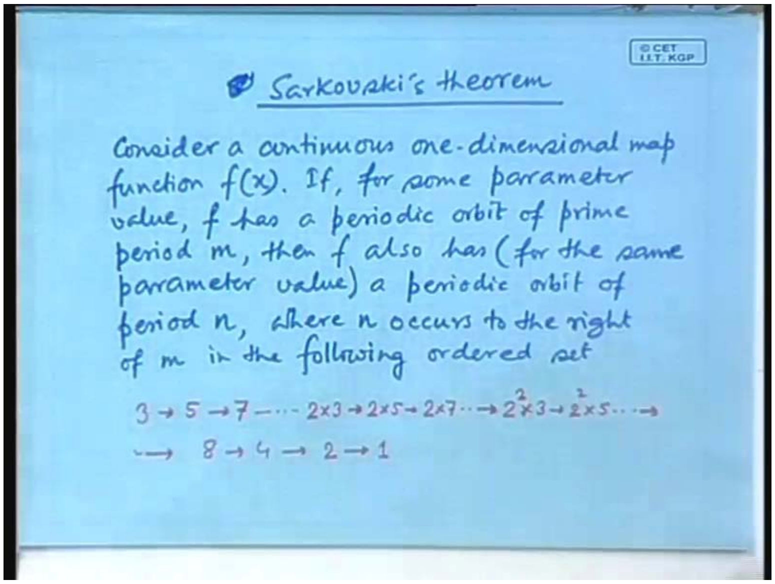

Now there is an extremely interesting theorem by Sarkovski's, it's called Sarkovski's theorem. I will write the theorem here and then explain what it is, so you also keep writing. Consider a continuous map function f of x, if for some parameter value f has a periodic orbit of prime period m. What is the meaning of the word prime period? For example if you have found a period 4 orbit then it is also a fixed point of the period 2 orbit, it's also a fixed point of period one orbit but we are not talking about that when you are talking about prime period. Prime period means where this is the largest number in the periodicity so period m, then [Conversation between Student and Professor – Not audible ((00:15:39 min))] If you get a period 3 orbit then you might say that it is the orbit that comes back three iterates later, so it's m is 3. The f also has for the same parameter value a periodic orbit n where n occurs to the right of m in the following ordered set and this set is important, it follows from number theory 3 to 5 to 7 then goes to infinity.

Now two times three to two times five to two times seven goes to infinity to two square times three to two square times five and this one continues. See what happens? It follows from number theory we are making ordered sets, the numbers. The first set is 3 5 7 9 11 so on and so forth, goes up to infinity then comes two times three, two times five, two times seven and that also goes to infinity. 3 5 7 9 [Conversation between Student and Professor – Not audible ((00:18:15 min))] Let it be because I am not talking about the branch and all, just as the ordered set. Then it goes to two square times three, two square times five and so on and so forth then it goes to two time three and so on and so forth. If it goes this way then after some time you will realize that we have covered the whole number set except the multiples of two. So then it continues and you have to say 8 4 2 1. So this is a whole set that covers the whole number theory, whole number set, natural numbers. [Conversation between Student and Professor – Not audible ((00:19:02 min))] Now it is actually here, it continues in the right hand side so this continues to this.

Just consider this number set, it is erased in the particular order first it goes up as 3 5 7 9 11 and so on and so forth goes to infinity. Then it goes to two times three, two times five, two times seven and goes to infinity; four times three, four times five goes to infinity and at the end of the day you have left these numbers, the set continuing the multiples of two and then it goes into the descending order, ultimately arrived at one. It finally arrives at one, the last number in this order set is one, the first number in this order set is three. Now you notice what this theorem say.

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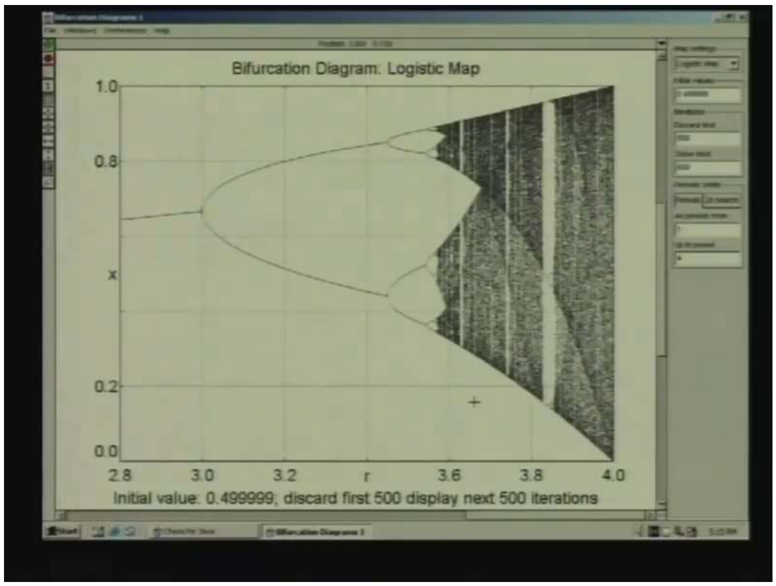

The theorem says; now look at the bifurcation diagram, it will be easier to understand here. Say at a particular point here, can you see this curser (Refer Slide Time: 20:15), if a periodic orbit of order n prime order, prime period n occurs here then suppose m is somewhere here, then all that to the right of it will also occur. So let's see if we ever find anywhere period seven then all the orbits to the right of it must also occur there. If you ever find period 5 all the orbits to the right of it must also occur there. You might ask where are they? They are all unstable periodic orbits because as it was going from the period one to period two so on and so forth, you found that they all became unstable but they continue to exist so all these orbits will be occurring. If you ever find a period one orbit what does it mean? Nothing is occurring, no other thing is occurring see that is here (Refer Slide Time: 21:42) that immediately brings to the conclusion that here there is no other periodic orbit in this part.

One is also there, period 2 the one fellow is also there, period 1 is also there. Now go to period 4, 1 and 2 are also there. So as you go this way, you will find that if you ever find when you ultimately arrive at this period three window, periodic orbits of all periodicity must occur by this theorem, all natural numbers are covered. [Conversation between Student and Professor – Not audible ((00:22:29 min))] because if we are talking about say period eight then that's also a fixed point of period four. So those things need to be eliminated. When I say period 8, I will say period eight only there should not be any complication as far as the definition is concerned.

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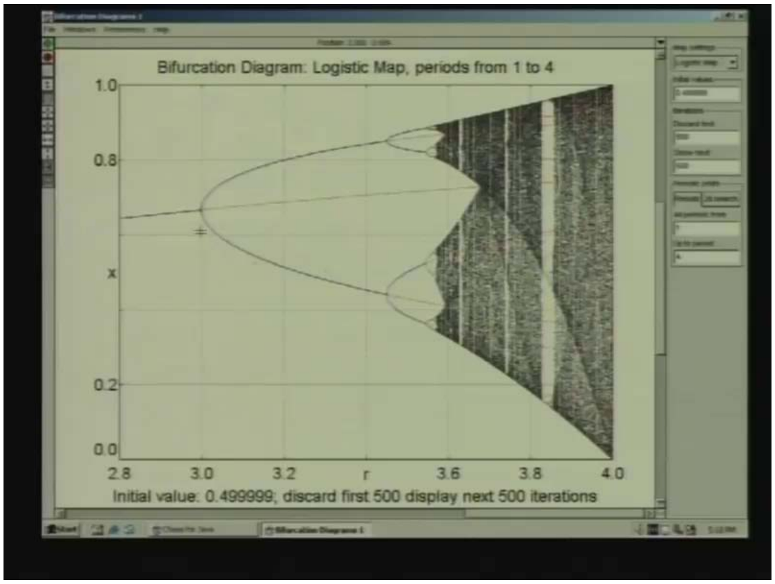

Period one this point, let me draw. This one is also the fixed point of the period eight behavior (Refer Slide Time: 23:21). So when I am talking about behavior of period 8, I am not talking about this one though that is also a fixed point of the period eight map that is the point of the prime period because those which are also the fixed point of any lower periodicity they are eliminated then only you got the… (Refer Slide Time: 23:43 min). It has nothing to do with the prime numbers. [Conversation between Student and Professor – Not audible ((00:23:50 min))] yes before that the system will be there, that come from infinity. That is the descending order. Here also it starts from infinity but in all this intervals you find things accumulated into infinity.

So if you ever find a periodic orbit with period three which means that all the periodicities must be there but all these periodicities are unstable. So if you have a period three orbit which means that at the same time for the same parameter value, all the other periodicities are existing and that is why the famous theorem by Li and Yorke period three implies chaos that can be said to be a corollary of this Sarkovski theorem but the historical fact is that Yorke and the other people they discovered it independently without any idea that Sarkovski's actually had this theorem already in place but Sarkovski preceded those people. In fact much of the developments in fields related to mathematics, you will find happened in the state space Soviet Union while people didn't know that these things already have been done. This is one of such theorems.

This means if in any system, if you ever find a period five orbit then except for period three, all the other orbits must be existing there. If you ever find period three all number of periodicities, including period infinity must be existing means there should be chaotic value orbit. [Conversation between Student and Professor – Not audible ((00:25:40 min))] His question is in the period three window, the chaotic orbit is there, all these periodicities are there existing only thing is that they are all unstable while the period three has become stable that's why you see all the initial conditions converging on to the period three orbit.

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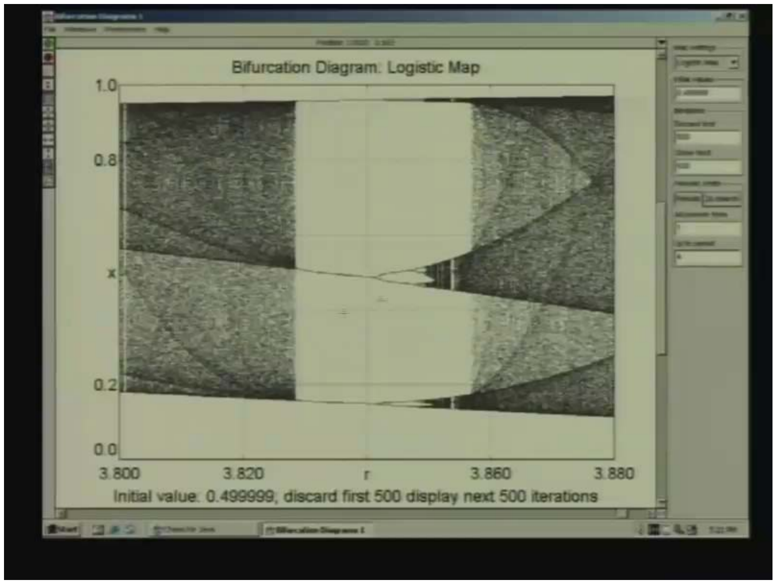

[Conversation between Student and Professor – Not audible ((00:26:10 min))] It is related to that but this phenomenon I will explain a little later. This only tells you that this orbit was actually preexisting and it became a part of the chaotic orbit at this particular point of value but this orbit was actually preexisting. One clue I can possibly give you, notice that at this period three point there was a saddle and a node. This is the node, this is the stable fixed point and this is the unstable fixed point saddle and that fellow where it intersects with the chaotic orbit and that is exactly where you find this sudden expansion 27:21). We will work on that line later when we talk in details about this particular phenomenon.

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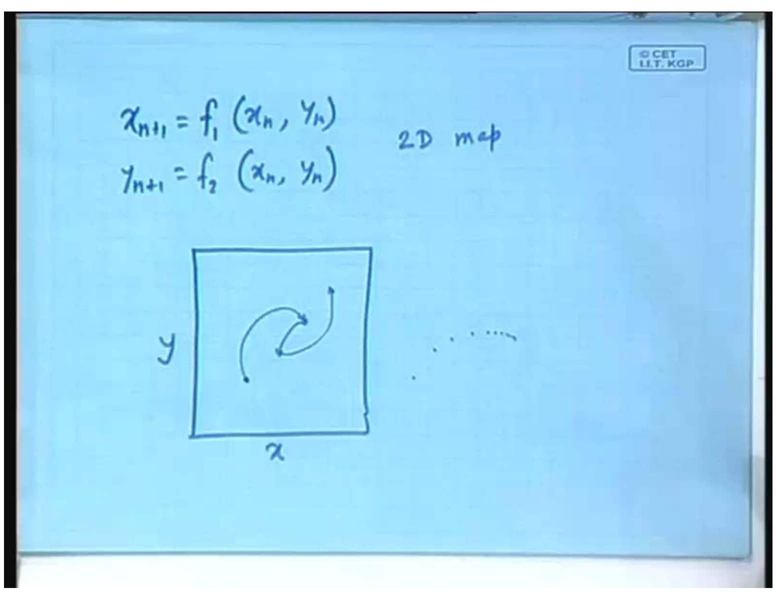

Let's come to what we actually plan to discuss today that is 2 D maps. so we see we ended the last class yesterday saying that let us consider x_{n+1} is equal to f_1 of x_n, y_n and y_{n+1} is equal to another function, so that will be 2 D map. Likewise there can be 3 D maps, 4 D maps on all possibilities but let us take the next step to the 2 D maps. So you have the x and y and then what this map doing? Usually starts from a point which means it has a particular value of x and y, x_n y_n by this it takes a jump to another point and if you go on applying this map again and again that means if starting point here we get the first one, apply the same map on this one you get another one point; apply the same map on this one you get another point and so on and so forth. So you get a succession of points, in other words you are iterating the map. You know this pronunciation is not iterate, its pronunciation is iterate many Indians make this mistake of wrongly pronouncing it.

So you are iterating the map like the pronunciation of Linux is Linux not line axe many people make this mistake of pronouncing it wrongly. So here you are iterating it and as a result, you will get a sequence of points. A sequence of points if they converge like so on to a point, there we would say the orbit is stable. So here is x_n, x_{n+1}, x_{n+2}, x_{n+3} and so on and so forth but ultimately if you observe them to be convergent sequence then you have a stable behavior. Can you picturize what is happing in the actual continuous time state space in which you have placed a Poincare section and on the Poincare section you are seeing this. A sequence of points that is convergent, what is actually happening in the three D convergent time space? It is actually converging on to a limit cycle, starting from an initial condition that is away from the limit cycle.

So if you cut it, it will intersect at points that are progressively closer to the ultimately the point at which the limit cycle crosses the Poincare plane. Then by this method you are trying to understand the stability of the limit cycle but you are doing it with a much simpler tool namely a map of this form. [Conversation between Student and Professor – Not audible ((00:31:00 min))] No, limits cycle can exist for a three dimensional system. Limits cycle can exists for a 2 D system yes but period two orbit can exist only for a 3 D system. That's why when we talk about the two dimensional map, you might imagine that we are considering a three dimensional continuous time system in which you have placed a Poincare of plane which is 2 D. [Conversation between Student and Professor – Not audible ((00:31:36 min))] not really meaningless because as I said there can be a limit cycle in 2 D like this and you can place a Poincare plane in that case it will be a line.

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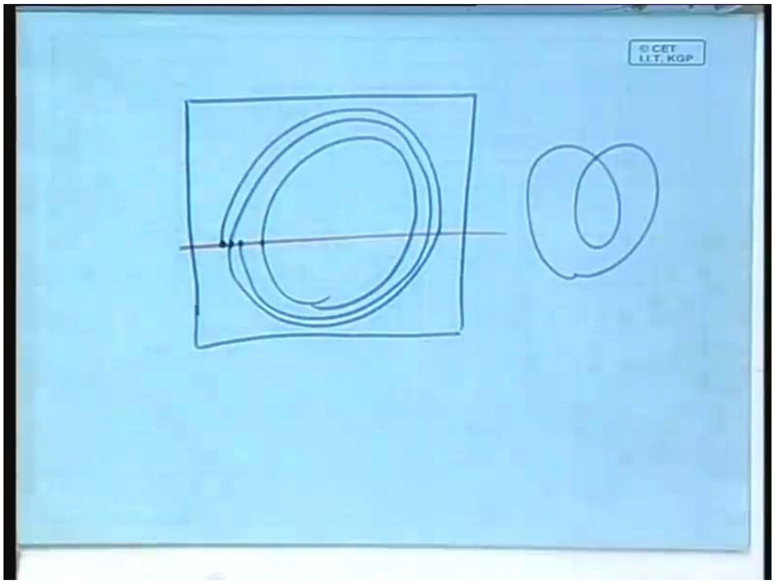

His question is does a one dimensional map meaningful in the sense of the Poincare section? Yes it is meaningful in the sense since of the following. Supposing there is a 2 D space in which you have got a periodic orbit. So here if you want to place a Poincare section, it will be no longer will be a plane, it will be just a section and if you start from an initial condition somewhere here, you will see it coming here and then again it coming here. So here you see a sequence of points, ultimately it will converge on to this point and that is exactly what we are studying as the map. So here is the one D map but this is another part of the story. Here if the system is autonomous then you cannot have a periodic orbit of periodicity greater than one, if it is autonomous system I explained that earlier.

If it is autonomous system, you cannot have an arbitrary like this in 2 D because then this would be an intersection point, you cannot have an intersection in an orbit. This argument will work only for autonomous systems, for non-autonomous system from where you have got a periodic forcing function in that case time becomes a third variable. So that you can have this kind of orbits with two D to special dimensions and one time dimension. [Conversation between Student and Professor – Not audible ((00:33:52 min))] Noise input is not considered to be an input because you are ultimately studying the oscillatory behavior. Noise is something spurious that is not predicted where we are talking about everything that is modulable that can you can write a differential equation for, you cannot write a differential equation for noise.

In any system if there is some kind of a periodic input from outside then it's non-autonomous, if it is an oscillator that oscillates by itself without any periodic forcing then it is an autonomous system. Naturally it brings you to the conclusion that you cannot have an oscillator without three dimensions. So you can still have a periodic orbit or a chaotic orbit or everything in 2 d, if the system is periodically forced, non-autonomous. In that case if you place the Poincare section it would still be one D.

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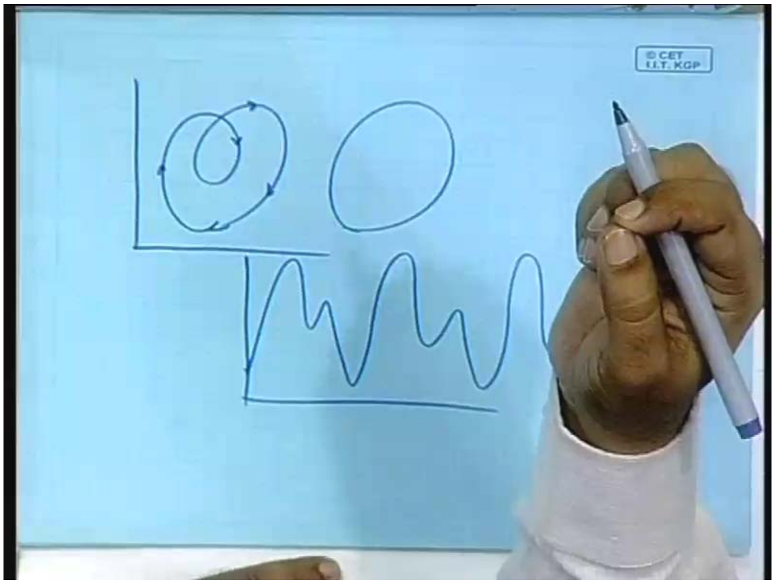

So in that sense one D maps are still relevant as a Poincare plane. Just if you have lc then it cannot generate the orbit like this, it only generates orbit like this (Refer Slide Time: 35:21). This one is also an oscillator that you can have this. An lc oscillator cannot have an orbit like this, it will have an orbit always like this and it is mathematically showable that it cannot have an oscillation like this, simply by lc, it cannot happen (Refer Slide Time: 35:33). [Conversation between Student and Professor – Not audible ((00:35:41 min))] No, if you are considering the autonomous system their time is evolving like this. As you are going around the orbit, time is going forward.

[Conversation between Student and Professor – Not audible ((00:36:07 min))] No, in that case what you will say as a periodic orbit? A periodic orbit means which comes back to the same value of the state after some time there is a periodic orbit. Now you would notice that if you start from here after this much it come back to the same state and therefore in every sense it is the same point but if there is a periodic forcing say a sin omega t kind of term then it might come back to the same point but it may be a different phase of the forcing, so it will be a different point effectively.

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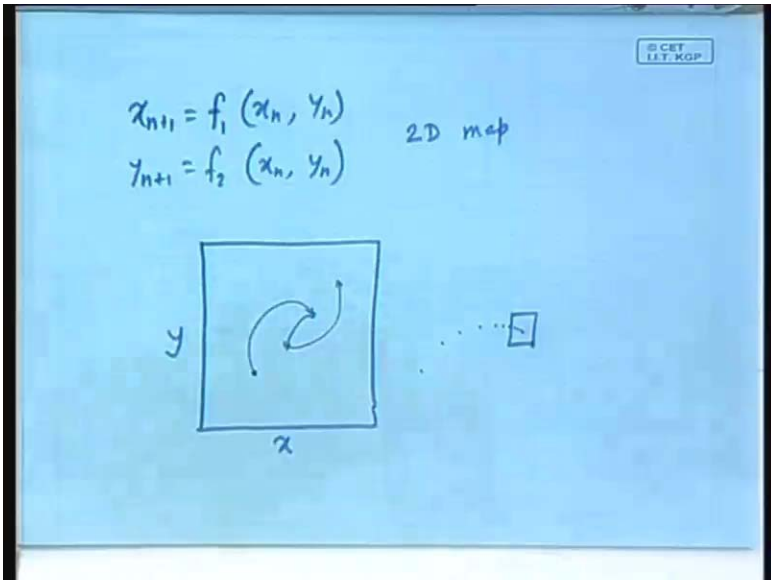

So you have this sequence of points ultimately converging and like in the two D systems or like in the continuous time systems when we had an equilibrium point, how did we study the behavior of that? By locally linearizing, here also we'll locally linearize along this part and try to study this behavior. Local linearization again will be done in the same way by taking the Jacobian matrix.

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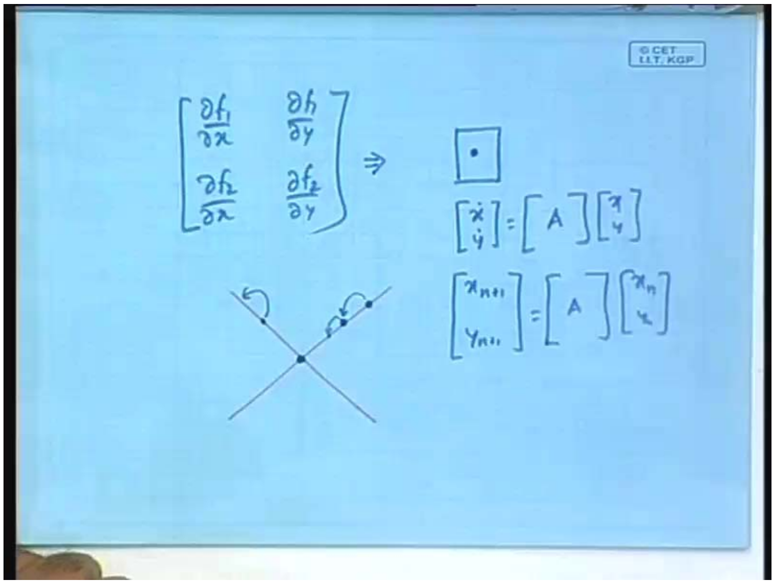

So if you have the f_1 and f_2 given then your Jacobian matrix will be given by (Refer Slide Time: 37:40). Now what are the possibilities out of this? As we have already seen in case of 2 D continuous time systems, this having obtained in the Jacobian this essentially tells that if this is my fixed point, here I am considering only in the local linear root. So I am taking a magnifying glass and look only looking at close neighborhood of it and the behavior around it will be given by its eigenvalues that's what we have concluded. I am not repeating that we have already concluded by some logical procedure that the behavior around it will be given by the eigenvalues. Here also it is and we have also seen what can be the different types of eigenvalues. It can be real, it can be positive, it can be negative, it can be complex so they are the possibilities. So initially let us see what happens if the eigenvalues are positive real, if the eigenvalues are positive and real then there should be identifiable, drawable eigenvectors. You can draw the eigenvectors, imagine that here is a fixed point and here are the two eigenvectors.

Now suppose I start from this point, here is a fixed point and here is my starting initial conditions. My question is where will it go? Obviously if this is the eigenvector then where will it land next, will be given by this vector times the eigenvalue. So if the eigenvalue is less than one then this distance will be multiplied by a number less than one, as a result it will land somewhere closer. So this will be the map (Refer Slide Time: 40:00). Notice one important distinction between what we learnt in the case of the differential equation and what we are learning here. In case of the differential equation, our situation was x dot y dot is equal to this Jacobian matrix times xy. Here our situation is that x_{n+1}, y_{n+1} is equal to that Jacobian matrix times x_n, y_n. So in case of the differential equation this right hand side tells the direction of its motion and how fast it will move and here really you have to solve the differential equation in order to see where it goes but here it directly tells where you would land next and that has a important bearing on our conclusions about the matrix A and the eigenvalues.

For example as you have seen that if the eigenvalue is less than one positive then it will go like this. Where will be the next iterate be? Again this distance will be multiplied by that same number so it will come, it will not move by the same extent, same length will be multiplied by the same factor so it will be closer but this distance will not be same and it is not difficult to see that in progressive iterates, it will progressively converge on to the fixed point. If the eigenvalue happens to be greater than one say along this direction it is greater than one, so where it will go next? This distance will be multiplied by a number that is greater than one, so it will map like this and it will never converge on to the fixed point. Do you see that we are now arriving at a condition for stability. System will be stable, if the eigenvalues are less than one which is a different condition from what we learnt for the continuous time system. There the condition was that the real part has to be in the negative side but here the condition is different condition for stability is different.

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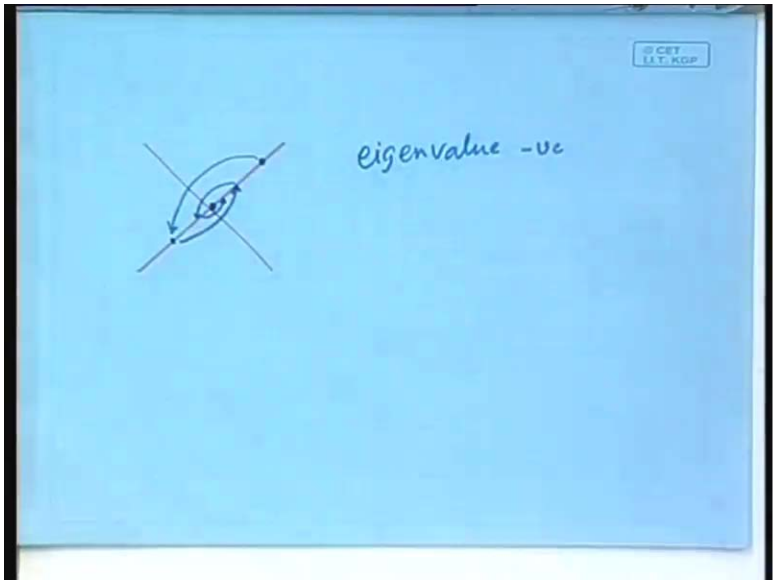

Now let us understand what will happen if you have a negative eigenvalues? Suppose this is the fixed point and these are the two, suppose you start from here, this is an eigen direction with a negative eigenvalue. So this distance will be multiplied by negative number which means that it will come somewhere here, so it will slip. [Conversation between Student and Professor – Not audible ((00:42:58 min))] No, e to the power things come in case of the continuous time systems. Here see the behavior is like this xn plus one is equal to that times the vector itself. Let me answer the question properly. When you do the diagonalization of this, here there was a dx dt and that led to the e to the power term, here there is no need that is why the maps are far simpler than the differential equations. [Conversation between Student and Professor – Not audible ((00:43:40 min))] once we have got the actual functional form or some kind of a representation like this, no its corresponding continuous time representation has to be greater than 2 D, greater than or equal to two D because in case of non autonomous system it will be the same dimension but once we have got this one, we will forget about the continuous time system.

Only in our understanding we will relate but we will not do the mathematics with the continuous time description any further because that is far more complicated. Once we have got this, we got a far more handleable description. So in this case what is happening? It is flipping to the other side, if the eigenvalues are negative. Next iterate it will again flip to this, next iterate it will even flip to this side so and so forth but ultimately it will converge while flipping between the sides that's why such a behavior is called a behavior flip.

Now we are in a position to sort of name equilibria. Notice that I am using two different words. In continuous time dynamical systems, we were using the word equilibrium point while in a discrete time dynamical system or a map we are using the word fixed point. So these are the nomenclatures though in some books, you will find that they are interchangeably being used, I don't prefer that in order to distinguish between the two types of systems. So we will talk about fixed points here.

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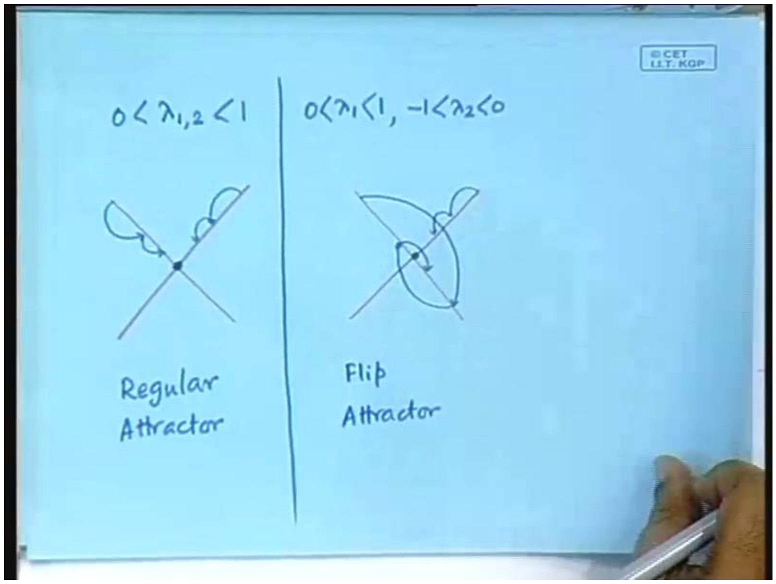

So first condition; both eigenvalues between 0 and 1, positive and between 0 and 1 then if you have the fixed point here and if you have the two eigen directions like this then what will be the behavior? In this direction also it will be like this. So the fixed point becomes an attractive fixed point, this will be called a regular attractor. Now suppose you have taken a situation where you have one eigenvalue between zero and the other eigenvalue between…47:23 then what will be the behavior? You will draw the positions like this and so there will be one direction along which it will be positive so let say this direction alone which it is positive. So it will be here like this but along this direction it will be like this. So it is a flip direction, it will be called a flip attractor.

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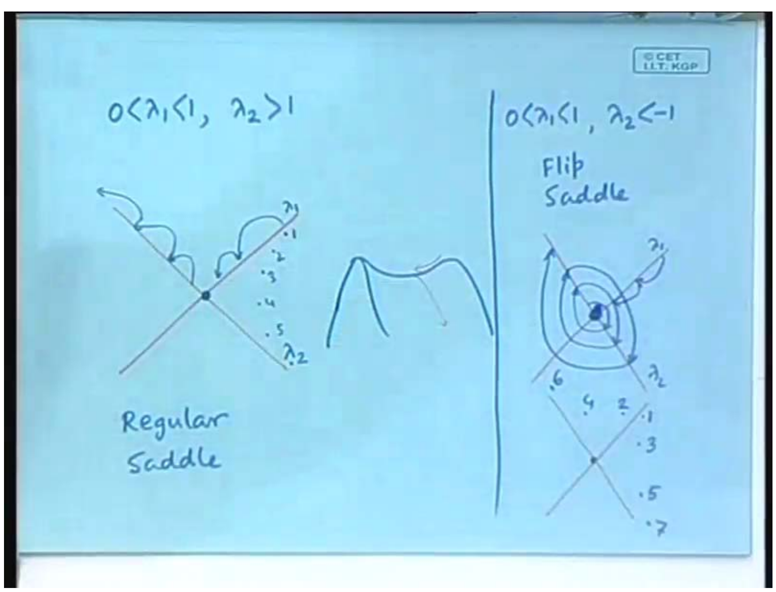

If you have a situation where and then it will no longer remain an attractor, its behavior will become like so there will be another direction of lambda two, along the direction of lambda one it will be so on and so forth. It will be converging but along the direction lambda two, eigen direction associated with lambda two it will be going out. If you start from an initial condition other than the eigenvectors what will happen? Say if you start from here what will its behavior be? As time progresses along this direction it will shrink, it will come closer to this, along that direction it will expand and it will go out. So you will get a succession points something like this so it will go out. Imagine it like this that it has a component along this eigen direction as well as along this eigen direction, the component along this eigen direction will decrease and component along this eigen direction will increase and so it will be one two three four five and so and so it will go out.

One thing that I find often students take time to digest is that if the moment will not be a continuous moment, it will be discrete jumps. The moments will be discrete jumps such a fixed point will be called a regular saddle because it also has the property of a saddle. I told you this saddle has behavior something like this. So its direction in which it is stable is this direction and the direction in which it is unstable is sort of this direction. So this is also a saddle behavior. We have come across the saddle behavior in continuous time system, similar behavior here. If you have a lambda two less than minus one then you will have the fixed point and the eigen directions.

Suppose this is the lambda one direction and this is the lambda two direction, what will be the behavior in lambda one direction? It’s same, so on and so forth. What will be the behavior in the lambda two direction? Flip but at the same time going out, so if you start from a point close it will go like this. The lines that I am drawing are not really lines, I am drawing discrete jumps but in order to show the discrete jumps I am drawing. Don’t imagine that it will actually continue to move in this way, it will take discrete jumps along this eigen direction. Can you tell me what will the behavior be if the starting point is away from the eigen directions? Say my starting point is somewhere here. It will come closer to this one (Refer Slide Time: 52:51).

No, there is nothing ellipse here because it discretely jumps and naturally it cannot go into electrical path which is a continuous time idea quoted here that will not work. Notice that in the next jump, the distance along this will reduce, distance along that will increase but at this stage it has to flip from this side to that. So the next iteration will be somewhere here, third iteration will be somewhere here, fourth iteration will be there, fifth iteration there, so 1, 2, 3, 4, 5, 6 so on and so forth, it will go out. Ultimately it is coming closer but it is flipping from one side to the other side. Such a fixed point will be called a flip saddle. It is a saddle of course but it has a flipping property from one side to the other, so it will be called a flip saddle. Ultimately it will go out into infinity but while it is flipping from one side to the other side and it will go out to along this eigen direction. It will converge progressively along this eigen direction and it will go out along that eigen direction and since we are considering it to be a linear system, we might say that it will go to infinity but actually since it is a non linear system, it might not go to infinity. We will come to those issues later but as far as the linear description is concerned it is convenient for us to understand that it will tend to infinity but along this eigen directions. So we will continue with this in the next class.