Chaos, Fractals and Dynamical Systems
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Lecture No. # 11
Intermittency Transcritical and Pitchfork Bifurcations

After the last class some people came to me and asked some questions and I want that to be shared on camera. So ask the questions now.

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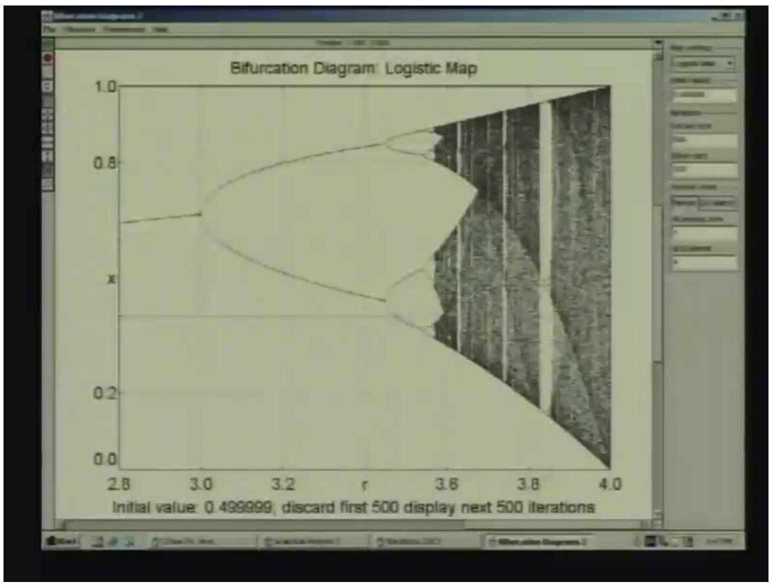

For example if you look at the computer screen. You might notice that this window is larger, this window is narrower, this window is narrower and as I showed you yesterday if you zoom any part of it you will still see other windows. Now out of this some are wider, some are narrower. In general the thumb rule is that lower the starting periodicity of a window, the wider will be the width of the window. That means this particular wide window is starting at a periodicity three and then it is going through a period doubling cascade through the periodicities of 6, 12, 24 and so on and so forth. While here it is another window, if you zoom you will be able to see.

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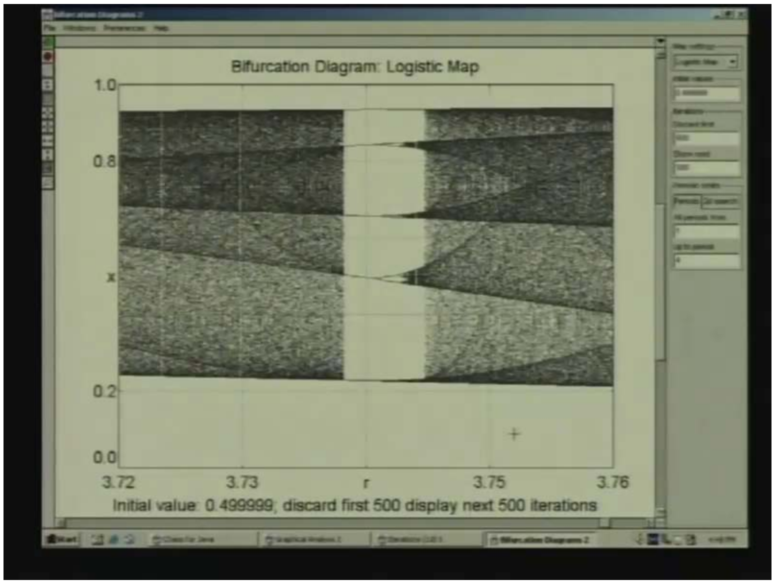

This window is period 5. It is higher periodicity, starting periodicity is higher and so it has a smaller width. Now why are these bigger or smaller? If I want to explain, it will require some bit of calculation which are not necessary at the moment. It will only complicate matters but it is because of that. Remember that inside the chaotic orbits when it goes to chaos they are all those periodicities ((Not understandable Refer Slide Time: 00:02:31 min)). So if I go back to the original one, yes it's here. So when it goes to chaotic orbit here, can you see the cursor (Refer Slide Time: 2:42), if it is somewhere here then the period one orbit has given rise to the period two orbit but this fellow is still there. If you want to see the existence of the unstable orbits then you can see that. Here the red one is the unstable orbit that continues to exist. At this point the stable period two orbit became unstable and gave rise to a stable period four orbit but the unstable period two orbit still kept on existing. So they go on continuing all through.

Naturally when we talk about this part here, there are all those unstable periodic orbits inside the chaotic attractor and then at this point the period three fellow comes into existence and that goes into chaos through the 6, 12 and so on so forth. So all those unstable periodic orbits will be existing after this, so on and so forth. Here essentially his question is concerned whether why is this wider? The answer is grossly that this is the largest window because this is starting at a periodicity three which is the minimum available within the chaotic region.

In fact there exists a theorem that says if we ever observe a period three orbit that means it is inside a chaotic orbit, it is inside a chaotic range of parameters. The famous paper by Li and Yorke proved this was titled period three implies chaos. So period three is always inside the chaotic range of parameter values but before that period one and period two are outside. Any other question? [Conversation between Student and Professor – Not audible ((00:04:45 min))].

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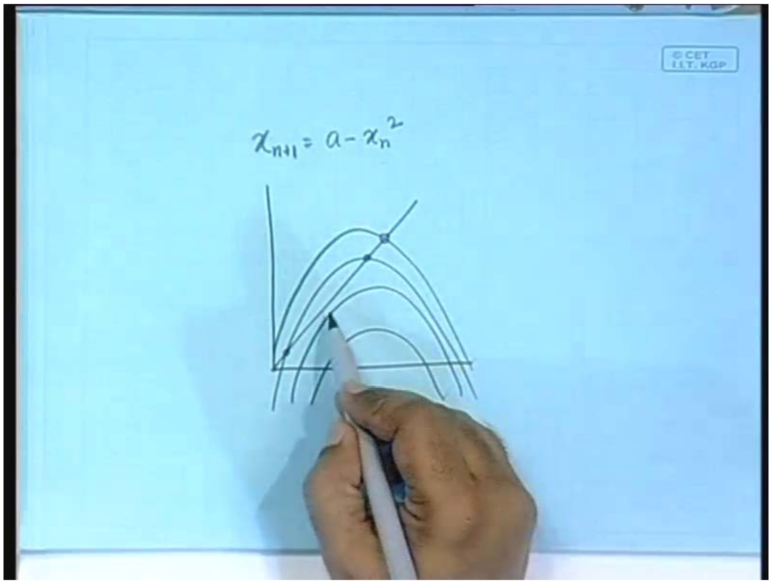

We have taken a minus x_n square. [Conversation between Student and Professor – Not audible ((00:05:03 min))]. We had obtained first with this a behavior something like this and it keep on changing like this. So we had illustrated the occurrence of what is known as the saddle node bifurcation or the tangent bifurcation with this, which gives rise to the starting or the birth of two fixed points at these two points.

Now after this as you change the parameter a farther that it goes up and up and up, you can easily see all that we learnt for the logistic map should also be here because here is the fixed point whose slope at some point of time will become minus one. This will give rise to period doubling and then if you have convinced yourself that the period doublings nature is the same irrespective of the type of the map so long as it is a smooth, one humped map. It will give rise to period 2 to period 4 to period 8 to period 16 and so on and so forth. This will also lead to the same behaviors. Why did I take this one not the other one? Because this one illustrates the saddle node bifurcation better, no other reason just to illustrate.

There was one question regarding whether there can be a trifurcation. Bifurcation is an English word that existed prior to discovery of these things. So that has a particular meaning, particular connotation. When it came to scientific usage that also thought with it a specific connotation which is that any change in the asymptotically stable behavior of the system is called a bifurcation even if it is something like this. So if it is something like this you would not call it trifurcation, you would call it a bifurcation from a period 1 to period 3 orbit but can it happen. It normally doesn't happen.

If the map has certain property for example, if it is a smooth map; smooth means everywhere differentiable, if the map is everywhere differentiable such a thing cannot happen but if the map is non-smooth, if is not everywhere differentiable we can show that it is possible to have this kind of behavior. I will illustrate when and how. [Conversation between Student and Professor – Not audible ((00:07:48 min))] Purely imaginary eigenvalues cannot happen in one dimensional map. Here we are talking about maps. Are you talking about differential equations? His question is when we were considering differential equations and we were considering linearization of the differential equations at an equilibrium point.

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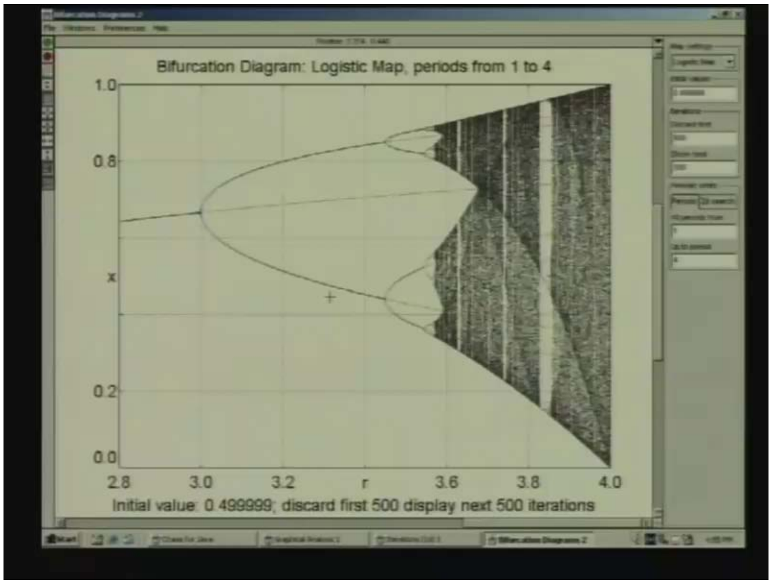

Suppose we have obtained the eigenvalues and found the eigenvalues to be purely imaginary. Then what do we say the behavior would be? It would be something like this. His question is how do you know there will be circles, there could be ellipses? What is this axis, what is that axis, are they in the same scale? No they are not, because one is the position and the other is the velocity or momentum. The units are entirely different so there is no reason to believe that really there will be circles. So all you can say is that the behavior will be topologically equivalent to circles by changing the coordinates, you can make them circles.

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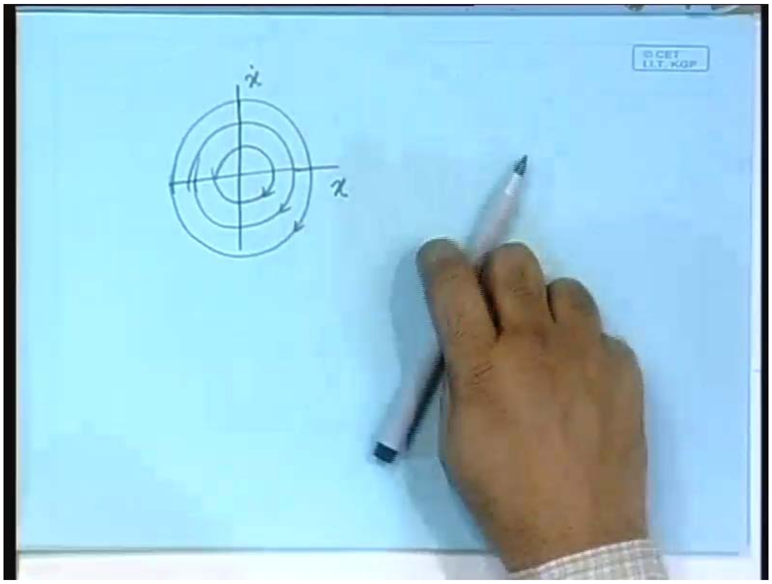

There was another question if I remember which is more or less illustrated by what you see on the screen now that when it goes into chaos; essentially it has a huge number of periodic windows inside. We can visually see the period window starting with the period 3, period 5 and here is period 6 but if you look closer, if you zoom any part of this bifurcation diagram, zoom that means you expand it with a smaller range of the parameter. For example I can show you a bit of it. For example let me show you the part that looks like chaos.

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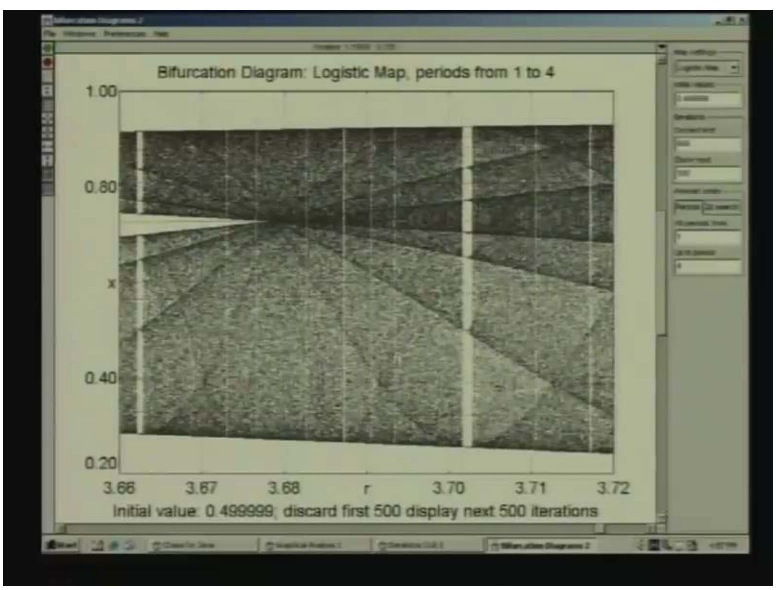

There are more windows. The part that we could not resolve any periodic window, there are steel periodic windows. Here you can see it is period eight windows, something that starts with period eight and then goes through the double periodicities and so on. So inside there are infinite number of such periodic orbits. In fact there are theorems that show and take any range of the parameter, you will find that in the neighborhood of that there would be some periodic window, may be very high periodic windows but nevertheless there will be. So if you are inside, if you are sitting in the parameter space at a parameter value for which the behavior is chaotic, look this way or that, you will find periodic windows.

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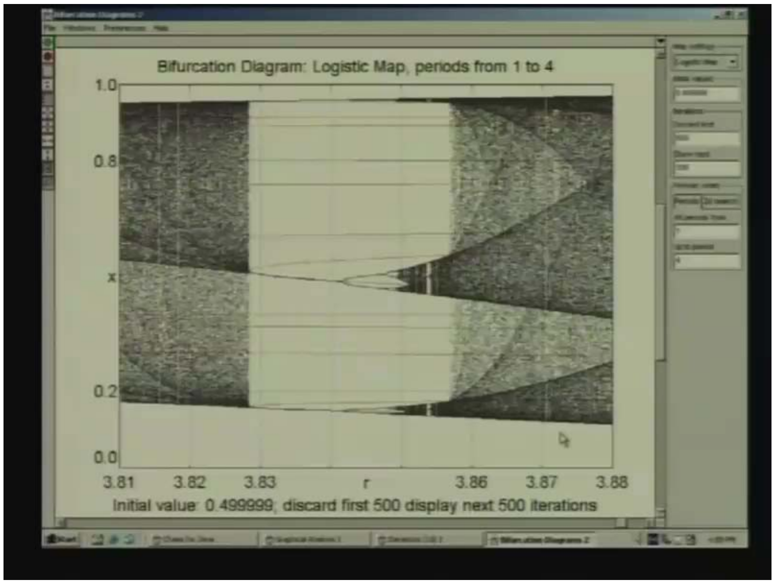

When we ended the lecture yesterday, we were talking about this window. So let us expand it and look at it carefully, here is the period three window. At this stage I do not want to see the unstable periodic orbits (Refer Slide Time: 12:21). Now let us look at this particular transition carefully. What is happening here? You have understood one thing that adds this specific parameter value, what happens? There is a saddle node bifurcation occurring in the third iterate of the map.

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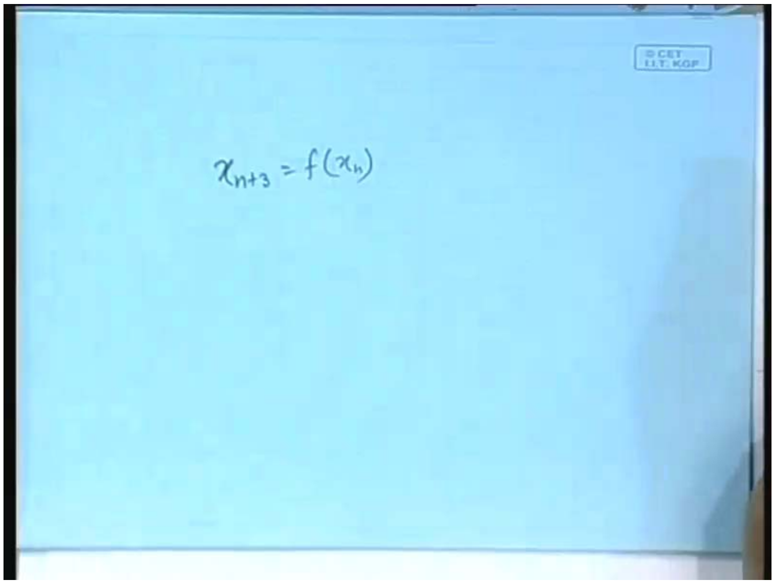

That means if you take x_{n+3} as a function of x_n and plot the graph of the map then we would find that graph is undergoing a tangent bifurcation, resulting in the creation of these three stable points. What is just to the left of that? Say we are here, take a parameter value somewhere there and let's see what happens. Something like this which is not periodic, which is reasonably chaotic but you would notice that it is coming again and again to a behavior say look at this to a behavior somewhat close to a periodic behavior.

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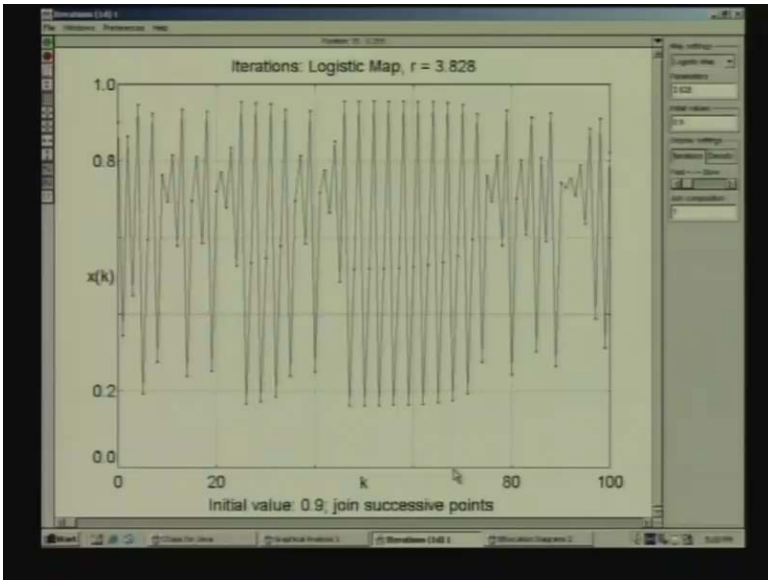

In fact if you take values closer then you see the amount of periodic behavior is increasing, as you come close to close to this point you will find the range of periodic behavior is increasing but it will not stay; not that it will stay in this behavior. As if there is a somewhat intermitting bus of periodic windows periodic behavior in the middle of chaotic behavior. The question is how does that happen? In fact in many experimental situations you will find that the behavior is more or less chaotic, you can see that it is chaotic but suddenly when you capture the wave form on the CRO screen you find that it is periodic behavior. What happened? It is just for that small span of time for which you did the grabbing of the wave form, at that point of time it was a periodic bust.

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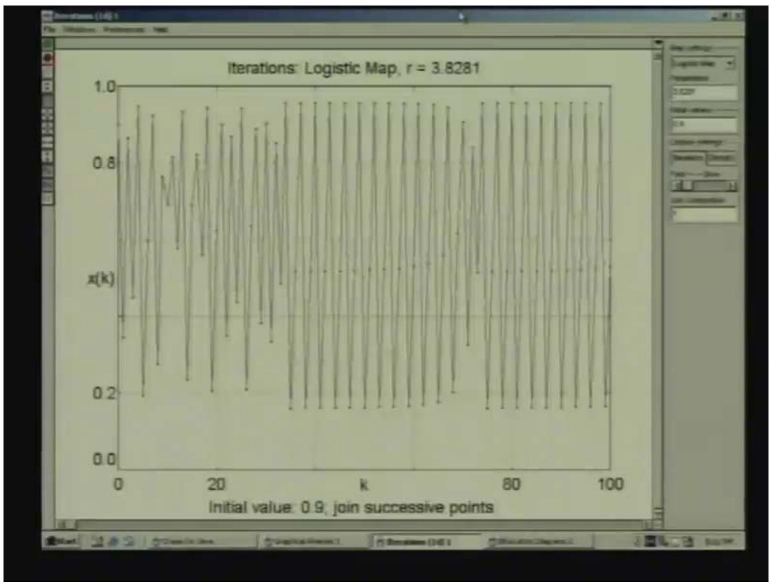

How does that periodic bust happen? In fact this phenomenon is called intermittency. The intermittency is intermittent bust of periodic behavior in the middle of overall chaotic behavior. So we are off to try to understand how does that happen? We have already seen that as the parameter approaches the value at which the period three orbit comes into existence.

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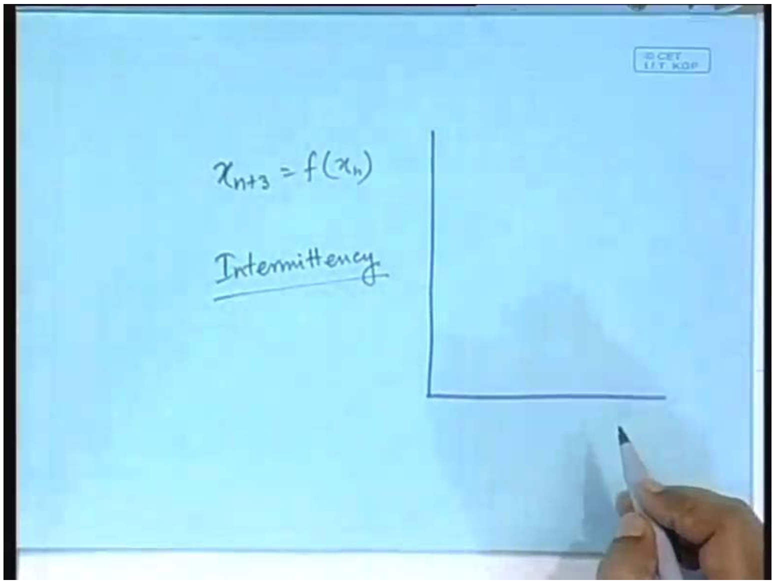

Look at the behavior say I will let it run. See it's the running chaotically but notice what is happening? Here there is an orbit that is visited more often and if it is visited, it more or less stays there for some time. That is why you see a darker line here. So in order to understand why this happens, let us draw the period three window, period three composition.

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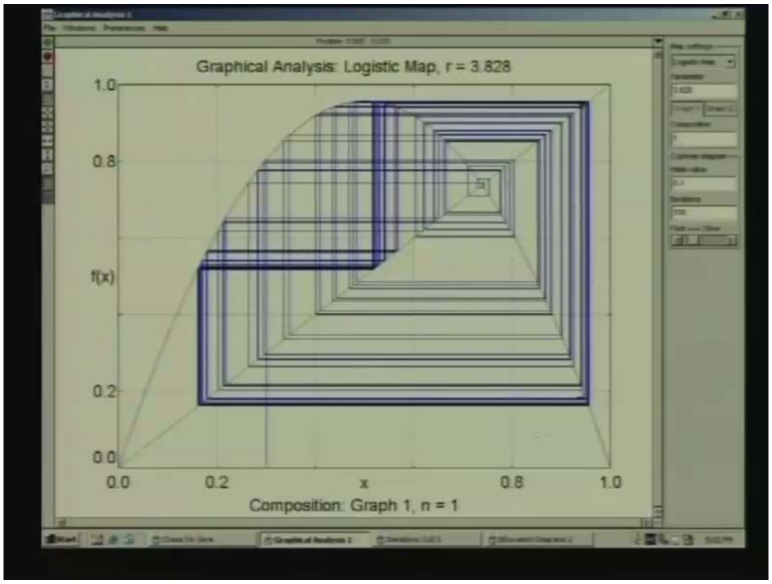

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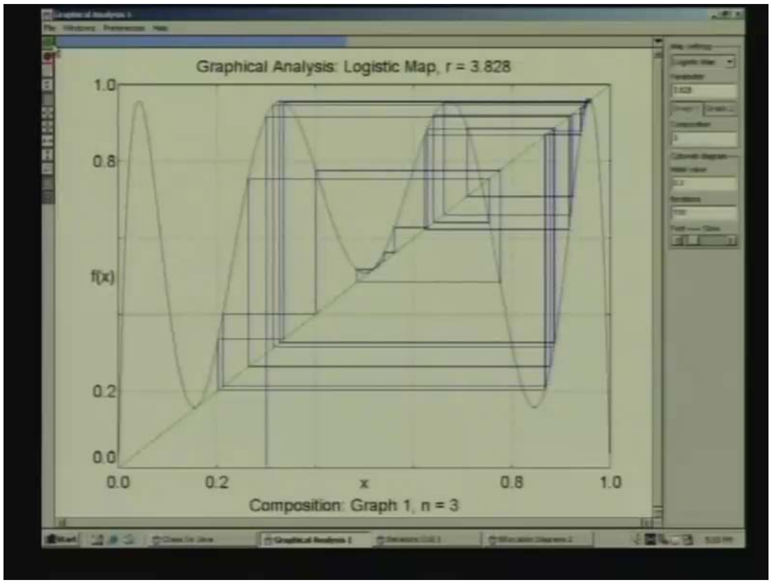

Notice what is happening, these chunks of the graph are coming very close to the 45 degree line. Here also it is coming very close to the 45 degree line and as the period 3 comes in to existence what happens it crosses the 45 degree.

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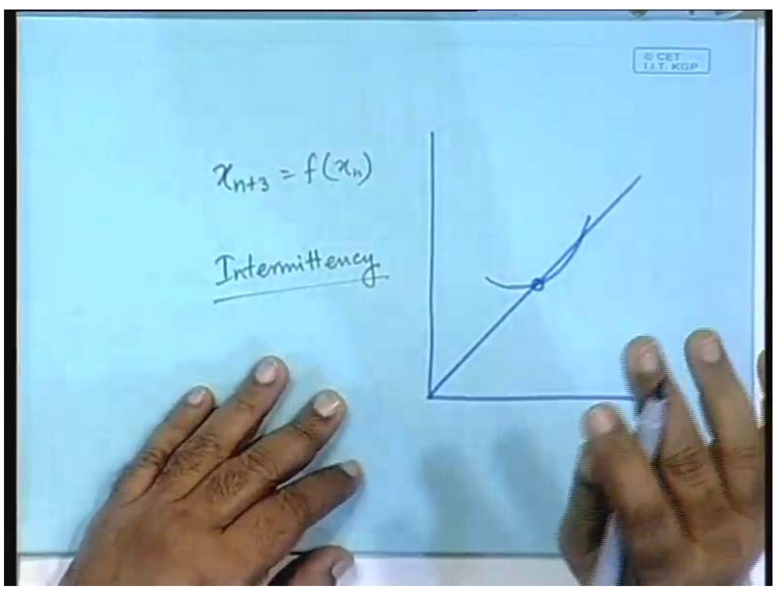

As it crosses you have this kind of a behavior and therefore that results in the creation of this as the additional fixed point but let us consider the situation when it is not gone into that state.

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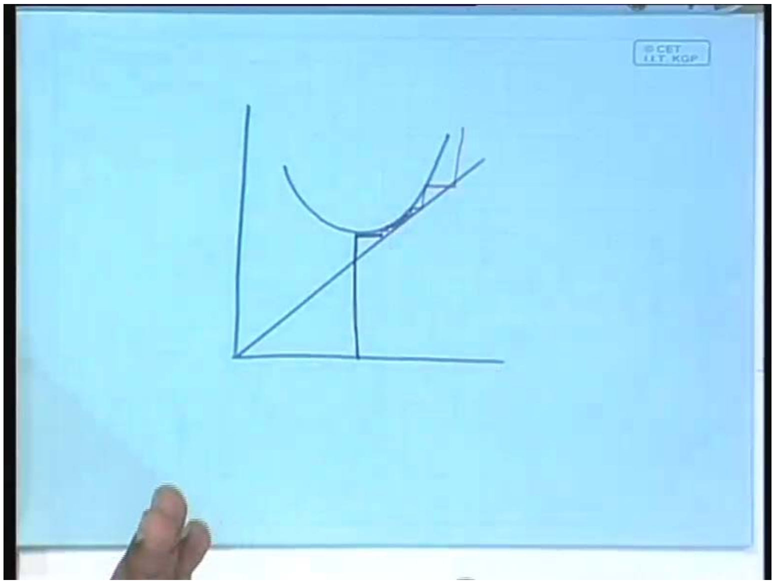

When the behavior is if you expand that part, it will be something like this very close but nevertheless not quite touching and suppose we start from an initial condition somewhere here. So we go to the graph of the map and we go to the 45 degree line and then if you keep on iterating it, I will do it in another color so that you can see it clearly. It will go like this (Refer Slide Time: 17:28) which means that this narrow passage between the 45 degree line and the graph of the map, it will take enormous time to cross the narrow passage, it's like a bottleneck. For a long time it will get stuck there, trying to cross it will move in that direction but it will take a long time to cross and after sometime it will cross like this and then it will go away.

So long as it is here if you look at it, it's behavior will be like a periodic bust and then it will again run around and finally if it is injected somewhere here, again it will go in to that periodic window. It is not really periodic windows though, it is just an intermittent periodic behavior in the middle of chaotic behavior which is called intermittency behavior. It precedes a saddle node bifurcation. This kind of a behavior precedes the occurrence of a periodic window.

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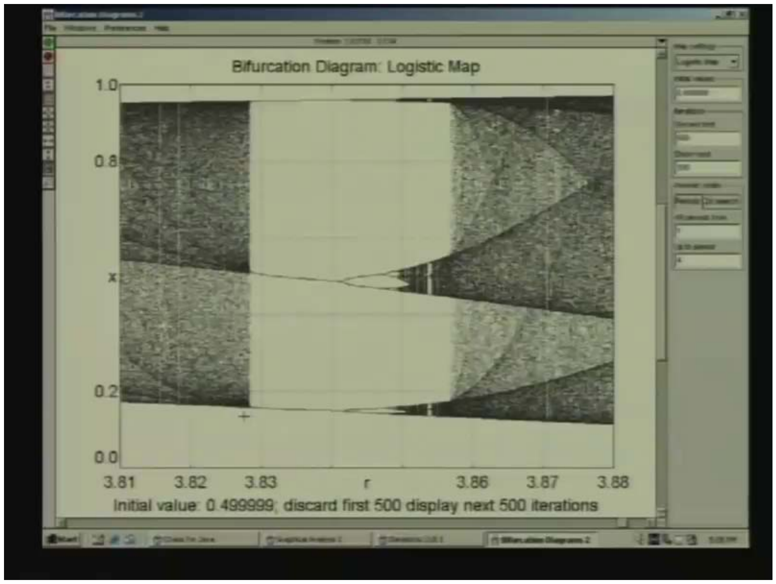

Where it is exactly happening? It is happening here inside the chaotic behavior, it is not here because here it is formally periodic period three (Refer Slide Time: 18:44). If you go this way then very close to this you will find that. Another thing noticeable is that notice this bifurcation diagram, the density of points are not the same everywhere. The density is larger here, larger here in this part and the larger here in this part (Refer Slide Time: 19: 10) which means that these parts were visited more often than the other parts. Why did that visit? Because of this phenomenon, if it is there then it gets locked for some time before it can go away. So that is the concept of intermittency which precedes a periodic window.

So as you can see in the whole bifurcation diagram there had been a large number of periodic windows and therefore that many number of situations where you are likely to observe intermittency but the intermittency will be of the same periodicity as the periodic window that is coming. In this part here, you are not yet in the periodic window, you are well inside the chaotic behavior but looking at the intermittent busts of periodic behavior you can say that another periodic windows is approaching. So that is the concept of intermittency.

Now let us understand somewhat rarer bifurcation that are observed in one dimensional maps. We have understood two things, they are very common and so you will have to have a very good idea about those two; one the saddle node bifurcation or tangent bifurcation and two the period doubling bifurcation but there are also some more which we will discuss now.

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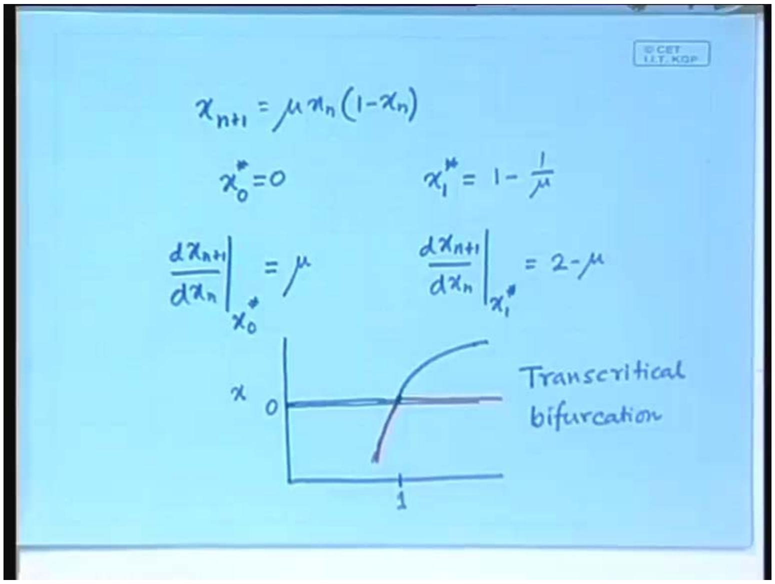

Let us start with the old logistic equation. We had decided that there are two fixed points of this, let us call them x_0 star and x_1 star. The x_0 star was at zero and x_1 star was at 1 - 1 by mu. The slope of this was dx_{n+1} / dx_n calculated at x_0 star was mu and dx_{n+1} / dx_n calculated at x_1 star is 2 - mu, this we have already done. Notice that when you are changing the parameter and it is say passing through one, the value of one. Let's see what happens. Notice here so long as mu is less than one, this fellow is stable and this fellow is unstable and the moment it goes through one this fellow becomes unstable and this fellow becomes stable but both of them were existing all through.

There is no reason to say that this fellow doesn't exist. The only way you would say something does not exists is, if you have a quadratic kind equation with a square root and inside the square root you have a negative number, so you said that it's yielding a complex fellow which cannot happen because I am dealing with a real numbers. So a real fixed point would mean that its position is given by a real number. If I get a complex number or imaginary number I know that it doesn't exists but here it's not so, therefore the fellows exist but their stability status change. If I now draw the bifurcation diagram, will it not look something like this. I will plot the zero line somewhere here and let's see what happens and this is say a value of one, so this is x equal to zero (Refer Slide Time: 23:42).

So the fixed point that is x_0 star was existing and was stable up to this point. After that it continues to exist at the same point but becomes unstable, so let's draw that with red color. What about the other one? The other one is given by this position, so at one it will have the value zero. So it will start from here and it will move like this and now it is stable. What is its behavior when the mu was less, it is unstable but it continues in the same. It is there in the other side also.

So you would notice that at this point what has happened, there were two fixed points which came close to each other, collided but they didn't vanish, as it happens with the saddle node bifurcation but instead they extends stabilities, while this fellow becomes stable this fellow become unstable, earlier this fellow was stable that was unstable. So it is like, this fellow was unstable now it is becomes stable this fellow was stable, now it is become unstable they extend stability. Such a bifurcation is called a transcritical bifurcation and they are somewhat rare. You won't easily find that in real physical systems but they do happen that's why I am teaching otherwise there has be no reason for me to teach but the frequency of occurrence of this kind of situation is relatively small. For example the transcritical bifurcation for this logistic map happened only once, while period doubling happens infinite times.

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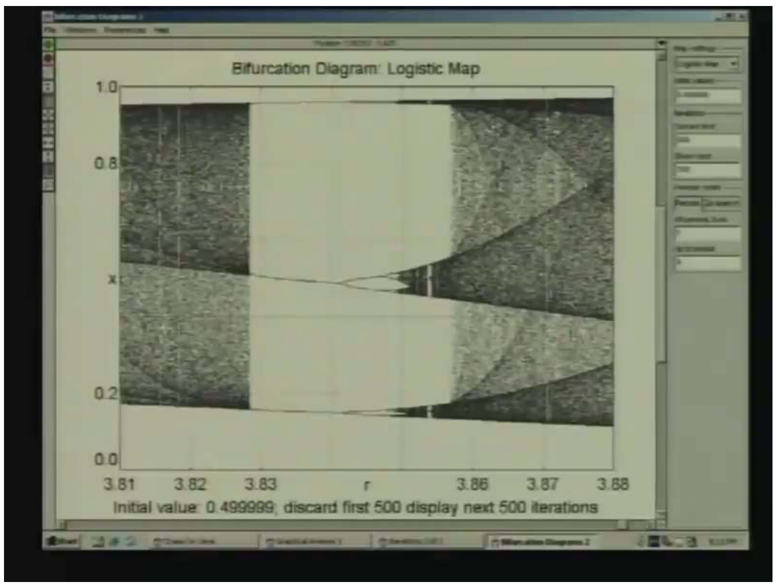

[Conversation between Student and Professor – Not audible ((00:26:32 min))]. No, frequency means it happened for this map, only for this parameter value and at no other parameter value. It happened only once while period doubling happens infinite number of times. Saddle node bifurcation happens infinite number of times but the transcritical happens only once, so you will find that somewhat rarely in real physical systems. There is another type let us try to understand, that will not be clear from the same map.

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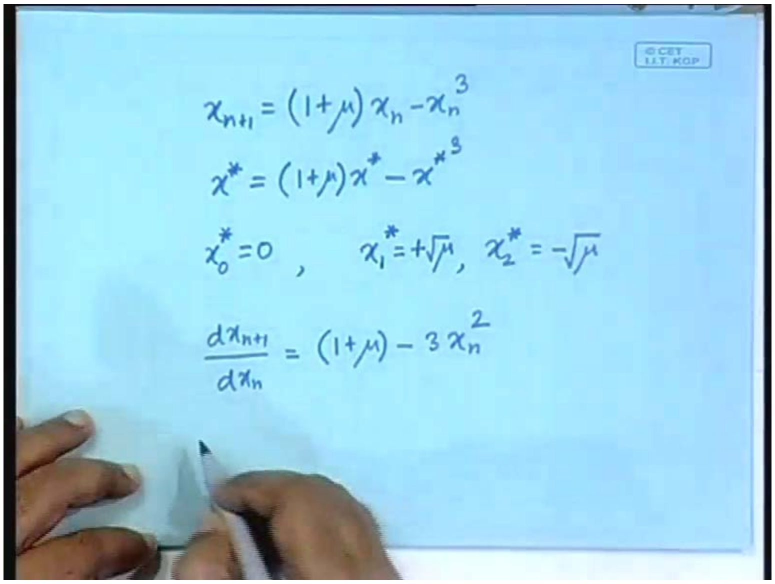

I will introduce a different map, here it is x_{n+1} is equal to 1+ mu x_n – x_n cube. Now I will allow you to do this problem. Can you investigate the stability of this? First where are the fixed points? In order to find fixed points, you have to say that x star is equal to 1 + mu x star minus x star cube, the left hand side is equal to the right hand side. Obviously there is one fixed point at zero, so one fixed point at let's say x_0 star is at zero and the other fixed point; there are two other fixed points so you have x_1 star and x_2 star. Let's call them like this and this is plus root mu and so plus minus root mu. So there are three fixed points.

What is the next step? We study their stability and in order to study their stability we differentiate this. We essentially study dx_{n+1} by dx_n as evaluated at these three points so this fellow is 1 + mu - thrice x_n square. So what will be the condition of stability of this fixed point? You substitute zero here, 1 + mu is less than one so mu is… [Conversation between Student and Professor – Not audible ((00:29:45 min))] 1+ mu should be less than one, mu should be negative but greater than minus 2.

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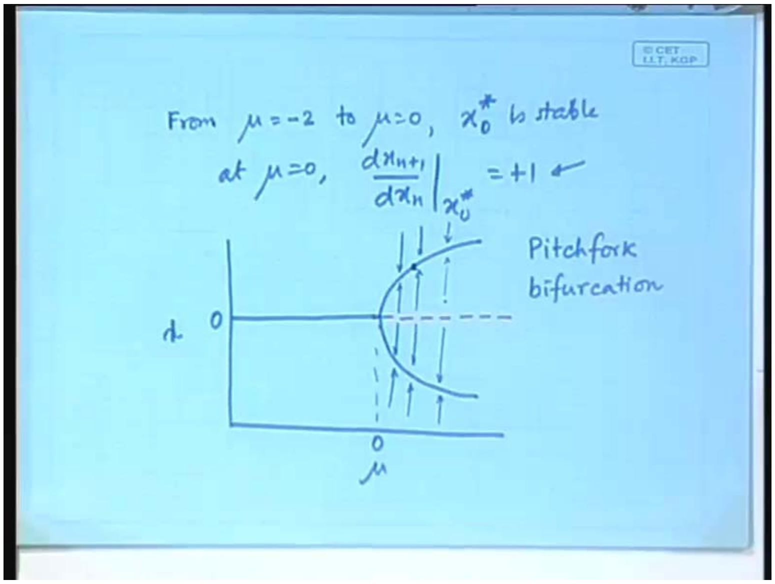

[Conversation between Student and Professor – Not audible ((00:30:11 min))] Yes we allow that. So we conclude that from mu is equal to minus 2 to mu is equal to zero, x_0 star is stable. What happens at mu is equal to zero? That means it is going to the positive side, slope becomes plus one, notice this plus one thing so at mu is equal to zero this dx_{n+1} by dx_n at x_0 star is plus one and when that happened this fellow become unstable.

Let us look at the other two fellows. What happens to the other two fellows? Since it is root over mu, so long as mu is negative these two fellows are not existing, they are not there because only after that they become real. The positions would become real only when mu is positive, so at mu is equal to zero they start to exists. What is the stability status, calculate from here and tell me. Root over mu is a substitute here. He says that if mu lies between zero and one they are stable. So we are considering even that happens at zero plus they are stable and both are stable because both will have the same stability status as given by this, plus root mu and minus root mu when substituted here will make no difference because x_n square. They will have the same stability status.

What does it mean? It means that if you plot the bifurcation diagram now, you would notice something like this that this fellow I will plot the zero here; minus 2 to 0 this is the zero position say, this fixed point was stable. So at this point say i reach mu, zero value and here is the x axis (Refer Slide Time: 33:14). At this point what happens? This particular fellow becomes unstable and continues at the position zero. What happens to the other one? Notice that they start to exist and they are having the value root mu, so it would be one like this, the other like that, looks like a period doubling bifurcation but it is not why because at that point the slope here is not minus one.

In order for a period doubling to occur, it has to have a slope of minus one. Now it is plus one and what has happened? It is not that a period two orbit has started to exist, it is that two individual separate fixed points have started to exist both period one fixed points. This one is a period one fixed point, this one is a separate period one fixed point, they have both started to exist and both are now stable. Before this point they both were not existing at all. So notice that if these both are stable fixed points, you might ask where will the actual orbit go? That depends on the initial condition.

If the initial condition is here, it will be attracted like this. If the initial condition is here, it will be attracted like that however if the initial condition is here it will be attracted like that. So you will have the orbits going like this, they are both attracting fixed points, they are both stable but the behavior would be something like this. [Conversation between Student and Professor – Not audible ((00:35:19 min))] No, it's not a period two because if you take the initial condition here, the next iterate will fall here it will not come here. In that sense it is not period two. If it had been period two then if you take an initial condition here either next iterate it will come here and then the next iterate it will come here and go on flipping like this. A period two orbit is always a flipping orbit. It flips from one side of the period one behavior to the other, here it is not and that is why it has to be given a separate name and that is pitchfork bifurcation, it looks like a fork.

This bifurcation is also somewhat rare as I showed that I had to introduce this map in order to illustrate this because it was not possible to have the bifurcation in the normal logistic map. So this map is also somewhat rare. This phenomenon is somewhat rare and it's not difficult to see that it is necessary to have the square term in the derivative which means a cubic term in the original map. So only if that has this kind of a structure, you observe this bifurcation else not. So this is the concept of the pitchfork bifurcation. So we have come across so far a few different types of bifurcation. One period doubling, two saddle node, three pitchfork, four transcritical.

Where does it land us, is it the whole story by no means but this is more or less the whole story as far as the first order maps are concerned that means we are considering only first order maps and there this is more or less the complete story as far as the local bifurcations are concerned. What do you mean by the local bifurcations? Here the prime thing in our hand was that either we were looking at the fixed point and we were looking at the local stability as given by the derivative, we were looking at only that and that is why these are called local bifurcations. All these are called local bifurcations. There are also some events that happen not depending on the local behavior but depending on some global behavior, I will come to that later.

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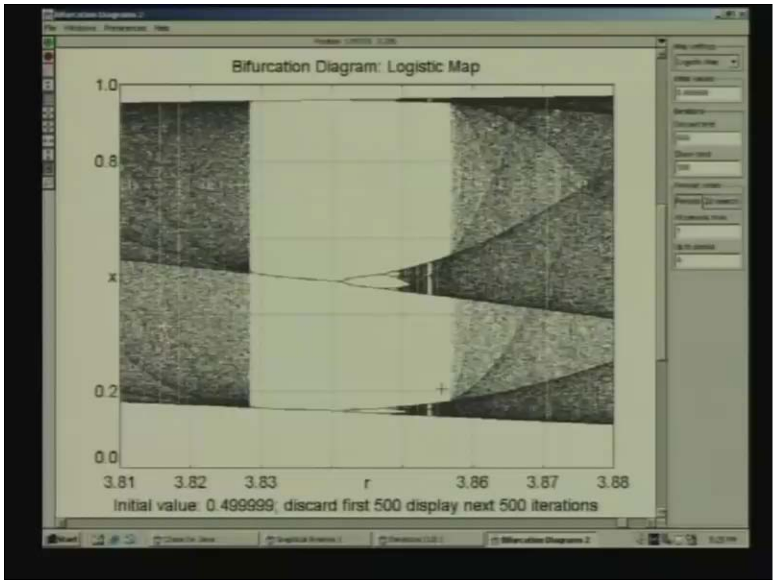

For example the event did you notice in this bifurcation diagram that here there is something happening. There was a period three orbit that went in to period 6 and period 12 and so on and so forth, it went on. Here also there is another window but here there is something happening. Suddenly there were a chaotic orbit with three chunks here and here (Refer Slide Time: 38:51) and suddenly they're all joined. The question is how does that happen? So these are somewhat global phenomenon, this cannot be explained by looking at the local eigenvalues or local slopes, the derivatives. So we'll come to that slowly when time comes.

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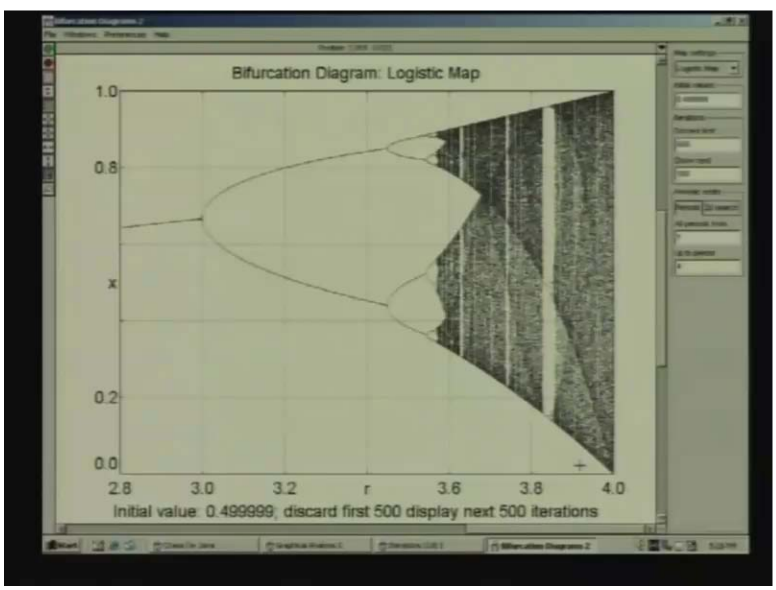

Now notice a few things let us go back to the original one. If you were working in the chaotic zone that means supposing you have chosen a parameter somewhere here, the behavior is chaotic. From theory you would know that there must be an infinite number of unstable periodic orbits there. Why because if you look at it this way when this fellow become unstable, the unstable periodic orbit still existed, it continuing. What do you mean by the unstable periodic orbit? You have come across once the situation of the unstable equilibrium point in the pendulum, we did the pendulum equation and there was an unstable point. What was the unstable point, like so (Refer Slide Time: 40:16). So slight perturbation it will go down but theoretically mathematically if it is there, it will remain there in that sense. In the same sense these fixed points are unstable in the sense mathematically if you place an initial condition there it will always remain there but slight perturbation it will go ahead. In that sense if they are unstable which means that inside the chaotic orbit there are really unstable number of such states which if you mathematically place an initial condition on that, it will remain there but they can not remain there because they drift away, they go away from there.

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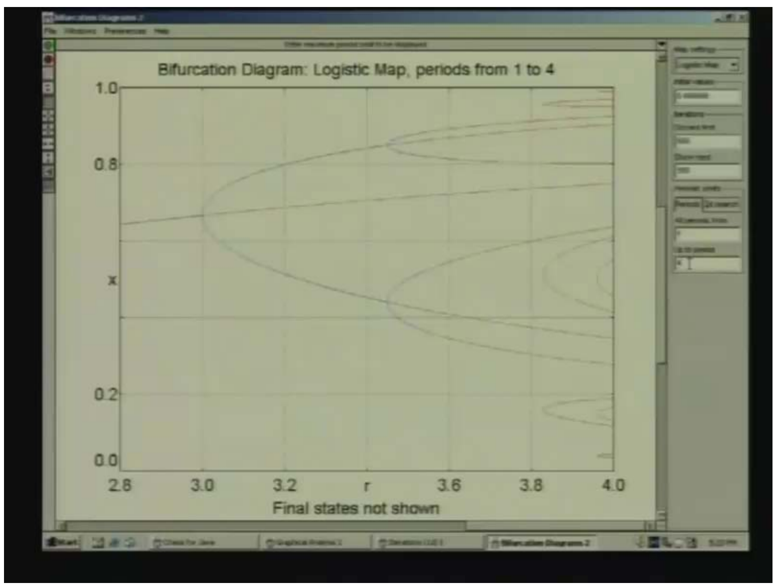

How many are there? I will draw some less, I have not shown the chaotic orbit. In the bifurcation diagram I have only plotted now the stable and unstable periodic orbits. The blue ones are stable and they have been plotted up to the period of 4 and then the unstable periodic orbits. At this point the period four orbits starts to exists and that continue, at this point the period eight orbit starts exists and that continue so on and so forth, this is period 3 orbit that that continues, so as it goes on this keep on multiplying.

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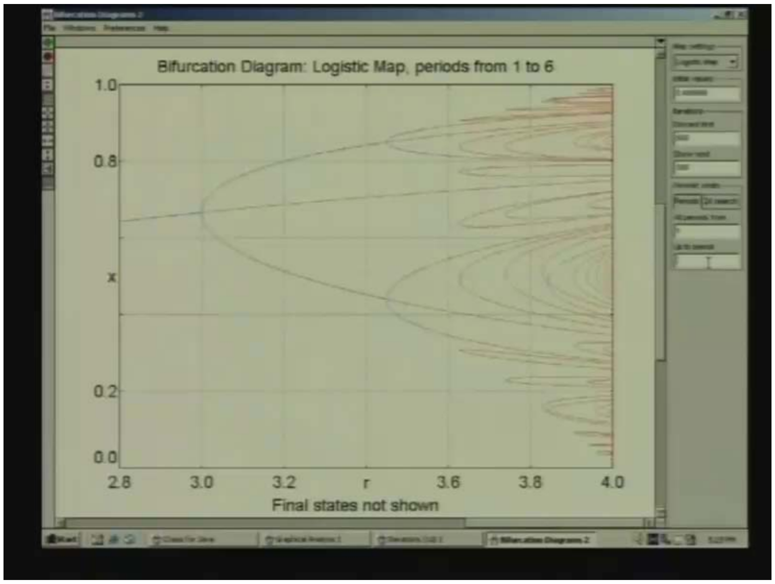

If I calculate up to a period of 6, you see more. So as it goes into the chaotic orbit there are all those red lines means there are all those unstable periodic orbits. Let us calculate it for some more.

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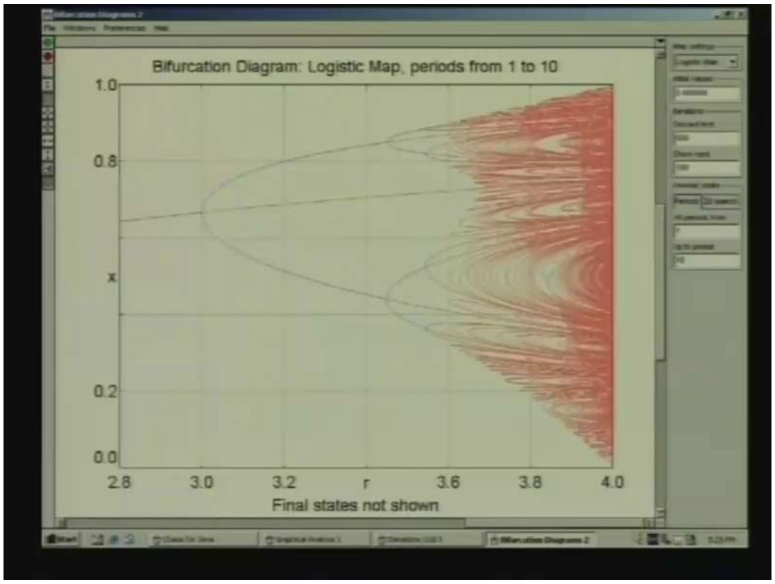

The period 10 orbit up to. Already it is becoming so clattered that it's not difficult to see that as it goes more and more into chaotic zone, there are more and more unstable periodic orbits available in the system. Each one you might visualize as a pendulum standing on your upside down. The interesting thing is that if the pendulum standing upside down it will normally fall but you might also hold the finger under that and with some small perturbation, you might keep it vertical. You all have dine that with sticks, all these are such things which can be stabilized in that way. So a chaotic orbit then allows far more freedom because a lots of unstable but periodic behaviors are now available which were not available in the system is stable because the system is chaotic a lot of unstable periodic behaviors are now available which can be stabilized the way you balance a stick. We'll come to this issue latter, how to stabilize them and what kind of work we can do with that but such flexibility we have in dealing with chaotic systems that you should understand. [Conversation between Student and Professor – Not audible ((00:40:24 min))] I said that there is a lot of flexibility in the sense that suppose a system is a linear system.

A linear system means its behavior are exactly given by the eigenvalues and that's it, you can do nothing about it. If you want to do something change the eigenvalues that is how the whole of control theory works. You put some feedback and that feedback essentially changes the whole systems, eigenvalues and that is how it works. If it is a non linear system then you have more flexibility in the sense that if something becomes unstable, it does not really immediately mean that the system will collapse because some other orbit may become stable. For example here you have seen that when the period one orbit becomes unstable, the period two orbit becomes stable so it doesn't collapse.

Now suppose in addition to I am saying that it is not only nonlinear but also I've chosen a parameter range in which the system is really chaotic. There are some applications where you might want chaos, for example spreading the spectrum and stuff like that, I will come to that latter but even if suppose you want to behave in a regular periodic way then the question comes which periodic? If it is linear you do not have the option because only one thing available to you. If it is chaotic you have infinity options, you can stabilize any of them. So you have a lot more flexibility in your hand. In that sense I said that dealing with chaotic system is a lot more flexible. That is why you might possibly know that wherever you want a lot of flexibility, you want to design the system in a sort of unstable way.

For example the big air liners, they have a control system, they will have an auto pilot where the pilot goes up and then he leaves it to the autopilot, the autopilot does it. It's a nice control system that has every thing very well set but you can not do that for a fighter pilots. For the fighter aircrafts you cannot do that. It has to have very large manure ability and very fast response and for that all fighter planes are therefore made open loop unstable and the loop is closed by the pilot, holding the joystick. Unless it is unstable you cannot really have that flexibility so that tells you some thing that in order to have flexibility you also want some kind of a instability and this is this instability. At every point it is unstable but with slight perturbation you can stabilize. So far we have mainly been talking in terms of one dimensional maps where you have the expression x_{n+1} is equal to some function of x_n.

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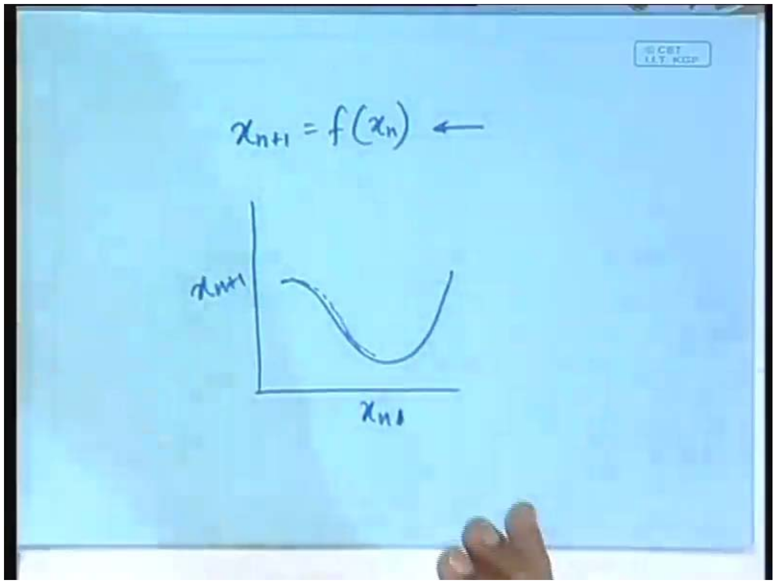

In general you wouldn't expect this kind of things. This is too simple but still we went into it for two reasons. First it gives a nice visualizable way of understanding dynamics because you can draw the graph of the map. You can draw those cobweb diagrams thereby understand the behavior of orbits but this really is too much simplification you might say, not quite because in many systems it has been found that even though the actual system appears to be far higher dimensional. For example there is no reason to believe that the dripping faucet experiment will lead to a one dimensional system. It's not one dimensional obviously, if you really want to model it, it will lead to a very high dimensional system.

Similarly there are many systems in nature as well as in engineering which if you write down the equations accurately they are very high dimensional very complicated things but when you actually observe their behavior and plot x_n versus x_{n+1}, if you get a nice looking curve what will be the conclusion? Even if the actual system may be very high dimensional but the dynamics can be model by a low dimensional model. For example this one was Poincare section for the Rossler equation. I want to plot large number of points so here it is actually three dimensional system. Place a Poincare section you are expecting a two d map to come up yet when you actually plot it you find that it is on a lies curve which tells you that even if the actual system is two d in discrete time, it can effectively model by a one dimensional map.

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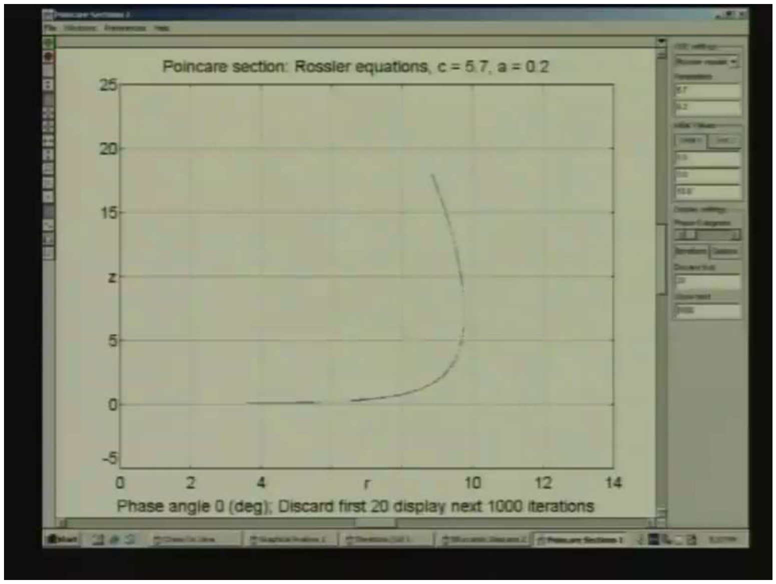

So in that sense the one dimensional map study is fruitful. You have also seen that in case of the dripping faucet experiment you yielded a behavior something like this, I have not brought that book today but it was more or less like that x_n versus x_{n+1}. Only it had a little bit of like this, remember yesterday I showed. this little bit I will tell you that in only that sense it is not exactly one dimensional but if you can ignore that part if you can approximate it by this then it is more or less one dimensional behavior. Again in that sense the study of one dimensional maps are fruitful but in the main, the study of one dimensional maps helps us in understanding this bifurcations in very concrete terms, mathematically most of these can be worked out by hand but then we have to take the next step. We have to go to the second order systems.

So we now need to study equations of this form x_{n+1} is equal to function one x_n, y_n and y_{n+1} is equal to another function of x_n, y_n. In this case how will we proceed? We will proceed by the same way though now we are no longer able to plot the graph of the map because we are constraint in our plotting to the 2 d thing and here the space itself is 2 d and therefore I cannot plot x_{n+1} or y_{n+1}, so we cannot plot. We are no longer have the option of plotting the 45 degree line and finding the intersection line, we cannot do that.

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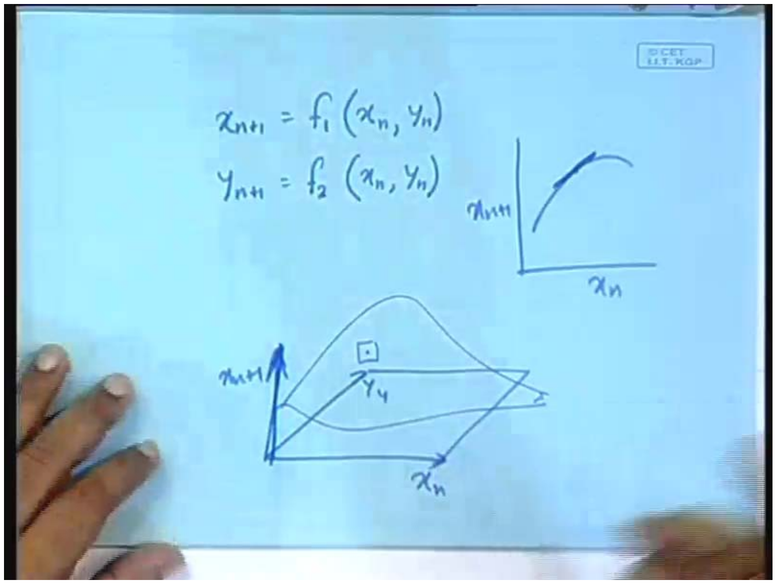

Still can we find the fixed point? Of course we can. How will you find? simply put x_n star in the left hand side and y_n star in the right hand side and then equate them, you will get a pair of algebraic equations, solve them you will get the locations of the equilibrium or the fixed points. How will you study the stability of the fixed points? We need then some equivalent of the derivative in 2 d. Imagine you have the x_n and y_n and suppose you have plotted the x_{n+1} here. It's not difficult to imagine that you can plot this way so you have got this horizontal plane x_n, y_n and you have some surface kind of thing and you are trying to find out the local linear behavior say at this point I want to locally linearize.

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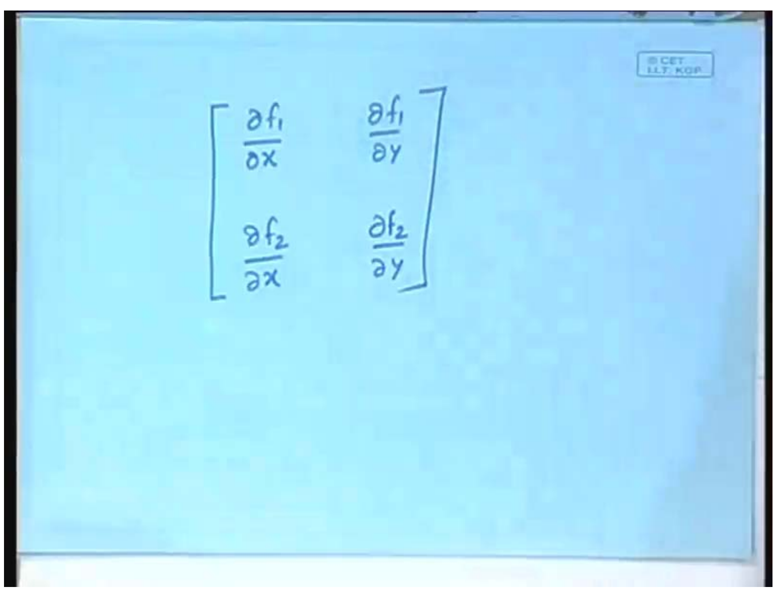

In case of a 1 d graph x_n versus x_{n+1}, a graph like this and this point and you take the derivative. The derivative was the local linearization. Similarly you want to take the local linearization here. This local linearization here where you are taking both x_{n+1} as well as y_{n+1} is given by the Jacobian matrix. Probably you have learnt that in the mathematics classes that is given by the Jacobian matrix and the Jacobian matrix is expressed as (Refer Slide Time: 54:55). So here also we will take the same root, we will look at the fixed point and look at the local linear neighborhood but the behavior in the neighborhood of that fixed point will be given by the Jacobian matrix and we will explain on this idea in the next class.