Chaos, Fractals and Dynamical Systems
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Lecture No # 01
Representations of Dynamical Systems

A source is also a dynamical system though it doesn't move physically because the voltage across the capacitor, the current through the inductor, etc. change. So that is also a dynamical system. In a chemical reaction, you don't see anything physically changing but there is a chemical reaction taking place. You have to mix hydrochloric acid with sodium hydroxide. Then there is a change taking place. Is that a dynamical system? Yes. In the sense that, if you dip a sensor of each of the constituents say, sodium hydroxide, over time the concentration of sodium hydroxide changes and therefore that's also a dynamical system. Similarly, you will find that anything in this world can be seen as a dynamical system. Our body is a dynamical system. Our heart is a dynamical system. Our breathing is a dynamical system. So everything that we can think of are dynamical system. We have found some specific ways of mathematically defining dynamical systems. How are dynamical systems mathematically defined? They are defined in the form of differential equations. This I suppose you have learnt in the mathematics class. Why are dynamical systems or dynamics-change represented as differential equations.

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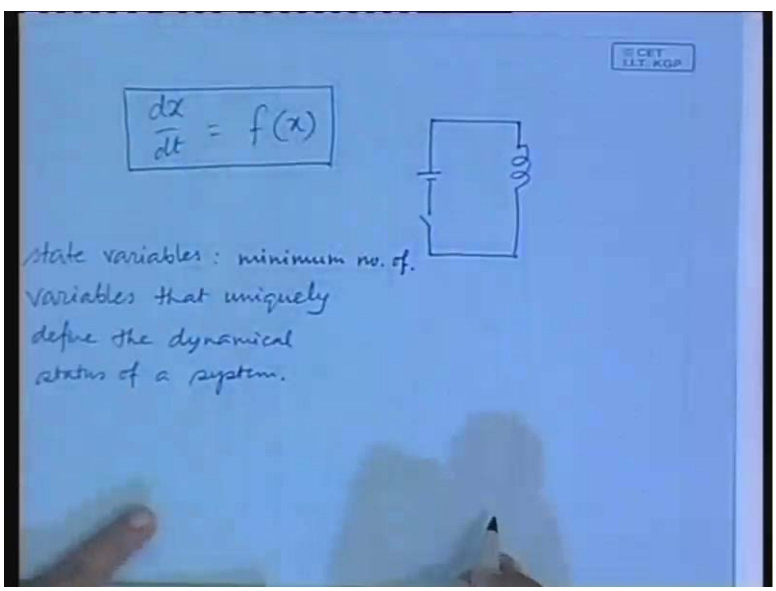

It is because in a differential equation you have the dx/ dt term in the left hand side and that is defined as function of x. that is a dynamical system. Here this is saying how x is changing with time and that is then expressed in the form of some kind of a function, f(x). Now can you think about one dimensional system? Pendulum is not a one dimensional system. It is actually a two dimensional system. In order to test how much you remember, we will today at some stage, derive this differential equation for a pendulum. but if you write this differential equation like this, there are situations for example, you have a voltage source, a switch and then here you have say just an inductance. So this will be one dimensional dynamical system but these are very rare. Normally, what we do is, given any system we try to define a set of variables which uniquely defined the dynamical status of the system. In case of the pendulum, what can you identify as the variables which uniquely define? Is it the angle? Just if you specify the angle, does it uniquely define? For example the angle is this much, in addition to that, you have to state theta dot. Is it going up? Is it going down? With what velocity is it going up? Only when that is given it says, it is in that particular state. That is why pendulum is really a two dimensional system. In general, we try to define a few variables in terms of which we will define the complete dynamical states. that means if these variables are given x with the value 1, y with the value say 1.5, z with the value of say 3.9 ultimately, given all these, the unit that uniquely defines the status of the system. Then these are called state variables.

The minimum number of variables that uniquely define the dynamical states of a system. In general, in mechanical system what are these? How do you define them? We essentially defined the different mass points, their positions and their momentum. Similarly, in electrical circuits we locate the storage elements. Storage elements are inductance and capacitance and the current through the inductances and the voltage across the capacitance uniquely defined the state variables. In certain situations, the number of states variables may be less than the number of inductance and capacitance in the system. However for this course we will not go into those issues. But essentially just remember that this is the definition of state variables. So in trying to understand any dynamical system, what we do is we write the variable equations as they are called. What does it mean? We write equations like these in terms of the state variables. So in that case, x will no longer remain a number.

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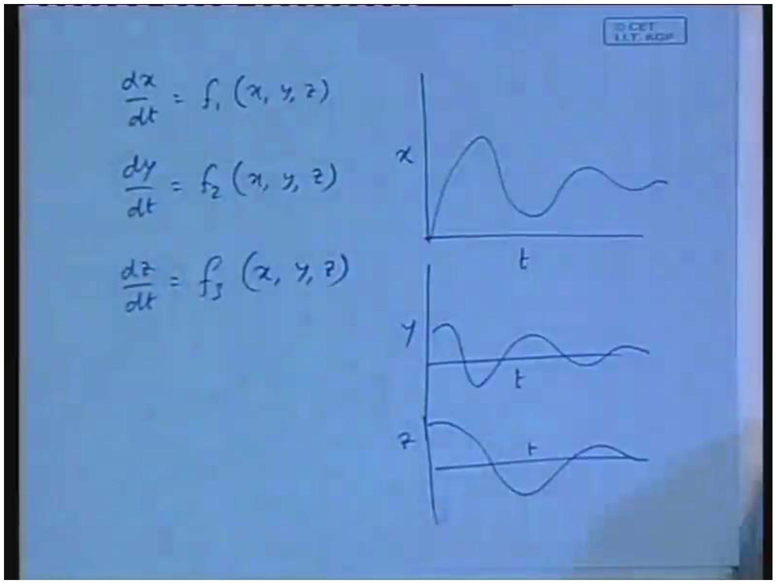

x will become a vector which means if say I have 3 variables in a system I will write dx/ dt is equal to something. dy/ dt is equal to something else. dz/ dt is again equal to something else. So this will be one function of xyz, another function of xyz and a third function of xy and z (Refer Slide Time: 08:21). Given this, the differential equations are completely defined. You have also learnt how to solve differential equations. At least you know how to solve a class of differential equations. Solving means that either we are able to explicitly write down the solution or we say that we cannot do that. But given an initial condition I can always tell where approximately the states will go by numerically solving it. That's how we finally do mostly but nevertheless, when we are doing that what are we in essence doing we are starting from an initial condition. That means here is my xyz at the initial time =0 and then it is evolving.

Evolving in what? Now the evolution, you normally would have seen something like this. you have seen a waveform something like time versus a variable x. similarly there will be a waveform of say, time versus the variable y. Similarly there will be a wave waveform for the time versus variable z. this is probably what you have already learnt how to obtain. As I told you at least for a class of system. What kind of class of system we have learned in all probability? We have learned how to do that in linear systems where the right-hand sides are linear functions. But in general, as I told you systems are non-linear. In fact all systems are non-linear and we will see what linear system means in the context of the general idea of non-linear system. We will see that. But in the sense we have this and you can obtain this. So what are we exactly doing? In order to understand that, it is necessary for us to understand or grasp the idea of what is known as a state space. In this what we do is, here we have the xyz. One of them could be a voltage across a capacitor. Another one could be a current through an inductor.

If it is an electro-mechanical system, the third one could be the even the momentum of a machine. So in general, these are those state variables. In a mechanical system these are related to the positions and momentum of the mass points. In electrical system these are related to the capacitor voltages and inductor current and in electro mechanical system it could be a jumble of all this.

So notice that the moment I have written in this form, we can, for all practical purposes set aside the actual system description. Forget about that. What we are talking about is a voltage or a position, current or momentum. We can forget about that because we have written down the equations and we can go ahead with that. May be at a later stage, when you have understood, it will have to translate it back in terms of what happens in the actual system but presently, we'll assume that we have abstracted the system and then we are not able to see the distinction between the x, y and z on the same pedestal. They have the same value for us in the differential equations and then what we do is we say, "Let there be a space where the xyz are the coordinates". It is quite unlike the space we live in because it is in this space in which we live. xyz are the special coordinates. There is also time coordinate't' but here, we are not talking about that. We are talking about the voltage being a coordinate. The current being a coordinate. The momentum being a coordinate. So that then assuming it is a 2D system.

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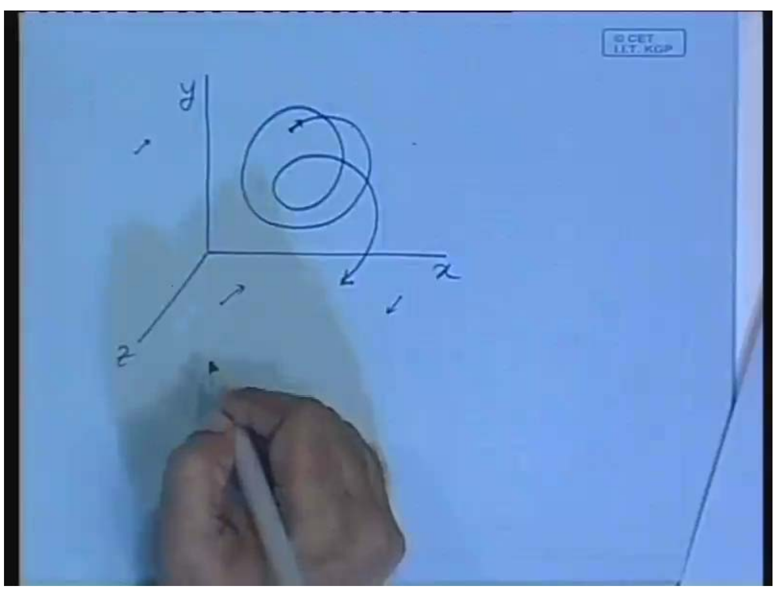

Then x and y are the state variables and such a space is called as state space. Now in that background then the differential equations give a meaning. What do the differential equation say? It says that if the x, y and z have a specific initial condition. Here I am talking about the x-y space. Nevertheless it could be xyz space, an initial condition given that initial condition if you solve these equations; we get some kind of an evolution in this space. So the differential equations essentially tell how the state point is going to evolve in time in the state space.

So from now onwards, all our views will be directed to what happens in the state space and we will try to understand visually in terms of visual impression. See these are not things that we really see before our eyes. But in the mind's imagination at the end of the day, after you complete this course, you should be able to visualize, "yes. It is happening like that". If you see a differential equation, you have a set of differential equations telling you how the state evolves. At every point the state is evolving depending on the set of differential equation starting from an initial condition. That is the setting. Now let us see how far this idea takes us till in the language of "Subramanyam Chandrashekar" till "the sun melts the wax in our wings". He ended a chapter in his book with this.

Let us see how for this takes us till the sun melts the wax in our wings. By the way, starting from here, how do we know where to go next? For example, at this point somehow this point knows that it has to go in this direction next. How does it know? It knows from the differential equation really because here means a particular value of x,y and z and if you substitute these values of x,y and z in the right-hand side you get numbers. You get numbers here. So this equation tells that in the next instant, it will move so much in x direction so much in the y direction and so much in the z direction and that together will define this vector. If the state point where here instead, same thing applies. You would have been able to calculate again by substituting particular numbers here. You would have been able to calculate what is dx/ dt, dy/dt and dz/dt which means that irrespective of where I am, the set of differential equations tell where I should go. This means that if I am here (Refer Slide Time: 17:09) the set of differential equation defines an arrow. If I am here the set of differential equation define an arrow. If I am here the set of differential equation define an arrow. At every point in the state space the differential equation define an arrow. So after you have written down the set of differential equations for any system, what I have essentially defined is, in the state space you have defined arrows at every location. If I am here I know where to go and here I know where to go. As from here I start, I go on. When I reach here, still I know where to go as it goes on. Still this fellow knows where to go and so goes on traversing the trajectory.

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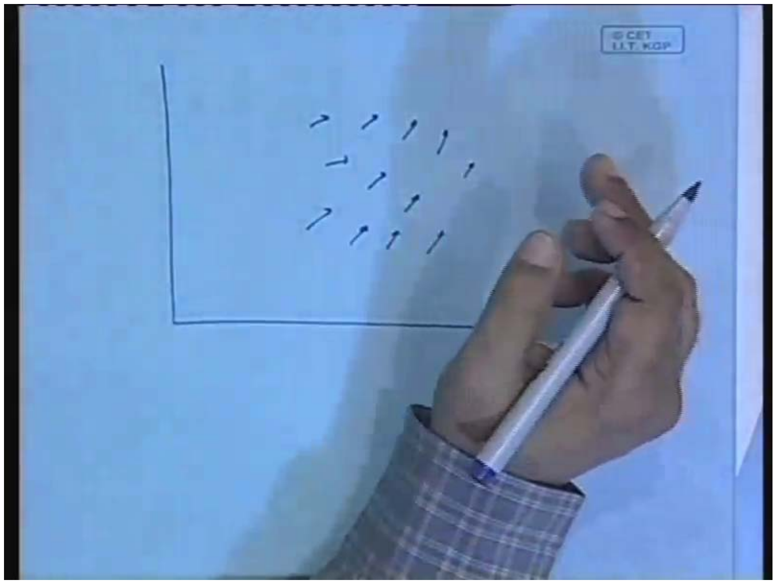

So, essentially the state space is completely defined in terms of the differential equations when you have defined the arrows at every point. This means that the differential equations define arrows at every point and notice that the state evolves along the arrows as if something is flowing and you put a drop on the floor, it flows around. The solution of differential equation is something like that. There is a stream and leaves drop on it and it flows along with the rest of the flow. it flows exactly like that. So that is why a set of differential equation is also called a flow. in literature we will find this word ‘flow’. You will see sentences like “let us define a flow”. It is nothing but a set of differential equations. It defines a flow and when a set of differential equations are given, it defines a vector at every point in the state space. That is why it is also called a vector field. So far the kind of impression that you had was that you were able to solve a certain kind of differential equation mainly linear differential equations. You were able to distinguish a particular integral and all other things which means that given an initial condition t, you are able to traverse for a certain specific set of differential equations. You cannot do it for everything. You have learnt in the numerical analysis courses how to solve this differential equations numerically. There also you have learnt how to start from initial condition and then traversed. But if I ask you what all can this system do, what are the different types of dynamics possible for this system? Can you see? No. then what will have to do/ start from all possible initial conditions and and go on computing or if it is a solvable differential, equation then do you know how how long it will take for you to tell what are the different types of dynamics possible for this system.

You have to solve it for every given initial condition which means in the solutions you will have to calculate this C one C two all that for every single initial condition. But the moment the states space and the vector field is pictorially in front of your eyes, you know what can happen. Now we will try to do that. For that purpose let us start with one example system. Let's take this example.

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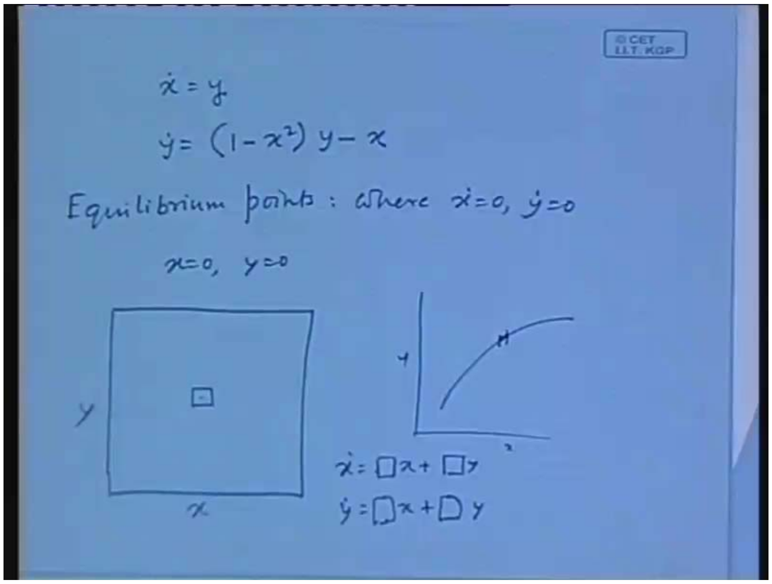

As we go on with this system, we will try to understand a few other concepts. Now given this system, can you draw the vector field? You will be at a loss. So start at the point that is simplest. What is the simplest point? It may not be (0,0). See you have to ask in which position in state space does the left-hand side vanish. Left hand side means x dot and y dot. x dot and y dot vanishing means if it is there, it remains there. These are equilibrium points. So how do you locate the equilibrium points? So let's write where x dot = 0. For this system can you obtain? Well this fellow says x dot =0, means y =0. Now y =0 if you put here you get a quadratic equation. You need to solve it and then obtain the possible values of x for which it will be zero.

So there is only one equilibrium point. If you multiply by y, it is it is zero sorry it doesn't get the quadratic here. So here in this case x =0 y =0 is the solution. Now therefore we have this state x-y state in which x =0, y =0 is an equilibrium point. What we do next is we take a magnifying glass and go very close to this equilibrium point look at it. What is the structure of states space or the vector field around it? That is the easiest to know because as you zoom closer and closer, the non- linearity in the system will slowly vanish. It is like you have got a curve and as you zoom closer and closer, it becomes more and more like a straight line. So as you zoom closer and closer it will more and more lose its non-linear character and you can you will look at the linear neighborhood of this point. So we are essentially zooming into this region. as you zoom if you are looking at the the linear neighborhood of that, the way you when you were looking very close to this, if this is your y and this is your x then what what are you looking at at this point? How do you represent the local linear representation? By dy/ dx.

When you when you are zooming into a two dimensional system and trying to understand the location, you still have a 2D equation. You should still have something like this. x dot is equal to something which is a linear combination of x and y. so x dot is something x plus something y. y dot is equal to something x plus something y. then it becomes a linear equation. What are this something's probably you have learned that. How do you obtain this something from this nonlinear differential equation? This has a name. The Jacobian matrix. So the Jacobian matrix is essentially nothing but if you are zooming closer and to closer to the equilibrium point, you get a local linear representation and that local linear representation. In the neighborhood of any point is given by the Jacobian matrix. So what is a Jacobian matrix?

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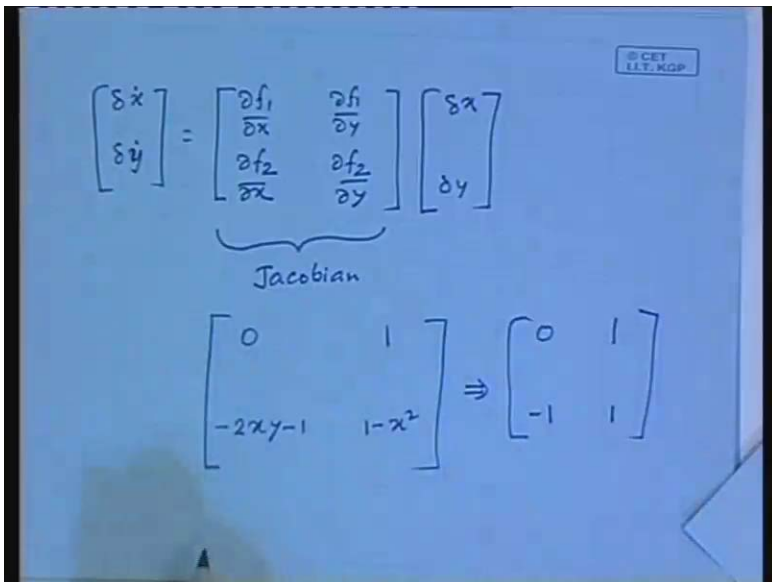

Then we will represent everything in terms of deviation from the equilibrium point. In this case it so happens that the equilibrium point is at the origin. But in other systems, it might not be so. in general we will write the deviation from the equilibrium point as delta x delta y and then this will be given as this (Refer Slide Time: 28:03). So this is the Jacobian matrix. So what have we done? in obtaining this, we have said how does that deviate from this equilibrium point vary. So we are saying that 'delta x' - the deviation from the equilibrium point in the x direction varies as this delta y varies as that. Can you obtain this for this system? So we will say this is f $_1$(xy) and this is f $_2$(xy). So the derivative of f_1 with respect to x is zero. The derivative of f_1 with respect to y is one because it is only y derivative of f_2 with respect to x is -2xy and -1. So let's write it. This will becomes 0, 1, -2xy-1 and here it is 1-x. also I have something that is containing all these things because you are trying to evaluate it at equilibrium point where x and y are zero. So substitute these values and then that takes the form 0,1,-1 & 1.

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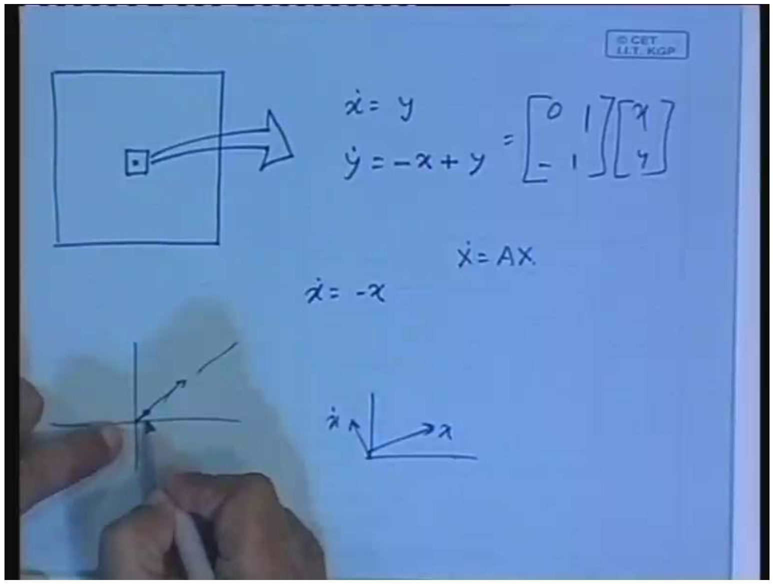

Then we will say that now in the neighborhood of this, my differential equations are the following. So this is the whole states space. Here is my equilibrium point. I am now zooming into this part and here the equations are x dot is y and y dot is –x+y. this is the method of local linearization. So the Jacobian matrix is essentially to obtain the local linearization and then when we look at this equilibrium point and the behavior around it, it is sufficient to study this differential equation. Now those of you who have gone through the course of contour theory and courses like that, we realize that in the contour theory studies, we essentially look at only this. We always take a magnifying glass and go close and close and say that “I am happy. I have got a linear equation and then what do we do? we say that now if we make a system work at the equilibrium point, I will make a tacit assumption that the deviations from the equilibrium point will never be sufficiently large so that it goes beyond the range where this local linear approximation is valid and then we work throughout our life with this equation.

We normally start with a set of equations like this but then it may actually so happen that in running in a particular system, it may really go beyond that because of either some kind of a failure or some kind of an overloading or some kind of a parameter that you really didn’t foresee. Of course that also takes into an account the possibility that you might actually want to use the rest of the system. You might actually not want the linear behavior to be used. I will come where these are useful. But then here we are. We have a set of linear differential equations given by this. Now you can solve it. How will you solve this set of differential equations? To briefly recall the essential method of solving differential equations, it says that this is a two dimensional set of differential equations. Suppose you do not know how to solve a 2D system of differential equations but then you are born with the knowledge of how to solve a one dimensional differential equation. So the one dimensional differential equation solution as you know is an exponential function. So we know that.

In case of 2D, there is a theorem that says that if I can look at a 2D system, just locate any possible two solutions then any of the solution is a linear combination of these solutions. We will not attempt to prove that theorem because I will assume that you have come across that in your differential equation courses. Essentially everything builds on that theorem which says that if I look at any two solutions, then any possible solutions starting from any initial condition is nothing but a linear combination of that. If it is a 3D set of equations, I need to identify three solutions and so on and so forth. That's what's work usually with two d so the the the mathematicians will say what is a simplest way of identifying a solution in such a system? Now the logic is as follows. Probably you have come across in the matrix theory. Here you have a matrix. so if I write it in a in a compact form, I will write x dot is equal to A X where X is a vector this dot is also a vector if I say it verbally, I will say A when operated on X gives me x dot. A is an operator operating on vector x giving another vector x dot. In general the vector on which it operates and the vector that you get are different in magnitude as well as direction. But then you have learned in your math courses that there exist in 2D system two directions such that if the initial vector is along that direction the x dot, the resultant vector is also in the same direction. These are called the Eigen directions. Any vector along the eigendirection is eigenvector and therefore in the Eigen direction, suppose this is an Eigen direction (Refer Slide Time: 37:58) and you take the x vector along that and you get the x dot vector like this, that means each has been multiplied by a factor to get x dot. The x has will multiply by number to get x term. What is the number called? It's called the eigenvalue. So the mathematicians would say, in solving this let's see if I can start with an initial condition on this line. Then I know that the resultant x dot will be along the same line and therefore whatever it does, its evaluation throughout will remain constant to this line. There is a one dimensional equation then. I know how to solve it. I can write down the solution. Again I can identify another Eigen direction like this. If I state an initial condition along that, I can again write down first order of equation. So the final solution is nothing but the linear combination of this and that. That is the essential way of solving differential equations. Probably you have all gone through.

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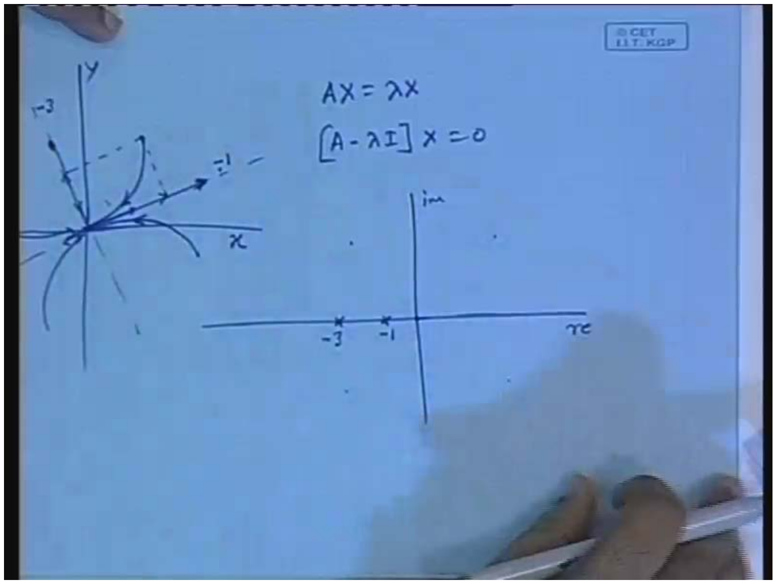

By writing see the the logic is that here I start from a vector and what I get is also along the same line. So A operated on X is lamda X. we essentially do this. Then we say (A - lambda I) X is equal to zero. So this fellows determinates should be zero that we obtain a quadratic equation for a two dimension system. Solving it we get two eigenvalues. Now the eigenvalues therefore are the solutions of a quadratic equation in case of a 2D system. In a 3D system, it will be a cubic equation. You need to solve. But then a quadratic equation may have both a real solution as well as an imaginary solution. A real solution can be positive or negative. For the purpose of this course it will be necessary. So there are various possibilities. in general we can say if we have this as the complex plane - the real and imaginary, then the eigenvalues can be on the positive real line, can be on the negative real line or anywhere and each case you should at least know you should be able to tell what will be the system's behavior like.

Here is something that is a concept that if the eigenvalues are such, the behavior will be so. That you should know by heart. What if the eigenvalues are say here (Refer Slide Time: 44:43)? The two eigenvalues are both real and negative. Try to understand the logic. so in a system if this is your x and y and say this is one eigendirection and this is another eigendirection and along this Eigen direction. Suppose along this we have got an eigenvalue at -1 and along this we have got eigenvalue of -3. What does it mean? It means that if the x is along that x dot will be -1 times this. 'Minus' means it will be in the opposite direction. It will decay its distance from the equilibrium point will die down and the more it comes closer the slower is the rate dying down. Because x times -1 is the rate. When it becomes very close then it also because very slow. So naturally it has to be exponential. You don't really have to remember. It's quite logical that it can be nothing but exponential. Similarly if it is here along this (Refer Slide Time: 46:53) line, it will die down it will die down faster because it's -3. So it will die down faster along this direction.

Now if you got the initial condition somewhere else, then when I say that the resulting solution is a linear combination of this solution and that solution, what does it means? It means that you can drop a component of this point on this and that. So this vector can be broken up into a vector along this and another vector along that. This vector will die down as e^{-1t} and this vector will die down at a rate e^{-3t}. As a result, this will finally come closer and closer to the equilibrium point. How can you figure out how will it come like this? It cannot come like that. It will die down faster in that the direction and therefore it will come like this. If it is here again you drop projections and you will see that it goes like this. If it is here it goes like this and if it is here it goes like this.

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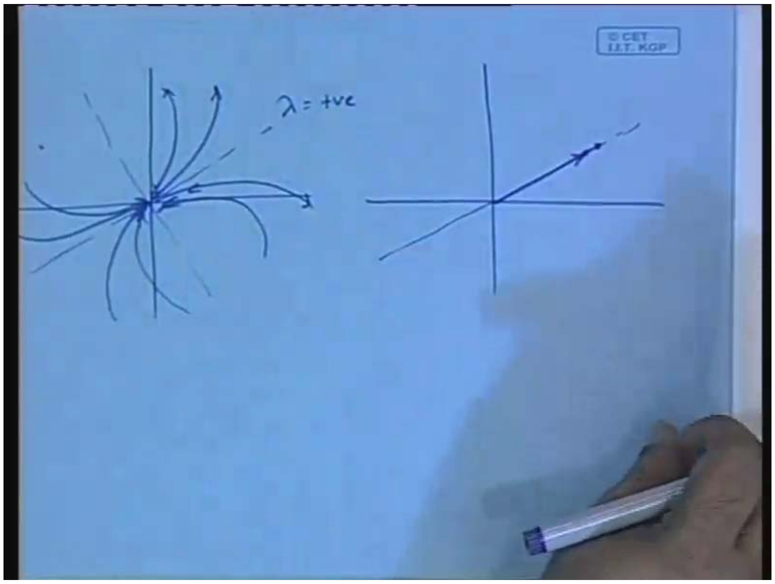

So you will see you essentially get an evolution like this. Do these arrows make sense? It says that if it is here, it will go like this. If it here it will go like that (Refer Slide Time: 49:07) which means that it will converge faster on to this eigenvector and then along that eigenvector it will approach the equilibrium point. It will decay in a specific way. Is that understandable? So if see you can then draw a picture of the whole state space. The moment you look at this state space you know all that it can do. So this is the diagram of the vector field. You don't really need to point at every place and then draw the arrow. That's not necessary. You know the starting from such an initial condition, you go like that. If you start from some represented initial condition and if you draw the lines along with it, it gives an idea.

So if both the equilibrium points are negative and real, it will go like this. If both the eigenvalues are real and positive it will be just the opposite. It will be similar only the direction of the arrows will be opposite. If an initial condition starts very close to the equilibrium point it will diverge away. Notice the logic you have got this you have got this this direction as the eigendirection starts from this initial condition if this is x vector where is the x dot vector? If it is real and positives then it will be in the same direction so that it will go further along that outwards. As it goes here it again has a vector pointing outwards so on and so forth. So similarly it will go out so it will go out. Therefore its vector field picture will look similar only with the arrows pointing in the outward directions for lambda – positive.

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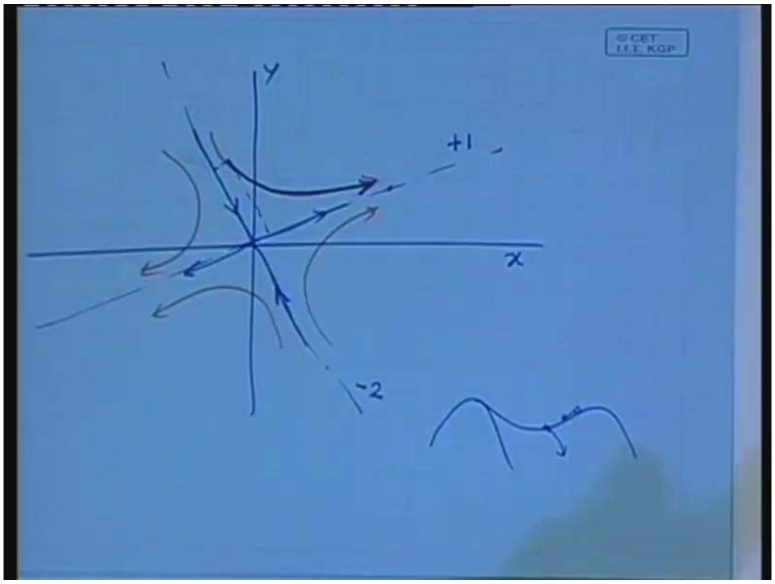

You have this state space picture – xy. Suppose this is one direction and this is another direction and here I have eigenvalue plus one and here it is minus two. What will happen? If there is any initial condition exactly on this point, it will diverge. If there is any initial condition exactly on this line it will converge. So this is the converging direction and this is the diverging direction (Refer Slide Time: 52:35). If there is any initial elsewhere say, here, how will be or big be like? Again drop a projection on this and drop a projection on that your argument should be that this particular projection should increase. This length should decrease and therefore it should evolve as this. If I draw the other side from here it will go like this, from here it will go like that, from here it will go like that from here it will go like that.

So there exist a very small region in this state space from which it converges everywhere else it will ultimately diverge and the state of initial condition on which if it is placed the initial conditions placed it will converge. How small is that spatial initial condition of measure zero. Out of this two D space it is just a line. Therefore for all practical purpose we will say this system is unstable. This is similar to the view of a saddle. If you have a saddle then it if you release a ball here, it goes this way, converges and if you release the ball here, it goes this way and diverges (Refer Slide Time: 54:30). So it has a structure like a saddle. That's why this kind of an equilibrium point it is called a saddle. Let's stop here and will continue in the next class.