Chaos, Fractals and Dynamical Systems
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Lecture No # 10
Flip and Tangent Bifurcations

In the last class we have seen that we have discussed mechanism by which a period one fixed point in a map gives rise to a period two fixed point. The mechanism was that we had talked about the intersection of the 45 degree line with the graph of the map and at some point this slope becomes less than minus one and at that point it happened.

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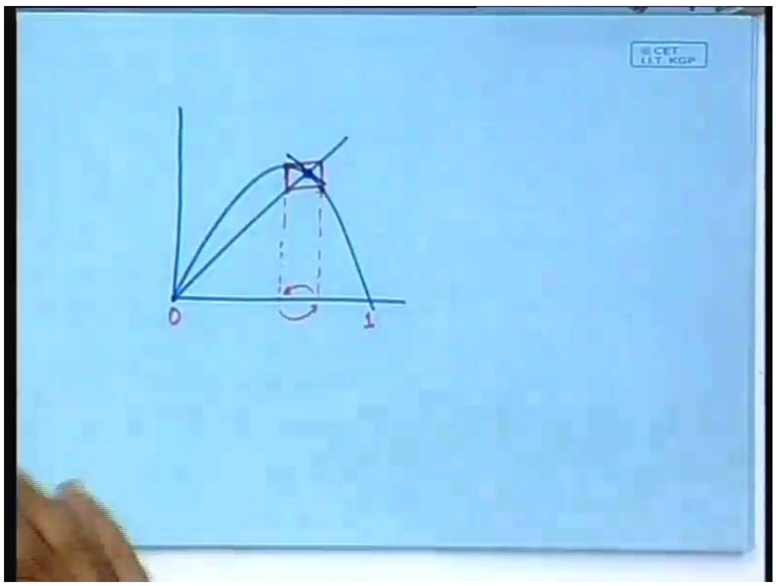

What was the result? The result was that this fellow became unstable still it existed but became unstable and at the same time an orbit something like this became stable. Do you understand what I am doing? I am saying that this point will map to this point and then for the next iterate I have to come down to the 45 degree line and come down to the graph of the map and so it forms a rectangle meaning that this point will map to this point and this point. What actually is happening is that this is the zero to one real line and this point is mapping to this point and this point is mapping to this point. That's what is happening on this real line.

What is it? This is something that is obtained by disceretizing a continuous time dynamical system that means we have placed a Poincare section and then we have seen the written maps and then we did all that and then we concluded that this is resulting in the destabilization, such a mechanism is resulting in the destabilization of a periodic orbit, resulting in a period two orbit.

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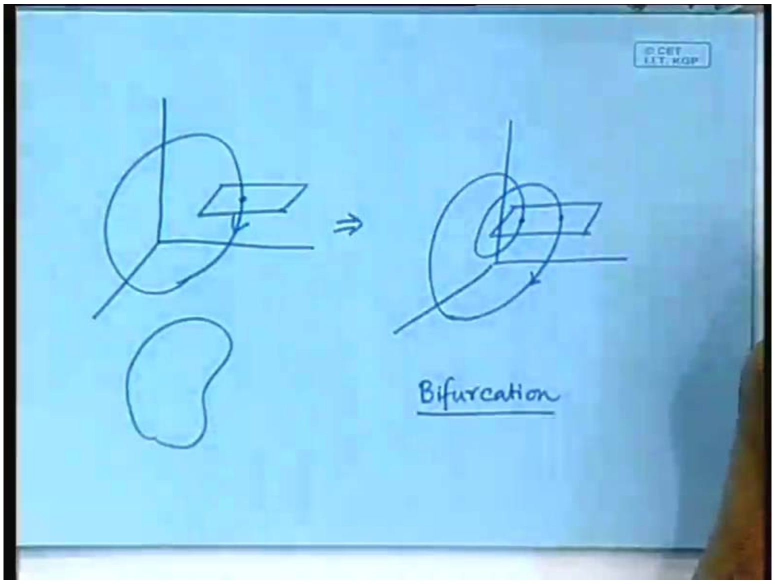

What is happening in the continuous time domain? Continuous time domain earlier it was an orbit something like this which is now turning into an orbit something like this because in this one, if you place a poincare section you will see a point and in this one if you place a poincare section you will couple of points. This one is changing to this one. Now there is a concept in mathematics called topological equivalence. Meaning that if you imagine this as a sort of a rubber band and in how many ways can you sort of distort it without making a fundamental change in it.

For an example if you want to distort it, you can push it here so that this orbit becomes something like this, it is possible but in order for this to be turned to this one it is not just pushing or pulling, something additional had to be done; meaning you have to take it and you have to twist it and then only you get something like this which means that it cannot be transformed into this one without by just pulling some part and pushing some part of this orbit. So these two are topological equivalent but these two are not. There has been a break of the topological equivalence as it came from this one to this one or there are most mathematically rigorous definitions of that but I am not going into that, for our purpose it is sufficient to understand that we are talking in terms of rubber sheet geometry and the way rubber bands can be turned and twisted that suffices in convincing ourselves that here we are talking about the breakdown of a topological equivalence.

The change over from a period one to two period orbit involves a breakdown of topological equivalence. Now I come to a definition, whenever as you change a parameter there is a breakdown of the topological equivalence of the asymptotically stable orbit then the behaviour is called a bifurcation. So what did I say as the definition of bifurcation? In bifurcation we are not talking about the transient behavior. We are talking about the steady state behaviour only. This is a steady state behaviour so is this.

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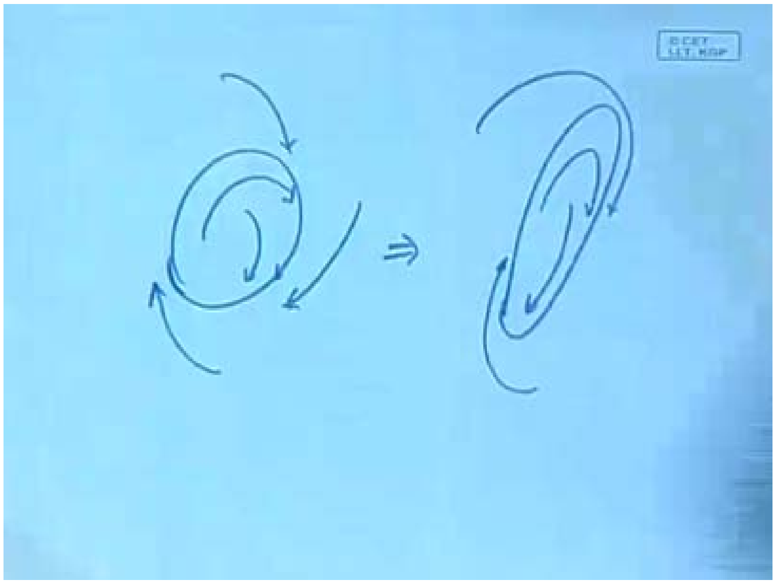

So when the steady state behaviour changes from one type to another type; type means there is a qualitative change in the behaviour when it changes from here to here, as you change the parameter that can always happen or you may say that there was an orbit something like this which was a result of an outgoing spiral inside and the incoming spiral outside and it changes to say a very longest stuff but the behaviour is more or less the same. This is topological equivalent to that one. So when that kind of a change happens we will call it a quantitative change, small change preserving topological equivalence but while a change like this happens where the topological equivalence is broken we will say it is a qualitative transformation, qualitative change and when that happens we have a bifurcation.

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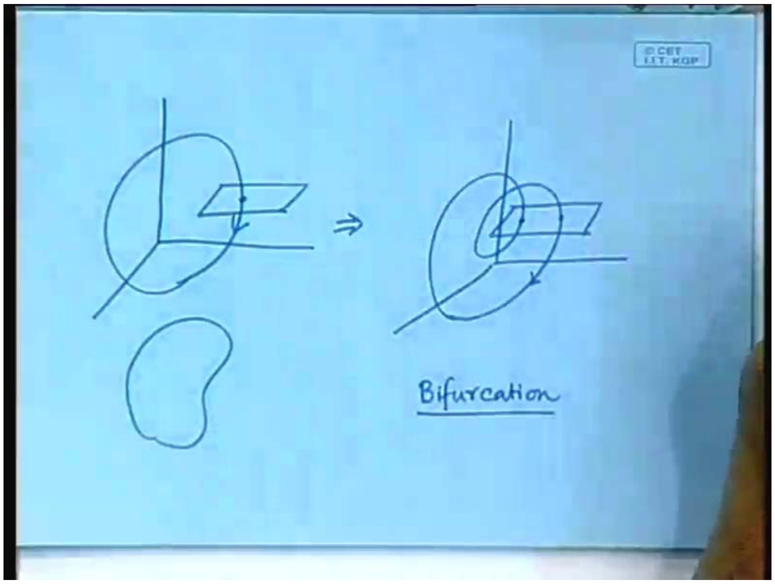

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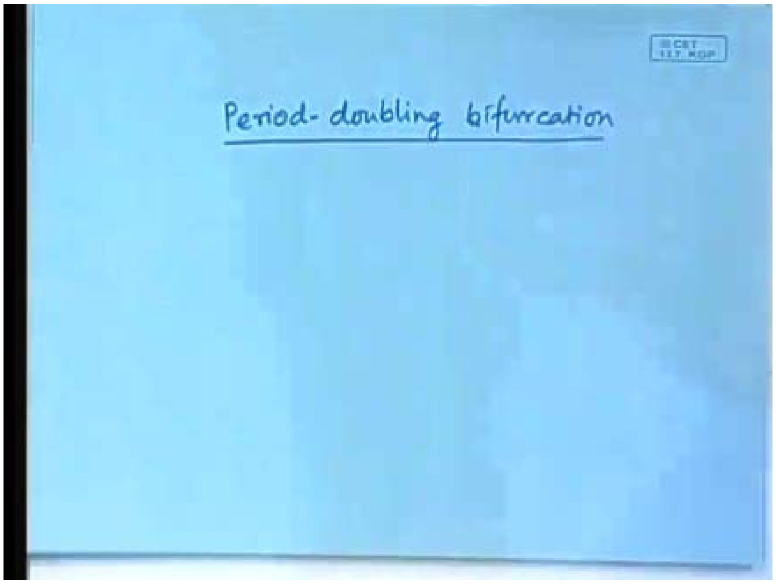

So in essence the last time when we're talking about the period doubling mechanism we are talking about a specific type bifurcation and that bifurcation where period one orbit was giving rise to a period two orbit and period two orbit was giving rise to a period four orbit and so on and so forth. That is called a period doubling bifurcation. There is another name and that goes with it. In order to understand the name we will have to take a relook at this behavior.

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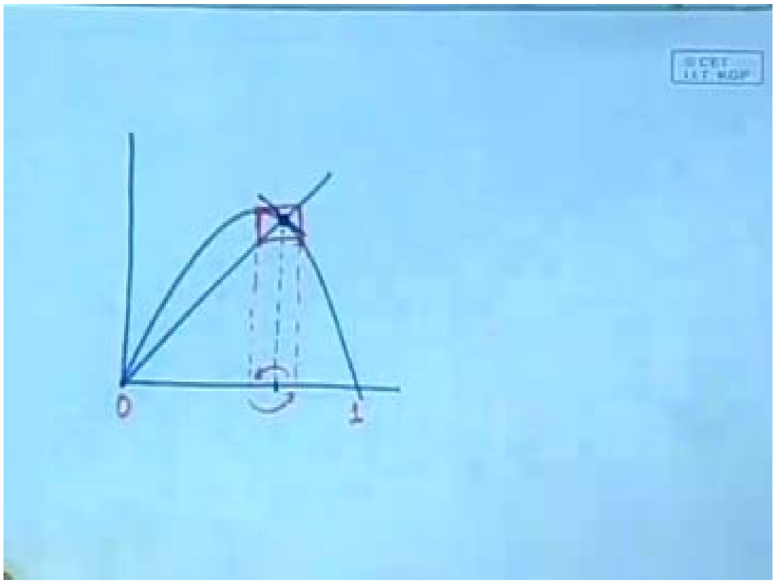

See here was a fixed point and as the slope became larger than or smaller than minus one because it is a negative slope then what happened, here the orbit changes to appear something like this with the preexisting fixed point somewhere here which means that the orbit actually flips between the two sides of the fixed point. Once it is here, another time it is so on and so forth.

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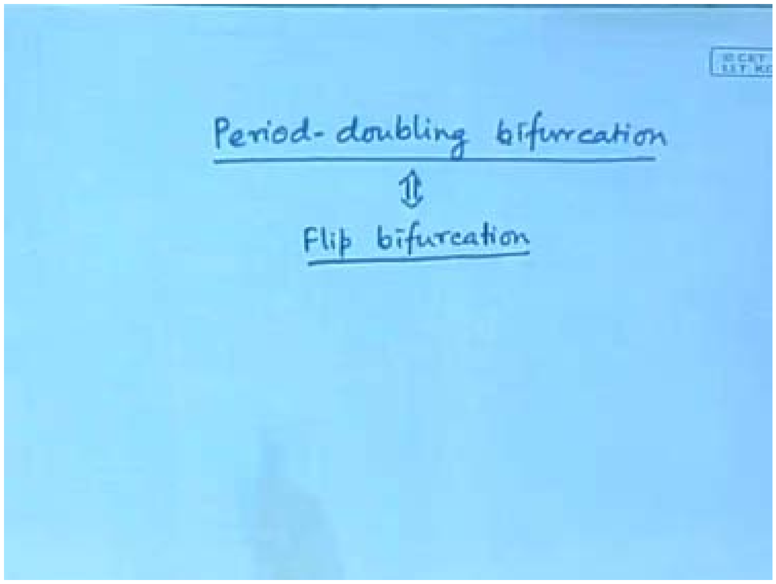

That is why there is another name for the same one, this is same as flip bifurcation. If you see in books or other literature this names just understand that they are talking about the same period doubling bifurcation. So let us revisit the period doubling bifurcation fast because we had spend enough time on that the last day.
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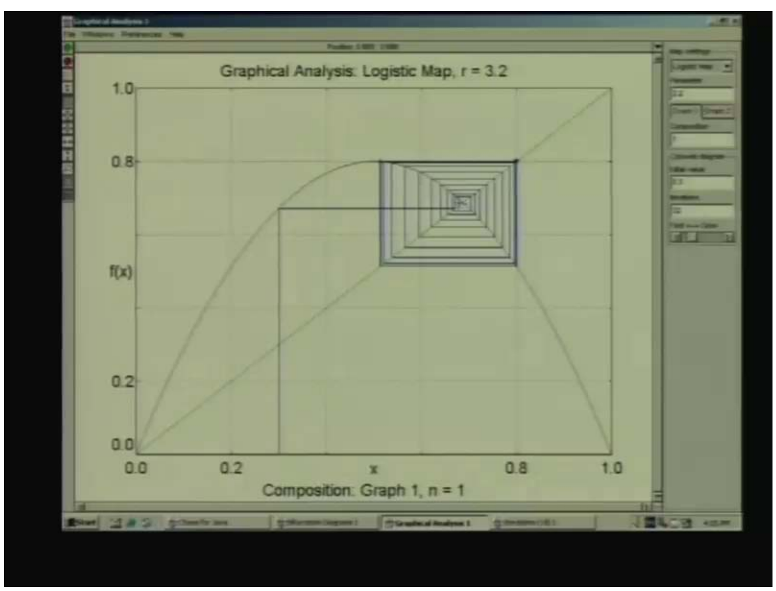

What was the parameter value at which it happened? So we have will have to come to a place that is very close to say three something like here. If it is say 3.2, so if you start from a point somewhere here that this point is the actual preexisting period one fixed point which has now become unstable and so orbits spiral outwards and what kind of diagram is it? This kind of a diagram, what is it called? Cob web diagram because its looks like a cob web. This kind of diagrams where we go from a particular value of the initial condition to the graph of the map, to the 45 degree line, to the graph of the map, to the 45 degree line so on and so forth. It takes the shape of a cob web and that is why such a diagram is also called cob web diagram and it should really cultivate the practice of drawing this cob web diagrams because that allows you to understand the behaviour of orbits qualitatively. We have really doing the algebra just imagine the algebra of doing this would be rather tedious and time consuming. If you simply do this graphically you can understand the behaviour.

Notice one thing as this fellow became unstable something here and there that became stable. If you change the parameter to a smaller value you can see that. So it is here, so this becomes stable. Now if I ask you what would be the stability status of the period two orbit, what would you do? You will plot x_{n+2} versus x_n, find out its fixed points. Drop the ones that are also the fixed point of the period one orbit. Find out the ones that are newly appeared and talk about their stability in terms of the slope and these are these two fixed points. What is a slope? The slope is nothing but the slope here and the slope here multiplied that would be the slope of the period two orbit. That is what we have already shown in the last class. So here the slope is negative, there the slope is negative and therefore the slope of the period two orbit would be positive. So as this fellow becomes unstable, a period two orbit occurs. Period two orbit is stable because at this point this slope and at this point, this slope is much smaller. If you multiply them you get a number that is smaller than one so it is stable fixed point but the slope is positive.

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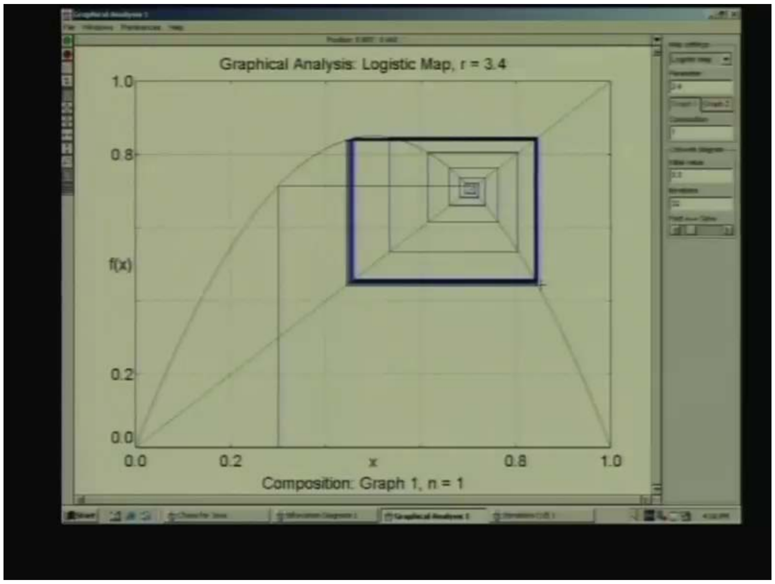

It is period two orbit still but nevertheless it has come to the other side. Therefore what does it mean? Here the slope has become negative. Here this is positive so product would be negative slope, so can you visualize that if you plot the slope then when the period two orbit fellow started, its slope was one. As you keep changing the parameter, it was reducing and inevitably at some point of time it has become negative and as it goes on inevitably at some point of time it will again become minus one.

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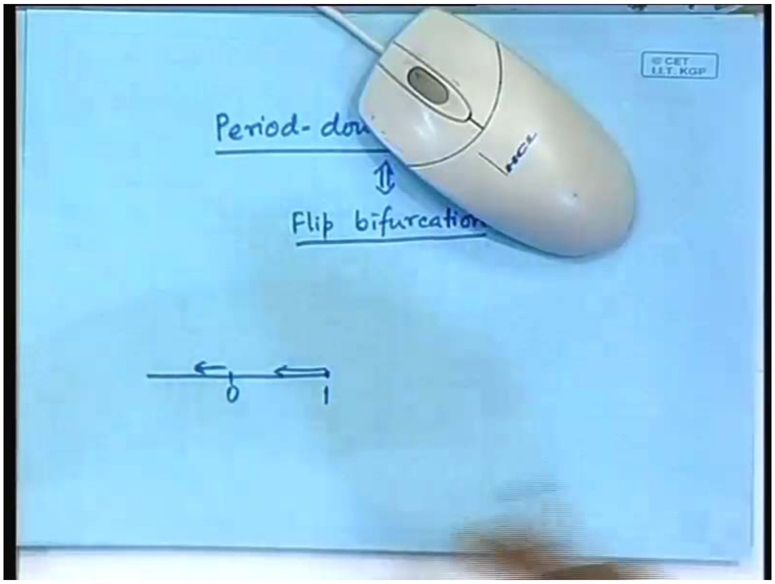

So the period two orbit will also lose stability through the same flip bifurcation. Continue this this argument you will find that the period 4 orbit will also lose stability by the same mechanism, period 8 orbit will also lose stability for the same mechanism. When each orbit comes into existence its slope is positive and plus one; as you change the parameter further then it goes from the plus one towards the minus one thing and then it goes out. Thereby that particular orbit becomes unstable. When it becomes unstable do the orbits lose existence? No they are still existing meaning that when you look at the bifurcation diagram like this of the same system, over the range 2.8 to 4.0 then here in your mind something that you cannot plot directly here but in your mind you should look at something like this that at a point, in this case 3.0, this fellow became unstable and to period two orbits immersed, it’s nice.

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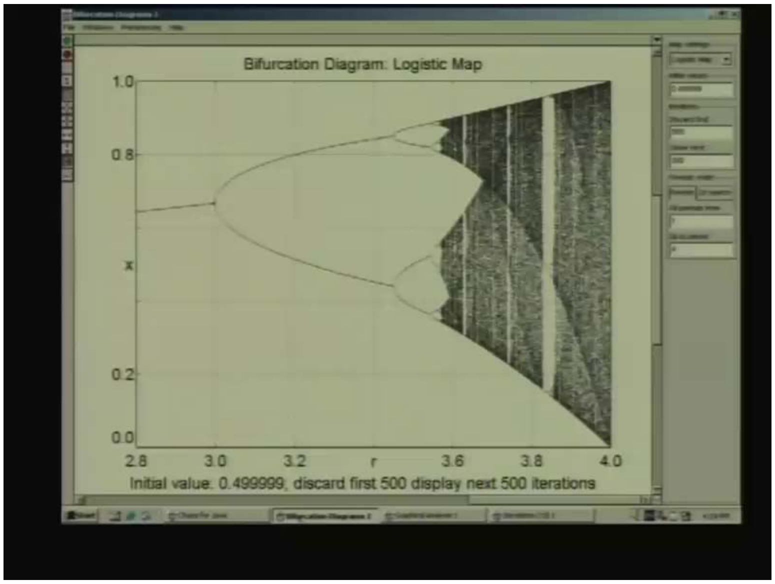

So as this one becomes unstable, this fellow continues (Refer Slide Time: 15:05) and after some time this fellow has also become unstable and you have the immergence of period 4 orbit and still this fellow is continuing and then this fellow also became unstable and again this fellow is continuing. As you proceed you can easily see that the lengths for which each of the periodic orbits are existing are slowly going down and that would ultimately accumulate if you think that as this by this again all these are ratios less than one and there will be an accumulation point. That means if you go on to this 1 to 2, 2 to 4, 4 to 8, 8 to 16, 16 to 32 and so on and so forth. It will accumulate to infinity but within a finite parameter range. That will happen within a finite parameter range that's exactly what you see here.

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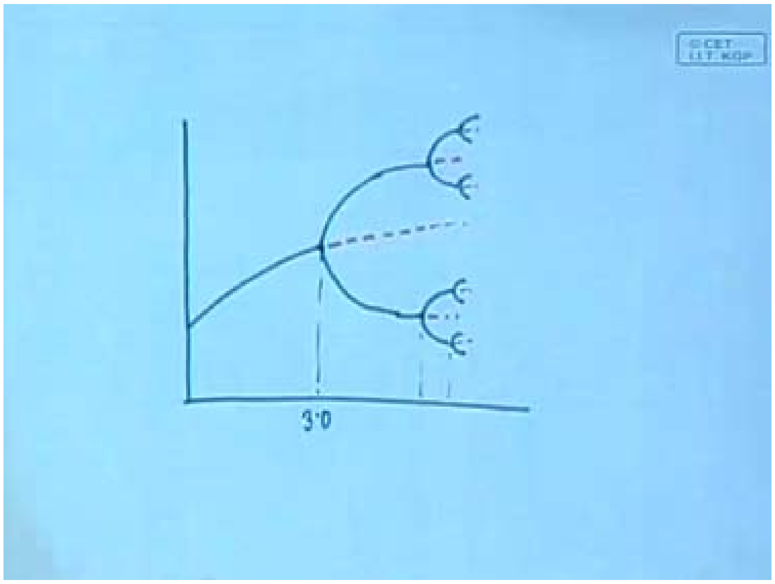

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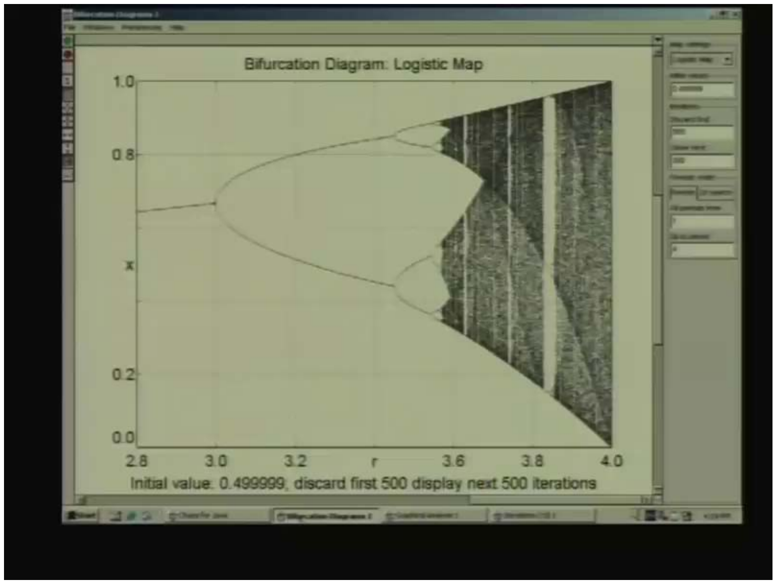

It has accumulated more or less here. Now there is a nice rule to it. The rule is that if you consider the range of parameter values for which the period 1 orbit existed and the range of the parameter value for which the period 2 orbit existed and if you take the ratio then you get a number. Period 1 to period 2 if you do that you will get a number greater than one. Again if you take the ratio between period 2 and period 3, you also get a number and as you progress you will keep on getting the number and it so happens that number that ratio always converges to a single number and that number is you can imagine it like this; limit n tending to infinity, the range mu_{n-1}- mu_{n-2} that means if n is say 3 then I am talking about the parameter value at which 3 that means mu_2-mu_1 and here it is mu_n – mu_{n-1} this range or if say mu is 3 then you have mu_2 – mu_1, here it is mu_3 – mu_2 which means the previous one is here and the later one is there in the denominator.

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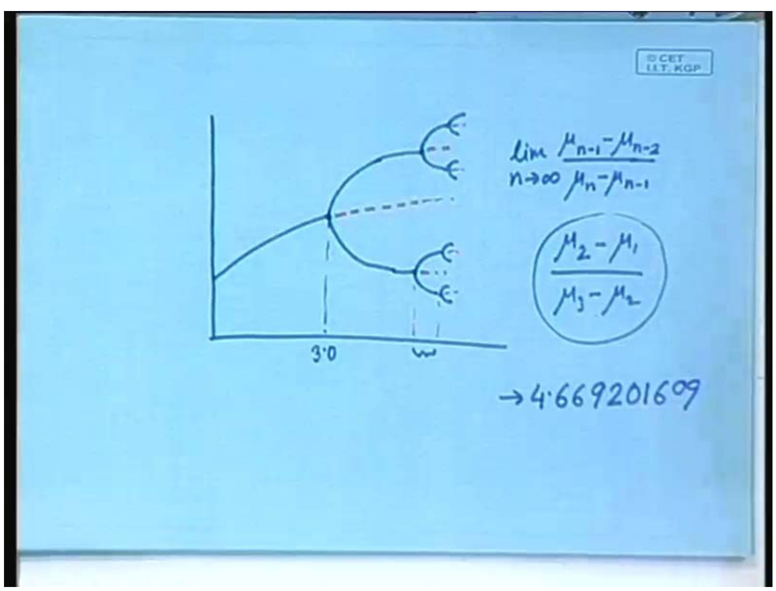

So this number as n tends to infinity should always converge to the number 4.669201609 so on and so forth. You might wonder why? This is a very interesting and famous result by Michael Feigenbaum. I will not exactly rigorously prove it here but I can sketch the proof how does it come about.

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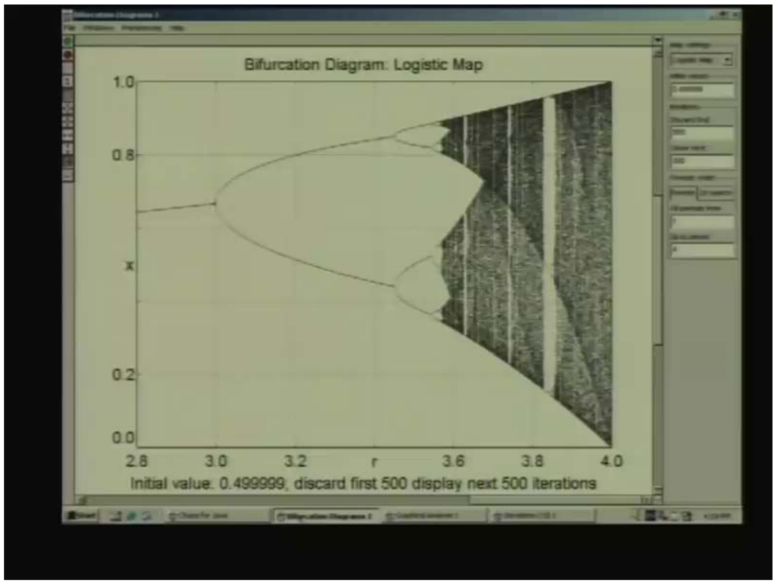

Here am talking about this range of the parameter divided by this range of the parameter then this range of the parameter divided by the next range of the parameter. So what will you do? You are going ahead, if you are going ahead then let us just blow up this part and see what is there. Looks identical. Let's just blow up this part, identical. Let's just blow up this part, identical. Let's just blow up this part.
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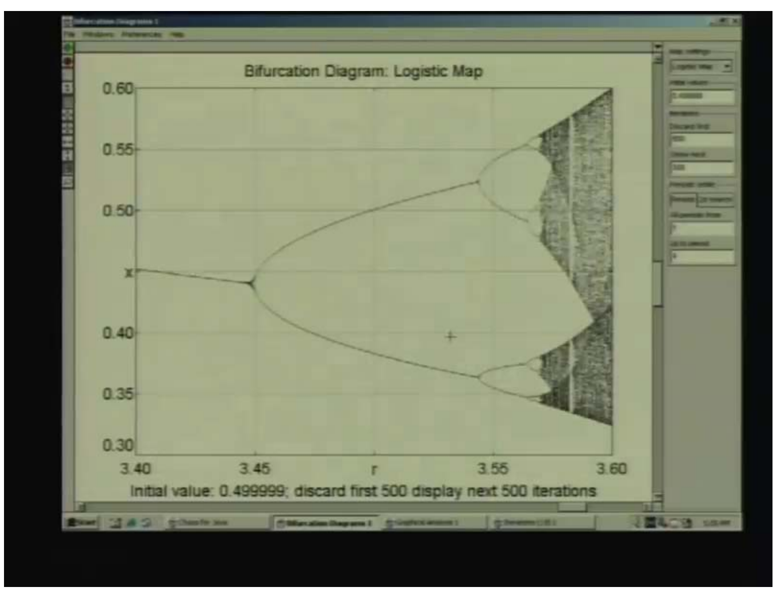

Now you understood more or less what is happening. So in the bifurcation diagram, you see what is known as self-similarity. If you take a part and zoom, it more or less looks likes the whole and that is why, as you go closer and closer, you find the same phenomenon happening at smaller and finer and finer scales.

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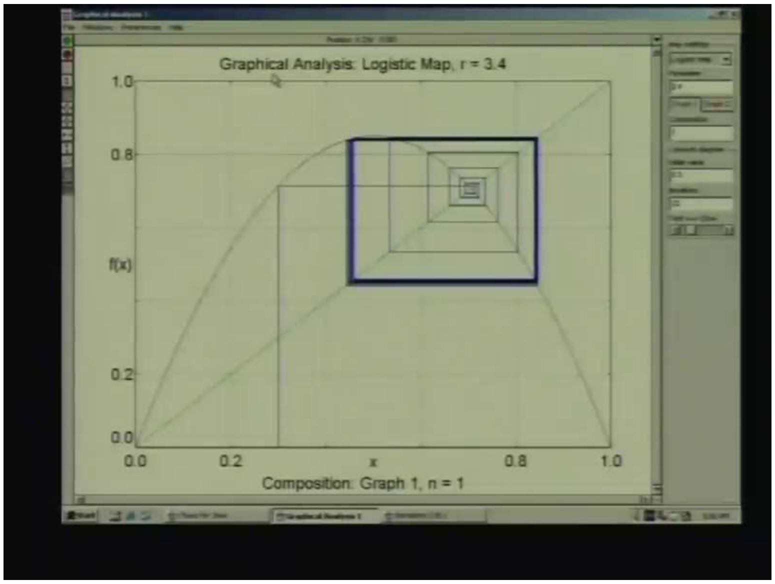

Now why does it happen? When we started looking at the first one, what did we do? We plotted the second iterate of the map that's what we did.

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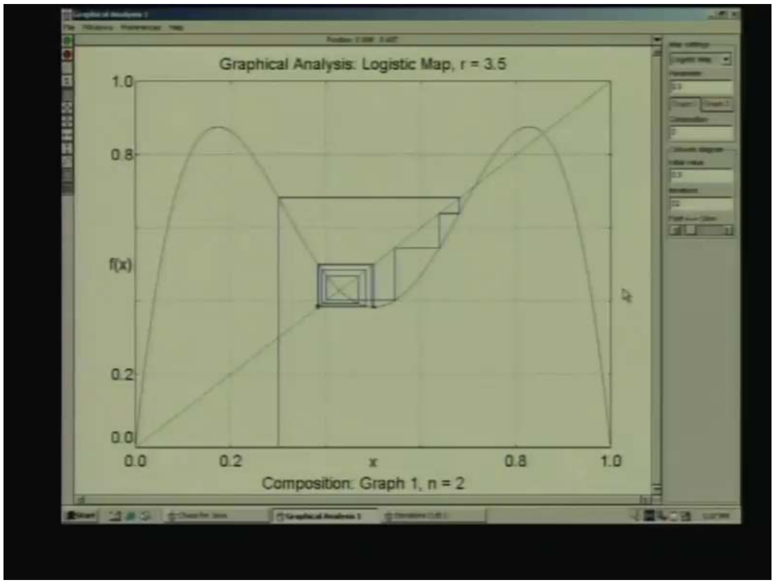

So this one was the fixed point of the original one and these two are the fixed point that had appeared. Now you see here there is a hump and here there is a fixed point that has now become unstable like there was a fixed point that became unstable and here this is a point that is now stable.

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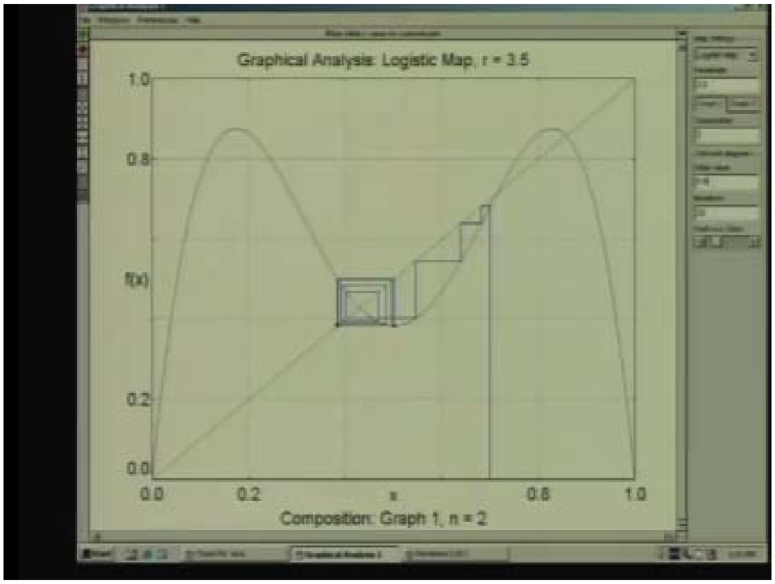

Now as you change the parameter further say the period two orbit also becomes unstable, let's start from a initial condition somewhere like 0.7.

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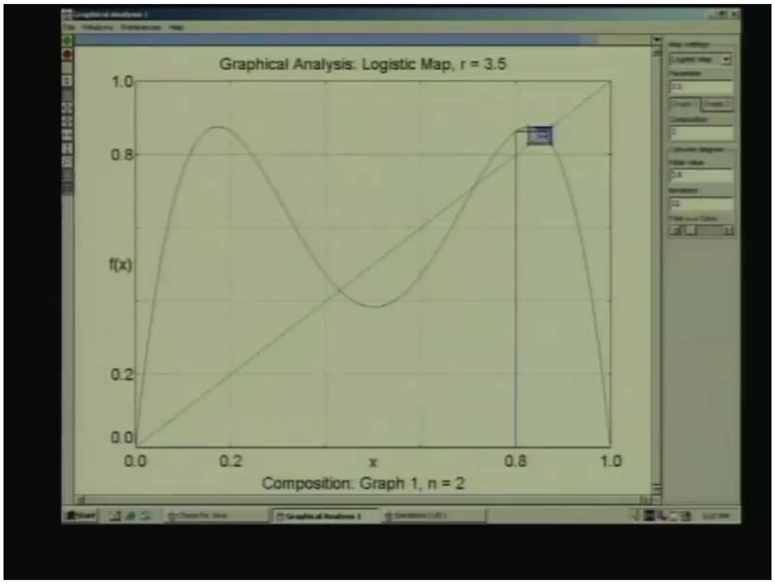

I want to show it here. It starts from here and it goes like this. Now you can see that this fellow has also become unstable. In order to study what will you do? You will plot the period fourth iterate of the map. So forth composition x_{n+4} as a function of… (Refer Slide Time: 22:38).

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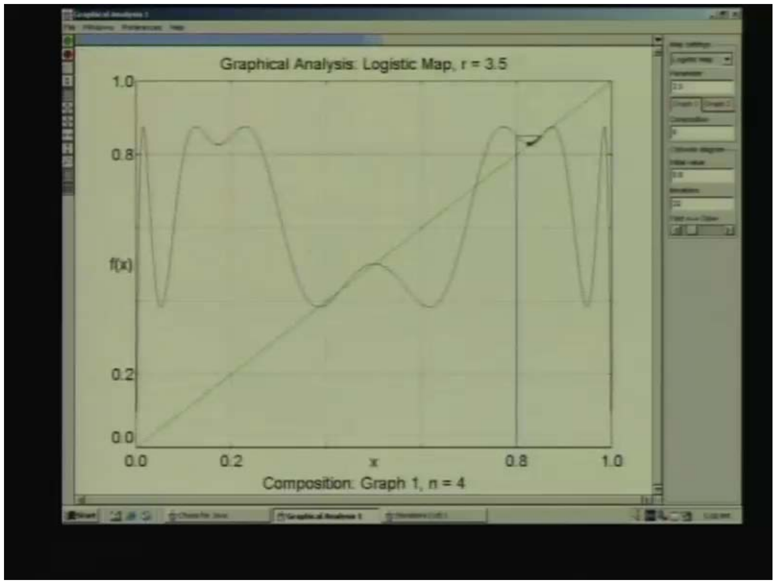

Notice here it is now having the same kind of behaviour that you saw earlier. Now this fellow is stable so if you increase the parameter further, you see here it has become a bit clattered. Let me reduce it. Can you see that here again the same phenomenon is happening and at some point, this parameter or this particular point will also become unstable and then you will need to look at a blow up of only this part.

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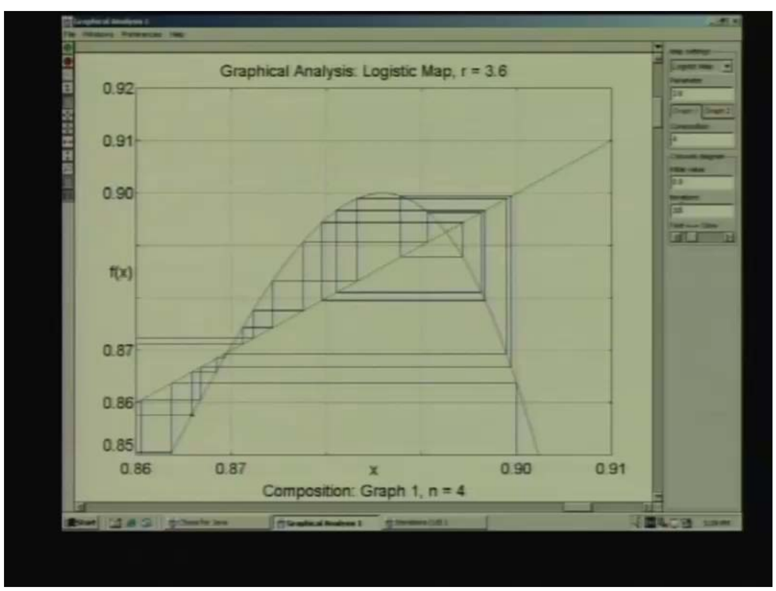

Then you will start from a parameter value say 0.9. So this point will become unstable and then in order to study its behaviour what will you do? You will again draw the eighth fellow here and this fellow has also become unstable and so on and so forth.

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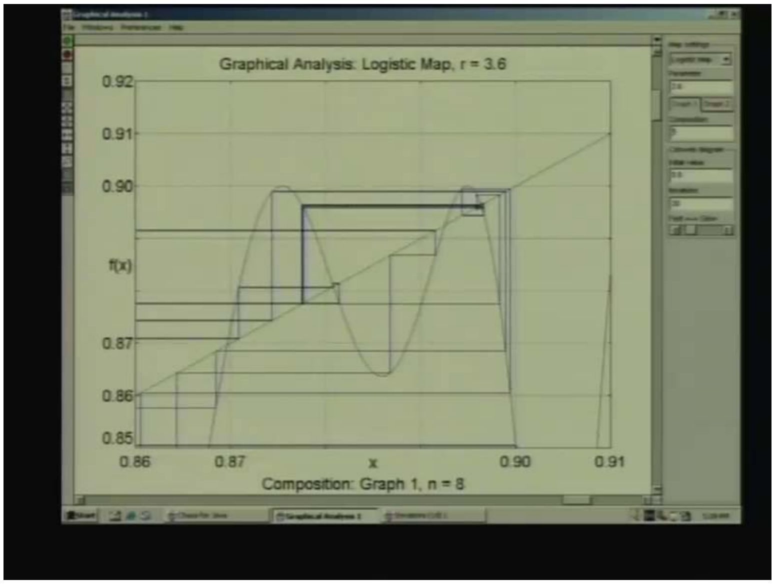

Now notice one thing I will reduce the iteration number, so that you can see clearly this part.

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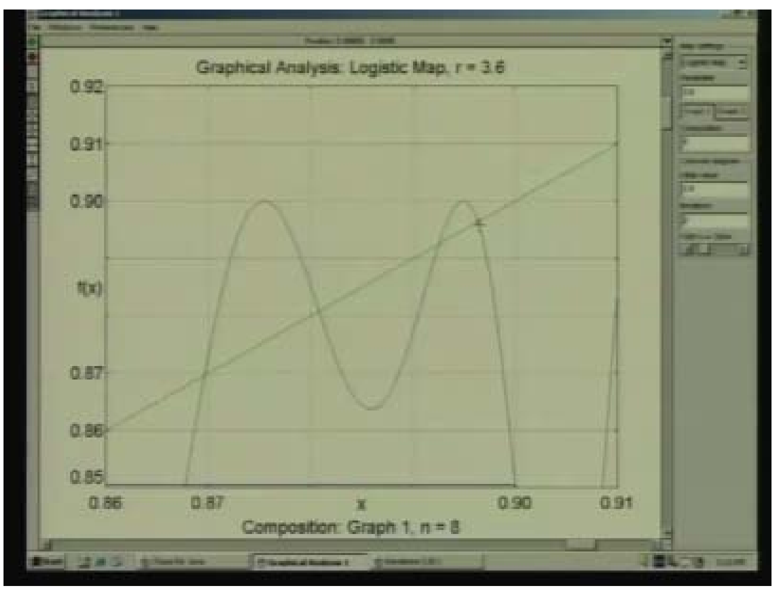

Doesn't it look like the same as the initial logistic map graph? It is basically the same. So what is happening is that as you go into higher and higher iterates, you are looking at a same kind of graph.

Only the individual differences between the graphs are now being sort of ironed out. So you might say that this kind of graph would result from an equation like this.

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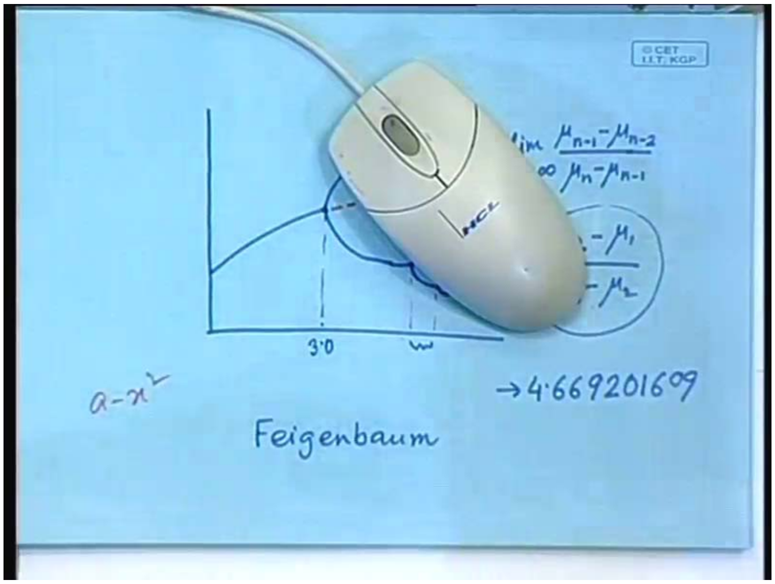

That's also a one hump map. It might also result from the equation of a parabola, it might also result from the trunked equation of hyperbola so on and so forth. Individual graphs are all different but you are zooming on to the top part of it. The more you zoom on to a particular part of it, the more individual differences of different maps are being wiped off. You are essentially looking at closer and closer zooms of only a part.

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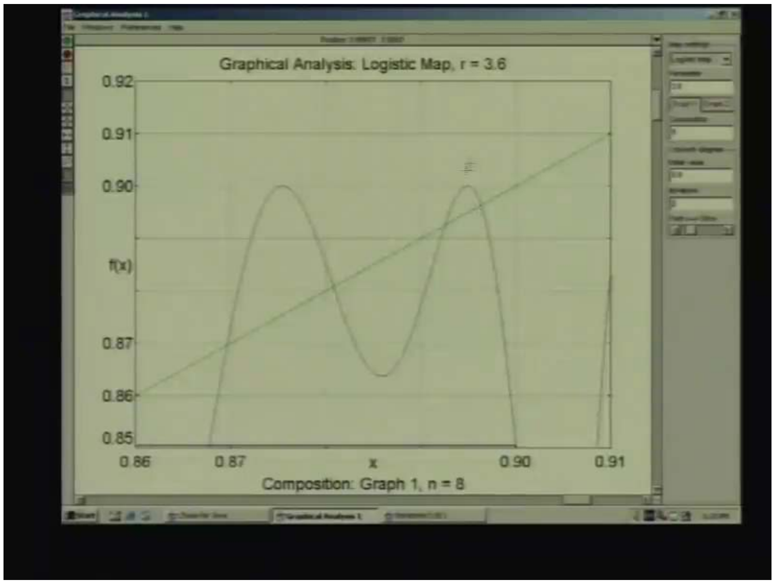

That is exactly the Feigenbaum's argument that's exactly why this number must be universal. That is why you do not get this number, if you take the low n value because if you take 3 and 2 then essentially you are still seeing the differences between different types of graph of the map. The more you zoom closer you eliminate the differences. You essentially come to some kind of a universal phenomenon and so this number is true for a wide variety of systems.

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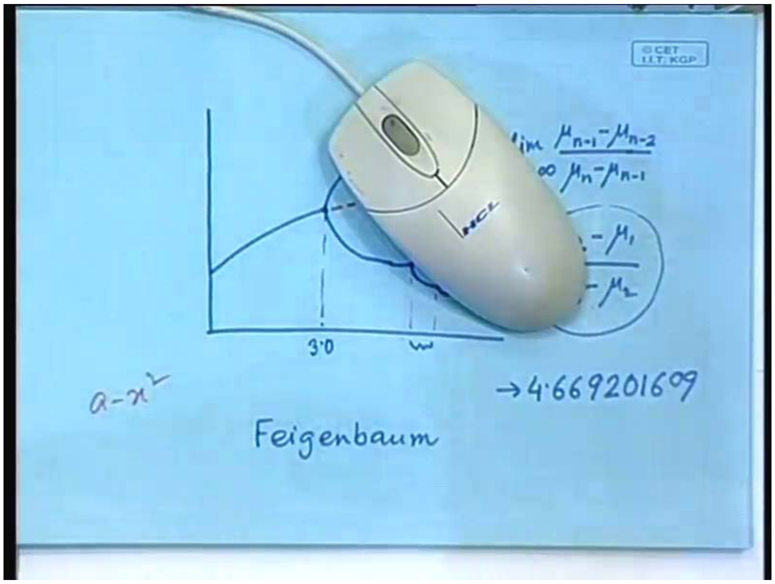

Notice the only logical necessity in my course of development was that it must be a one humped map, it must just have one hump. So just one maximum so that a graph like this will not work but any system that ultimately maps into a one humped map should give rise to the same kind of behavior. That was the concept of universality. In fact widely different systems; the systems that have no connection with each other have all found to exhibit period doubling cascade and in the period doubling cascade this number appears, this number to this extent of accuracy appears.

The mathematical logic is that the more you are going into the cascade, the more you are eliminating the difference between the individual functional forms. You are essentially looking at this particular part which take a hyperbola, take a parabola, take any type of graph it will remain the same and that is why ultimately you will land up in the same number. This number is called the Feigenbaum number or Fiegenbaum ratio. They might prove it that this number you get involves renormalization group method that are now very common in mathematics and some disciplines in physics that has been used here but for our purpose, since most of you come from engineering disciplines it is not all that necessary to go into the detailed proof. Just get an idea why it is true. Now how different are the systems in which it has been found to be true? I probably talked about the experiment by the Doyne Farmer.

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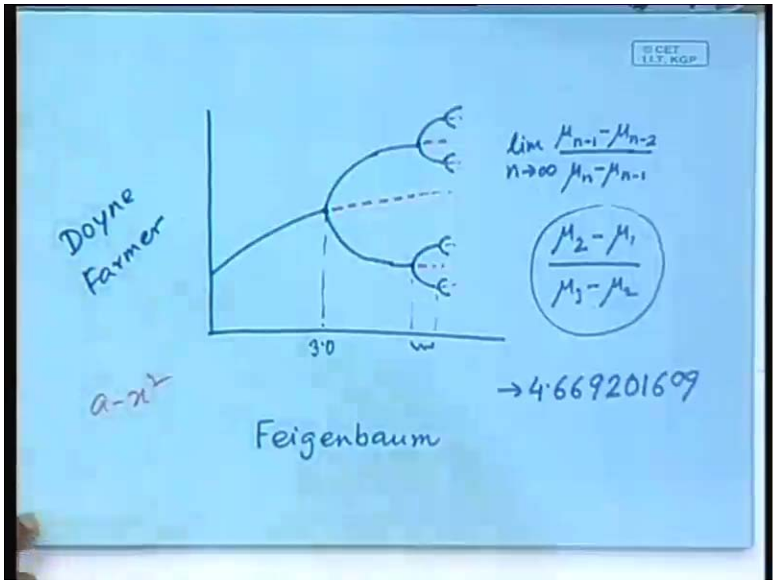

Doyne Farmer was a student and he was staying in a hostel which had an attached bathing. So they have mostly hostels with attached baths which you are not blessed with but they have. It was not to his liking why because at night he found the faucet was a leaky faucet and drops of water would fall with a irritating sign tip, tip, tip, tip, tip, tip and all that so he couldn't sleep and if creative guys cannot sleep they do something creative and that's what he did. what he did was he tried to study this tip, tip, tip, tip, tip, tip and he found that depending on the amount of opening of the faucet, you can have a completely periodic dips tip, tip, tip, tip.

Change it a bit, it will become tip tip, tip, tip tip, tip, tip tip, tip and then he wanted to find out if there is any rule behind all that. So he set up a very elaborate experiment because there was a tip, you could put up a micro phone, collect the information and then on the graph it would show like a regular peaks. You can find out the time difference between the peaks, estimate the time differences, use it at x_n and then that max to x_{n+1} and so on and so forth which means that you can draw a graph. He found that for small values of the parameter that means the opening of the faucet, it was a period one orbit. I probably have the graphs, I will show you if it is here.

For a small opening all the points collected at one place, this is time n, T_n, T_{n+1} collected at one place which means that it is a period one orbit, increase a parameter they are collected in two places; increase the orbit further, increase the parameter further 4 places and if you change it further it was a perfectly chaotic orbit. So it went into same period doubling cascade and if you notice when it is chaotic points fall on every part of this graph of the map so this can be taken as the graph of the map and it is a one humped graph, inverted nevertheless but one humped graph. This completely unmodelable system, you cannot really do a modeling of it.

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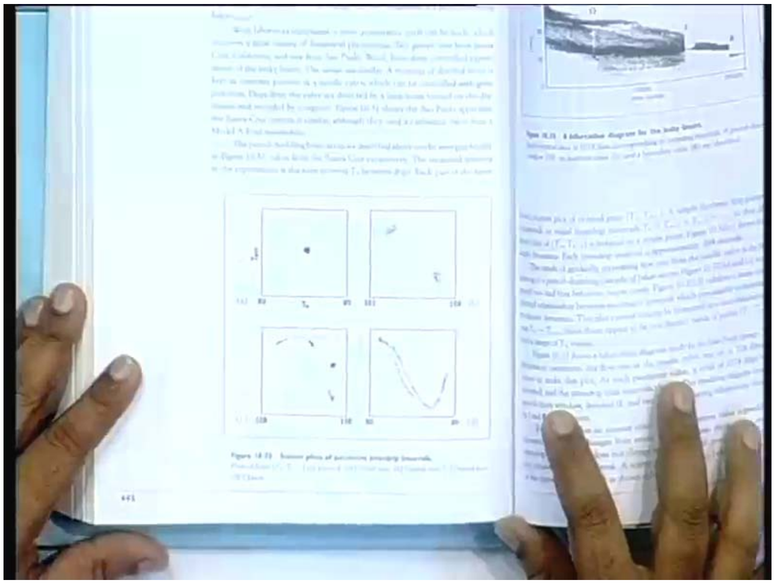

Such completely unmodelable system also had a one humped map and the moment you know that there is a smooth one humped map, you know that the Feigenbaum numbers will also valid here and it was. So you can do the mathematics on the simple system like logistic map. Derive conclusions that would be applicable to completely unconnected system like this. There have been experiments, the Feigenbaum number experiment has been also confirmed in very peculiar systems. For example there was a leaf cover experiment where he took a very small cylinder in which there was a bit of liquid helium.

Why liquid helium? Because he could do the experiment with any kind of substance but liquid helium is liquid at a very low temperature and thereby the environmental noise would be minimum at that temperature and there was a heating from the below and cooling at the top and that would give rise to a set of convective currents and he was looking at the behaviour of the convective currents and there was measuring instruments and when the report came he showed clear period of doubling cascade and here also low and behold there was a nice one humped map. In electrical engineering and mechanical engineering and in all such places we find this kind of one humped maps appear and in all there would be a period doubling cascade with this particular number as in organizing the whole period doubling cascade. What we are talking about? I am not special to this particular map that we are taking, it is general it is universal. That’s why it is said to be a universal mechanism of period doubling cascade.

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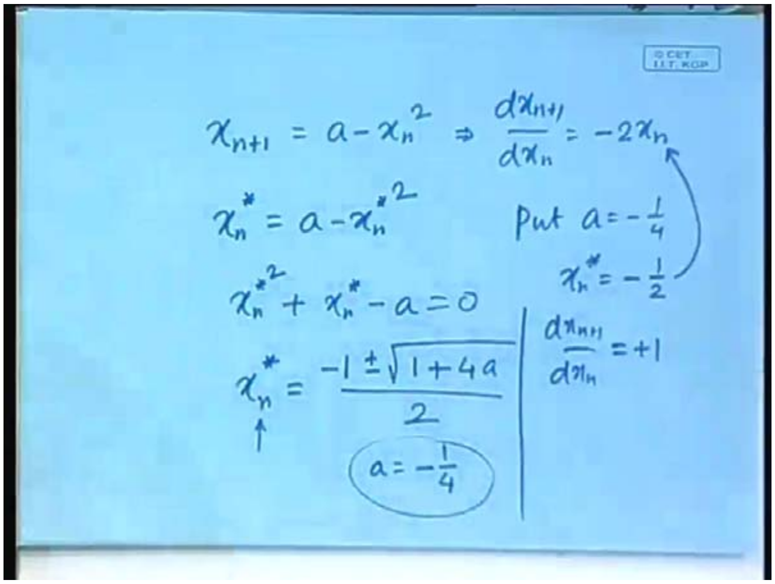

Now let's come to something more. Imagine a map given by, I am using this because here a is the parameter because this is simple. It is simpler than the logistic map that is why I took this. Can you find out the fixed point of this one? Very simple, because in order to find out fixed point you say that the left hand side is equal to the right hand side and then you would say x. so you say the fixed points are located. Now you notice that for a this particular fixed point begins to exist at a specific value of a. Where is it? So a is equal to minus one fourth is a sort of a critical thing below which there is no fixed point above which there is a fixed point. In order to find out what is the behaviour here, you take a derivative of it and try to understand how it is behaving. So if you take the derivative you will find, what is a derivative; is equal to… (Refer Slide Time: 35:55).

If you substitute this minus one fourth here, what you have here? Zero and therefore you have minus one by two. So minus half, substitute it here, you have plus one. If you put a is equal to minus one fourth so that x_n star is equal to minus half. Substitute it here, you have dx_{n+1} dx_n is equal to plus one. I will write plus one with the reason, I will come to that. So from this can you infer the shape of the graph? These were on the computer so without really thinking you could see what is happening here but now I am deliberately choosing a different map so that you are forced to think, what will be the shape of the map what is happening here.

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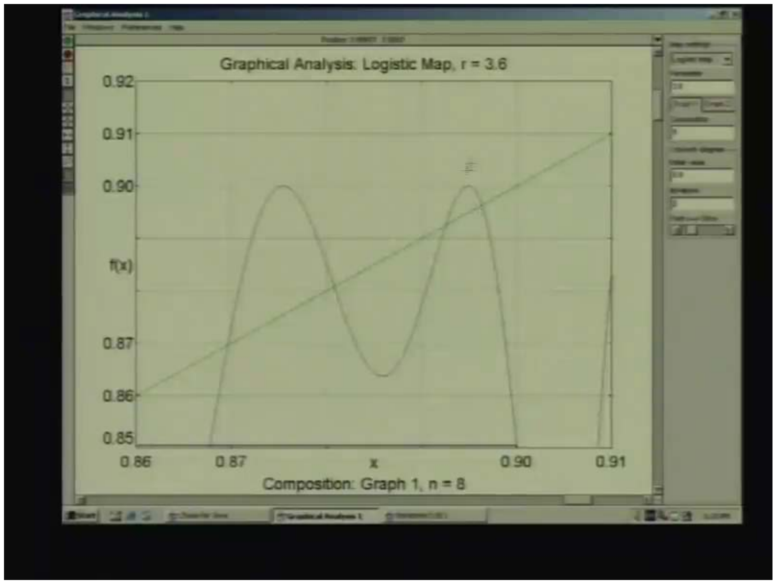

What is happening here is that below a certain value of a, there is no intersection with the forty five degree line. That's the meaning of the non-existence of a fixed point, there is no intersection of the 45 degree line, at that particular value there is an intersection and when there is an intersection, the point of intersection has slope of one. From this can we not infer the behaviour would be something like this that initially it was something like this and then as you the change the parameter would be like this and then it would be like this. That's the only way it can happen. Following this what do we expect the behaviour to be?

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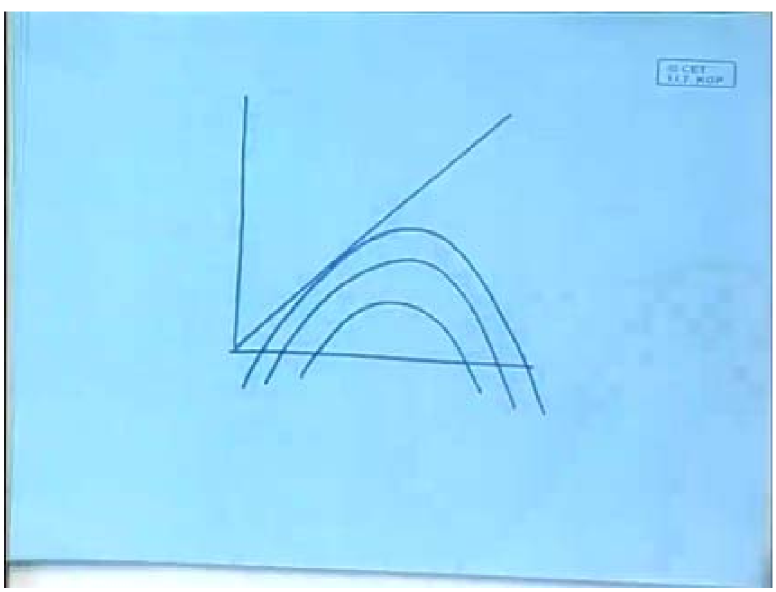

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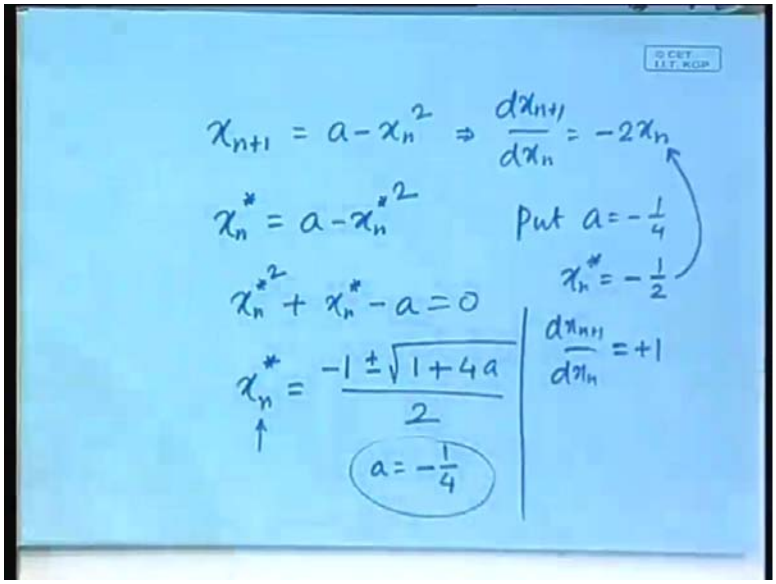

Following this there would be two fixed points, if you change the parameter even further there will be two fixed points one with plus one and this plus component and the minus component and these two fixed point will be here and here. Notice that always one of them will be unstable and the other one stable, there cannot be any other way. Always one of them will be unstable and the other one will be stable and in the actual system what do we observe? There was no stable behaviour earlier, if there is no intersection with the 45 degree line there is no stable behavior. Suddenly a stable behaviour is appearing that's also bifurcation. That is also a qualitative change in this asymptotically stable behaviour of the system. Now this bifurcation is happening with the graph of the map becoming tangent to the 45 degree line. That is why this is also called a tangent bifurcation. This is also called a tangent bifurcation.
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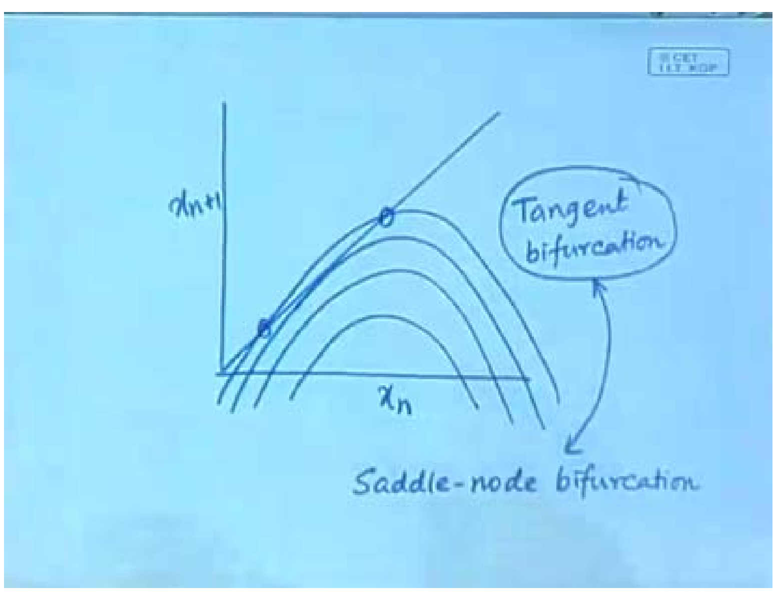

In books however you will find another name for it more common. The tangent and being able to see the 45 degree line applies only to one dimensional maps. If it is 2 D then you cannot really draw a graph like this and therefore we cannot really see a tangent or something. So the general nomenclature of it is saddle node bifurcation. They are the same phase. Only in 1 D systems, one dimensional maps where I can plot x_n versus x_{n+1}, only such systems the word tangent bifurcations are valid. In higher dimensional system there is no concept of tangent and therefore you cannot really call it as a tangent bifurcation, the more general term is saddle node but also in 1 D this term is also valid. Though you cannot really see a saddle or something like that I will come to why this name came, when I treat high dimensional systems but presently just remember the name. I may be interchangeably using the word saddle node bifurcation in place of tangent bifurcation. So you should not be confused because I am also more used to using this term because that is more in literature.

What has happened was in the system there was no fixed point, no stable periodic behaviour and suddenly this fellow is stable. That fellow is unstable, this fellow is stable that has appeared. So a tangent bifurcation or a saddle node bifurcation results in the birth of a stable periodic orbit. That results in the birth of a stable periodic orbit that was not there earlier but whenever there is a birth of a stable periodic orbit, you should know that even if only this one is visible. Why, because start from any initial condition it will go there not here. Only this one will be visible but even if it is there from theory you should know that there is a also an unstable fixed point existing. That is very important but why i will come to that later. You should remember the one that you can see is not the whole of the story. Whenever there is a saddle node bifurcation in fact this fellow is called a node and this fellow is called a saddle in more general context.

So a node has appeared and attracting fixed point is appeared but also a rippling fixed point has also appeared. You cannot have anything otherwise. So a tangent bifurcation is associated with a birth of an instrument, just contrasted with the period doubling case where there was no birth of a fixed point. It was something becoming unstable. At a period doubling bifurcation, the period one orbit became unstable but it is still existed. At a tangent bifurcation it ceases to exist. Imagine that there was a graph of them of like this and as you change the parameter is approaching this way. What will happen? These two points will come close to each other, collide with each other and then it will annihilate. It will no longer exist. So it sort of makes a pair of fixed point of n.

Imagine a practical system such a thing is happening means that you have a stable periodic orbit, you are happy that my unit system is working fine but as you change the parameter such a thing is happening. What will happen? They need a catastrophic because at this point suddenly you will find that it is no longer existing, something that was there is not only loosing stability it is just ceasing to exist, it’s a catastrophic situation. So a saddle node bifurcation seen in one direction it is a birth of the fixed point, seen in the opposite direction it is a death of a fixed point.

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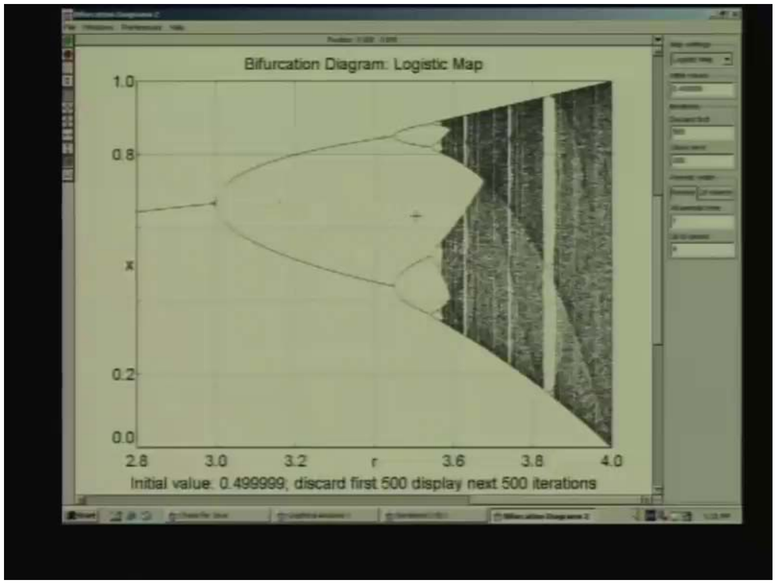

Now this also allows us to explain some of the things that you have seen in this bifurcation diagram. I have talked about the bifurcation diagram earlier. So you have this period doubling cascade and all that are going on but do you see that here is a opening, there is a something known as a periodic window. it is not continuously chaotic, in between the chaos is broken by some range of the parameter when the behaviour is periodic and for example here I can say that there is a one iterate here and the iterate here and the third iterate here so it is a period three window. How does this come over, how could this come over? I can see that the value of the parameter is slightly greater than 3.8.

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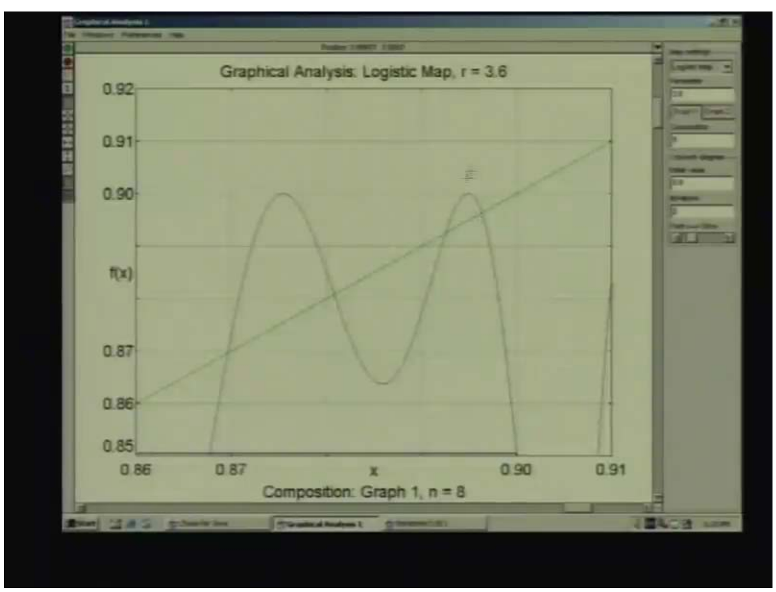

So let us look at the graphical analysis at a parameter value say 3.8 but now I want the first compositions to start.

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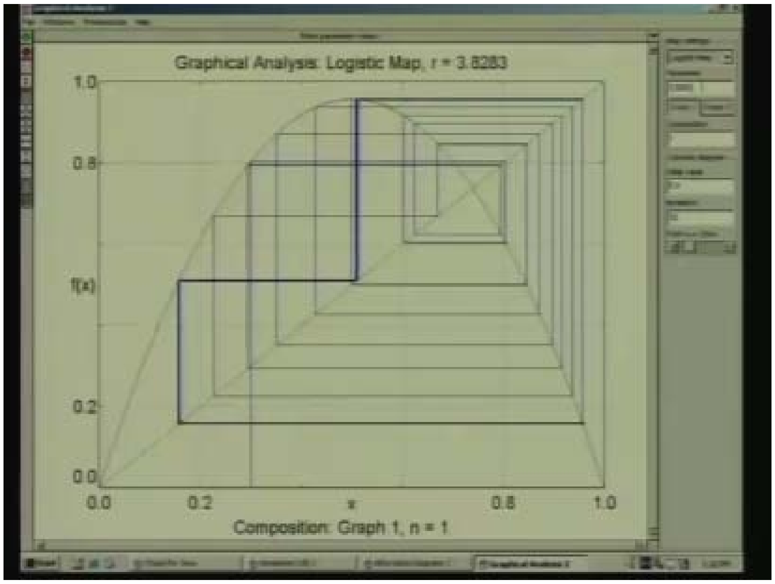

Here ultimately when it has to come to period three orbit, you can see this period three orbit. So start from here, it goes to the graph, comes here, goes to the graph, comes here, comes to the graph of the map and it looks which means that it is a stable period three orbit. I will note down the parameter value for which it happened 3.8283 but I will start from slightly less value. It's still chaotic.

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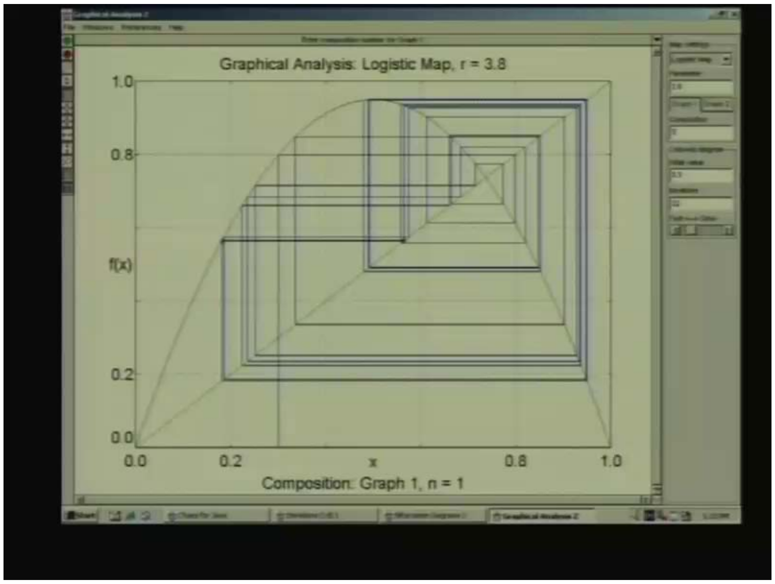

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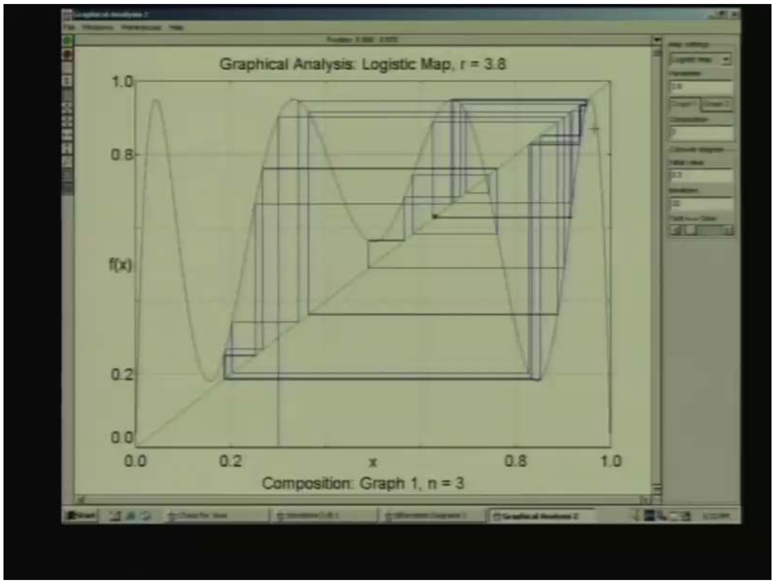

Now in order to understand it let me plot the period three behaviour the third iterate of the map x_{n+3} plotted as a function of x_n. Notice this is the graph. Have you seen that? Now keep noticing as I increase the parameter slightly.

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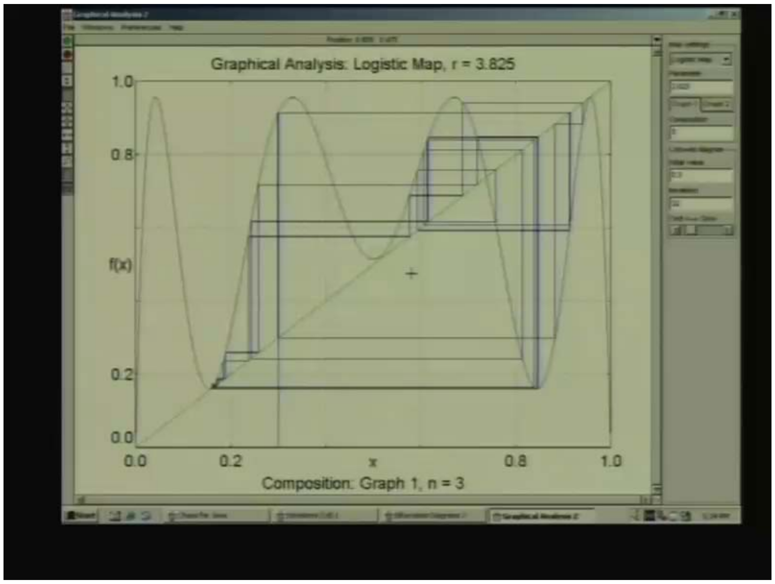

Again slightly more, do you see what is happening? these points are coming closer and closer to the 45 degree line, even closer to the 45 degree line, 328 now become tangent.

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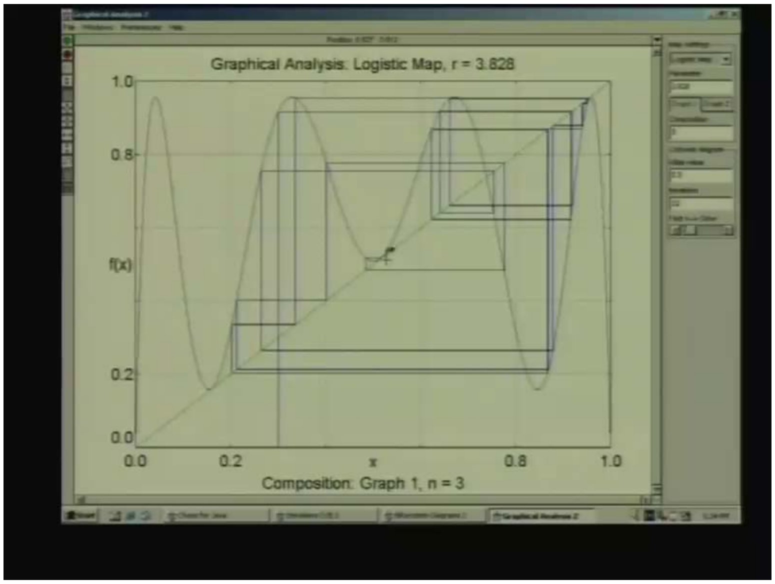

So what we are talking about here in this direction, do you notice that the same thing is happening here?

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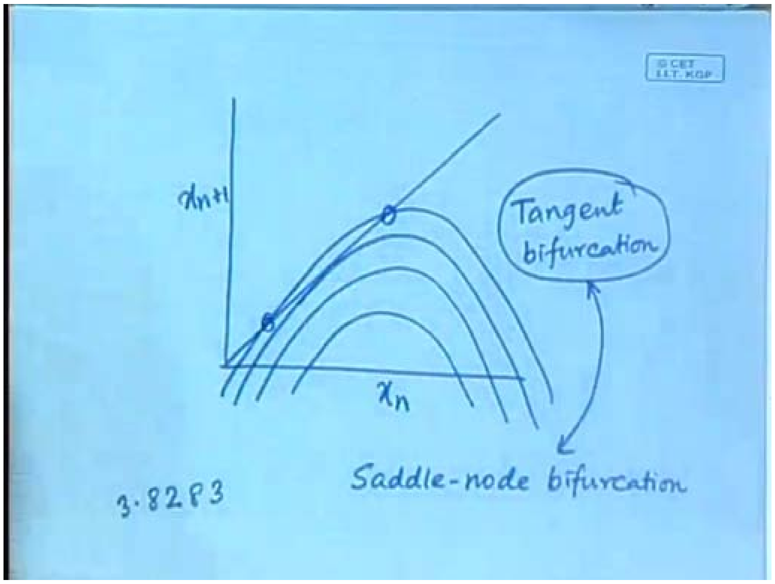

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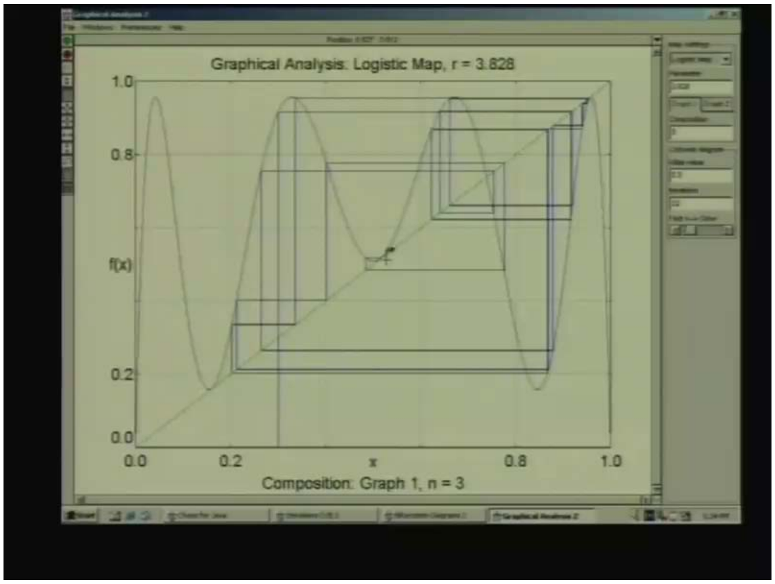

It is becoming tangent so what is actually happening here at the birth of that period three window is nothing but a saddle node bifurcation. So if I ask you what created the periodic window, what will be the answer? a tangent bifurcation or a saddle node bifurcation through which a new fixed point was born but now this new fixed point see if it is I will increase it slightly further, it has now crossed. I will make it a little more visible.

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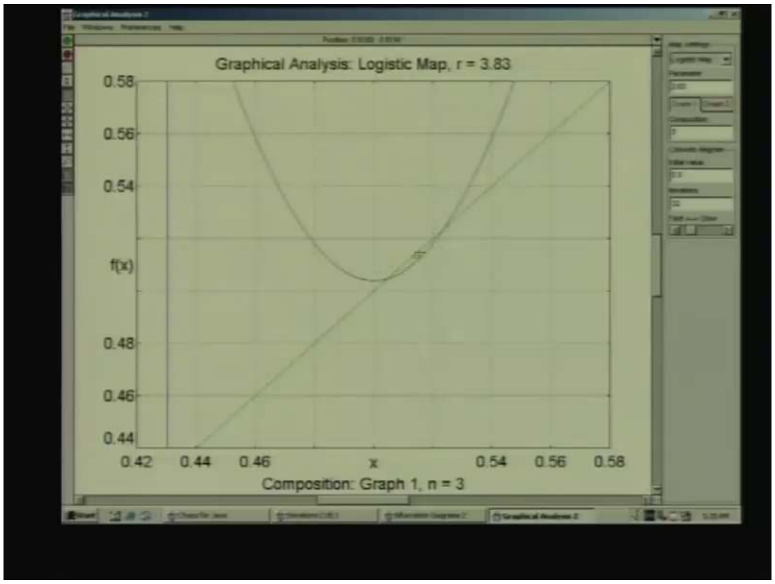

I will show only this part, see it is crossed. So the phenomenon that we are talking about here now has happened here but obviously it has resulted in one unstable fixed point here.

A new pair of fixed point one unstable another stable and this is what we are looking at and this is a fixed point in the third iterate of the map, x_{n+3} is equal to x_n which means it is a stable period three behaviour. That is what we have seen in the bifurcation diagram.

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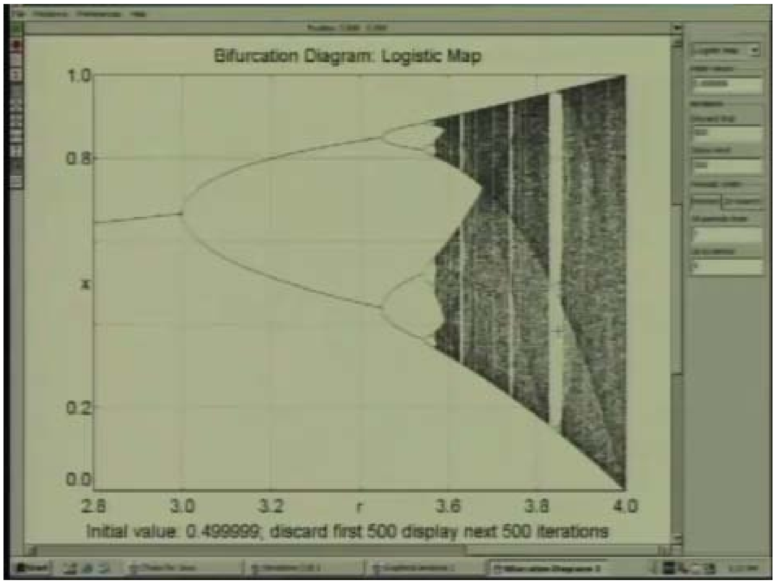

It has resulted in the birth or it has resulted in the chaotic behaviour becoming unstable and the period three behaviour becoming stable because of this. So at that point a new period three window appeared because of the saddle node bifurcation.

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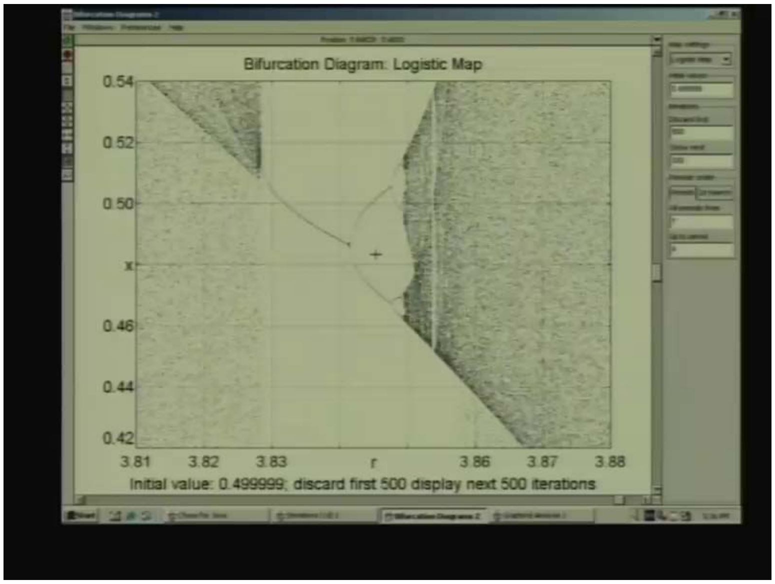

Now notice if I expand only this part, it is again the period doubling cascade because what has crossed here is also the same thing and as it goes on, you can expect the same thing to happen.

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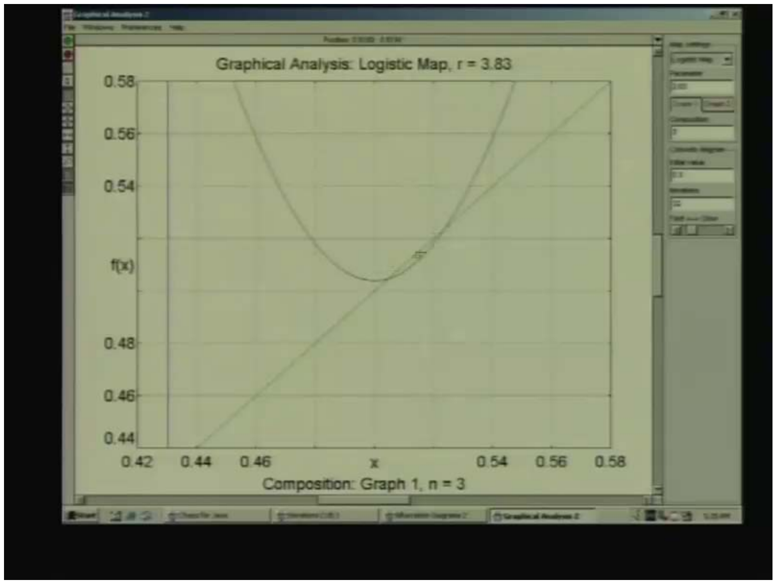

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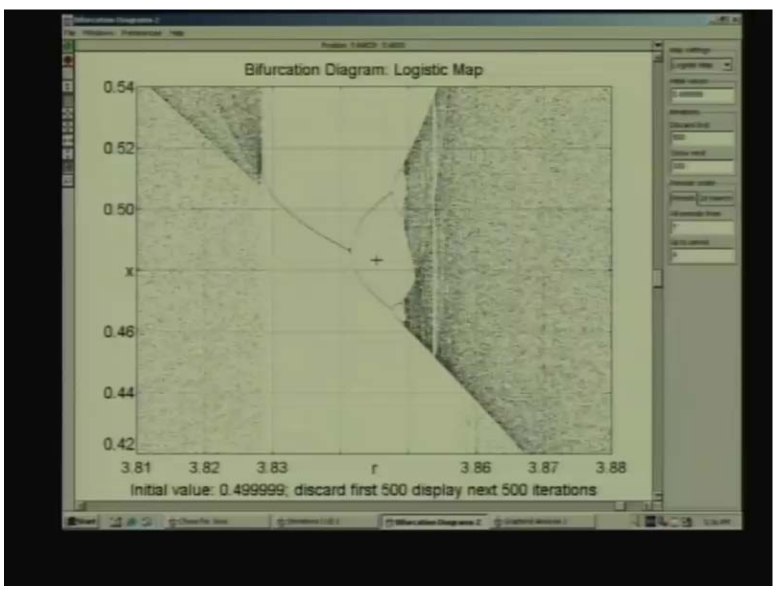

So the period doubling cascade will again appear in this small window and again if you keep on enlarging it, you find same thing and all that will again show the same Feigenbaum number. Let us clean it and let us start it all over again.

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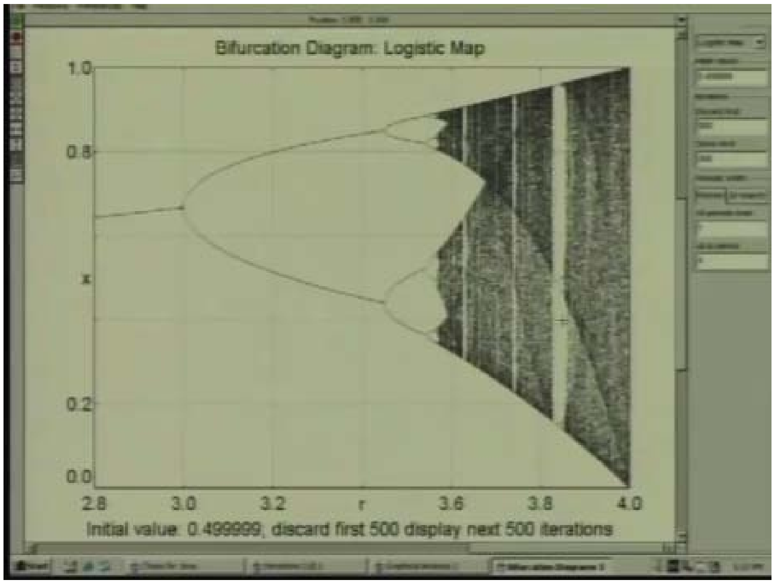

Not only the period three, this was the large periodic window of period 3, period 6, period 12, period 24 and all that. It is also a period doubling cascade but here you can also see a gap, here we can also see a gap. Let us explain and check out what this fellow is. This is period 6.

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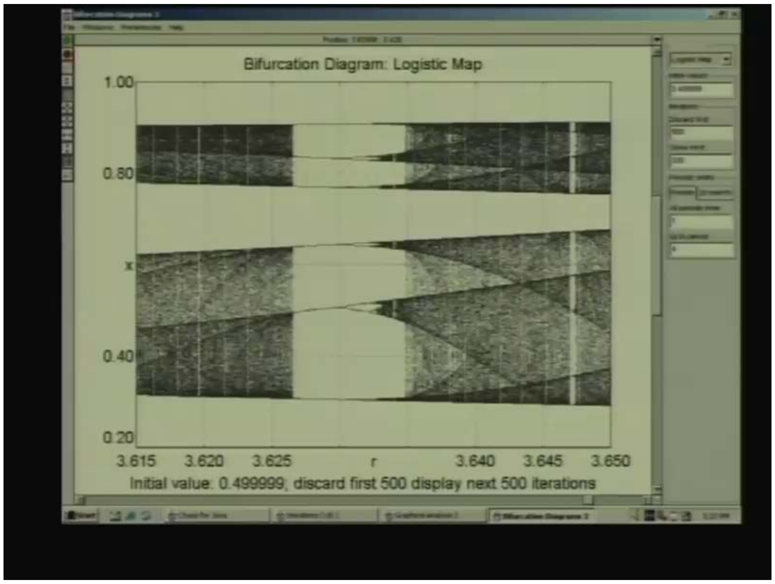

That is also going through a same period.

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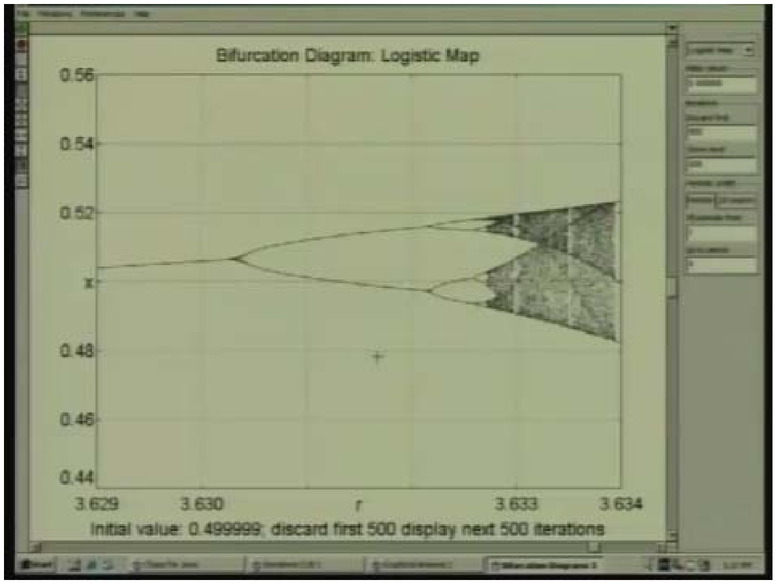

If you look carefully then you see other periodic windows here and here. So that as you zoom into the bifurcation diagram, you keep seeing small periodic windows, each one created by the same mechanism of saddle node bifurcation. Each one ultimately undergoing a period doubling cascade, finally merging into the chaotic behavior. So in the whole bifurcation diagram therefore you can see not only a period doubling cascade once but infinite number of period doubling cascades and another important point is that while you see something as chaotic, in fact there is a theorem to prove it that there are periodic windows at every range. For example suppose here you might think that this range is very chaotic, let us expand it.

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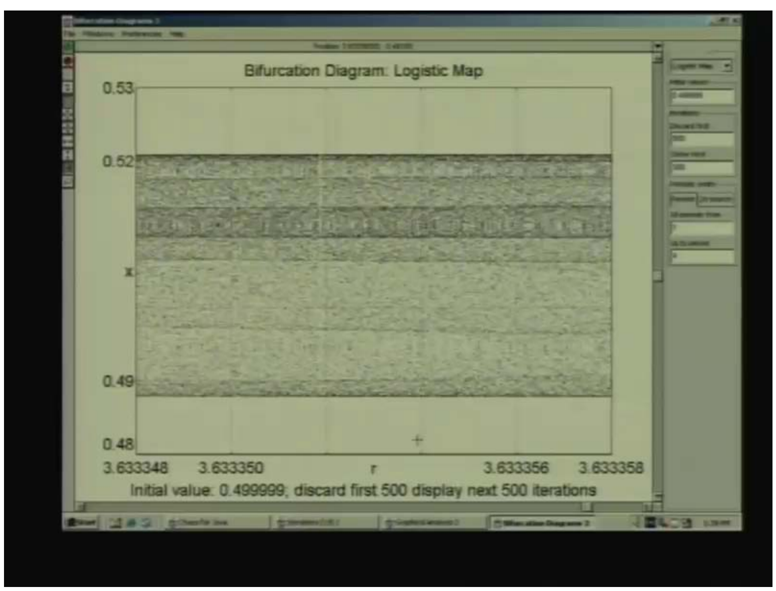

There is something, you might think that this range is very chaotic let's see expand it. Let us further expand it, further expand it, it is allowing for the expansion but there is a theorem that tells at every range there should be some periodic window. So chaos in this case is there but at every range in the parameter space, you can expect a periodic window nearby. All that created by this saddle node bifurcations. So in the bifurcation sequences, as you change the parameter you see a inter play of these two kinds of phenomena, the period doubling bifurcations or flip bifurcations and the tangent bifurcations or the saddle node. In the one dimensional maps you only see these two types. Try to understand the reason. A bifurcation cannot happen unless there is an instability. So all bifurcations are related to instability.

Most of you are from engineering back ground, so it will be easy to understand from that point of view. A linear system with which we are so very accustomed, if that becomes unstable what happens? The system collapses, state runs to infinity but in a non linear system then there is no reason for that to happen. It might go to another stable behaviour. So in a nonlinear system an instability results in a bifurcation and instability happens when the graph of the map becomes either minus one or plus one when it becomes minus one we have period doubling bifurcation. When it becomes plus one we have saddle node bifurcation, this is the only two things that can happen.

So there are essentially two different mechanisms of the loss of stability. Though there are more names, in books you will find more names of bifurcations. I will come to what those names imply but in a sense you can easily understand that there are essential two types, the tangent bifurcation or saddle node bifurcation and the period doubling bifurcation. Through this a specific period of orbit can become unstable. That's all for today, tomorrow we will continue with it.