Optimal Control
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Lecture - 49
Solution of Minimum - Time Control Problem with an Example

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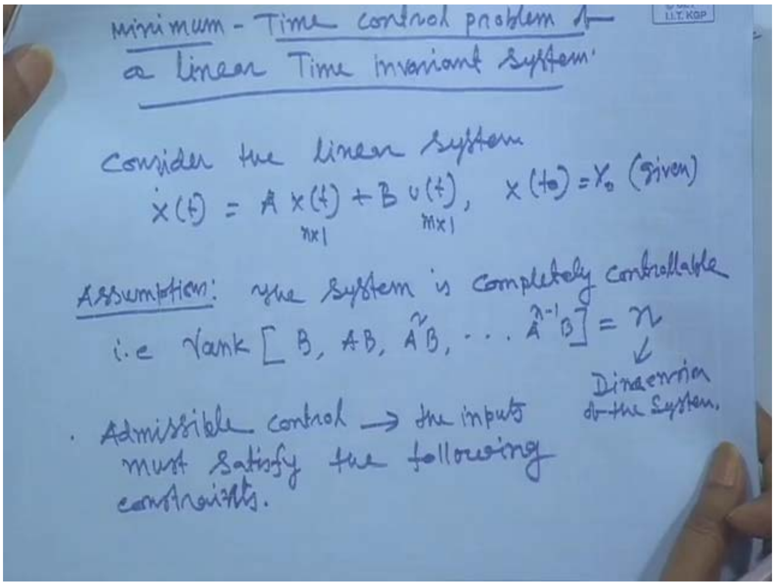

So, last class we have discussed that statement of the, what is call minimum time control problem of a linear time invariant system. Let us recap this statement of the problem. Then we explain the solution of minimum time optimal control problem with an example. So, let us call the linear dynamic system is given by x dot is equal to A x plus B u, u is the control input and x of t 0 at time t 0, initial time t 0 is x of 0 is given. So, our problem is to drive the state from initial time t is equal to 0, the state value is at 0 drive this state to a final state at t is equal to t f, where x f is given in a minimum time. So, that is our problem.

So, first we have to check whether the system is completely controllable or not, by the what is call these rank condition we have to satisfy B A B A square B A n minus 1 B, n is the number of states involved in the system derivation. That rank must be n then all the states will be controllable and not only this we have a constant on the input most of the physical system having a definite input to the systems agree.

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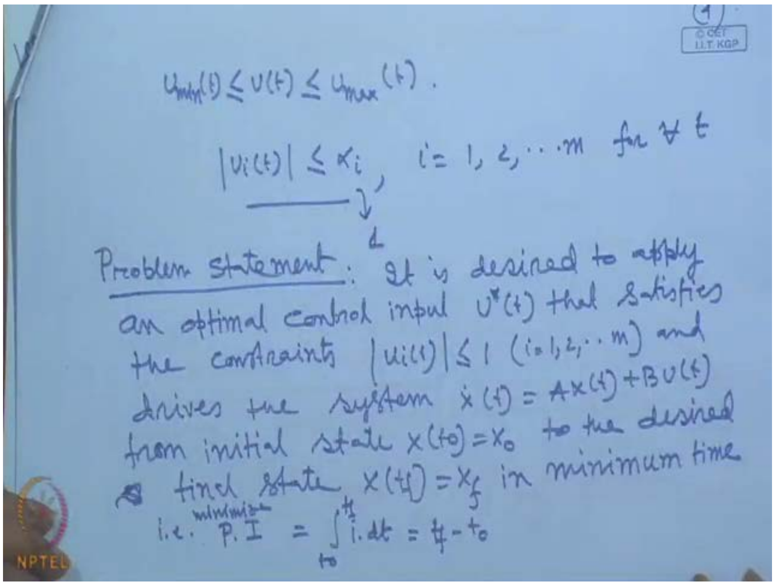

So, that constants are given maximum value of input to the plant u max and minimum value of the plant is U min. Generally this upper and lower limit are same with the positive sign and negative sign this one. So, the our problem statement is like this way, it is desired to apply an optimal control input U star that satisfies the constraint. Now, in case of this alpha 1 we have constrained the 1 unit you have to input for all the input i is equal to 1 to m, m is the number of inputs drives the state of the system initial state X 0 to the desired final state X f agree in a minimum time.

So, our performance index, this performance index we have to minimize that means we have to find out the minimum value of the time t f for that one. That is been solved by using what is called ((Refer Time: 03:00)) minimum principle PMP. When the x, when our terminal state or the final state is 0 at the origin, we want to drive the initial state to the origin, then the problem is called time optimal regulated problems. So, this is our problem statement towards that and we have shown it last class derived it how to approach this 1 by using the ((Refer Time: 03:28)) minimum principle.

Let us take our example. So, that will drive the state from initial state x t 0 u is equal to x of 0 to the final state x t f is equal to 0 in a minimum time t f by using, what is called switching the control law. In two state either plus 1 or minus 1. So, that is called bang-bang control.

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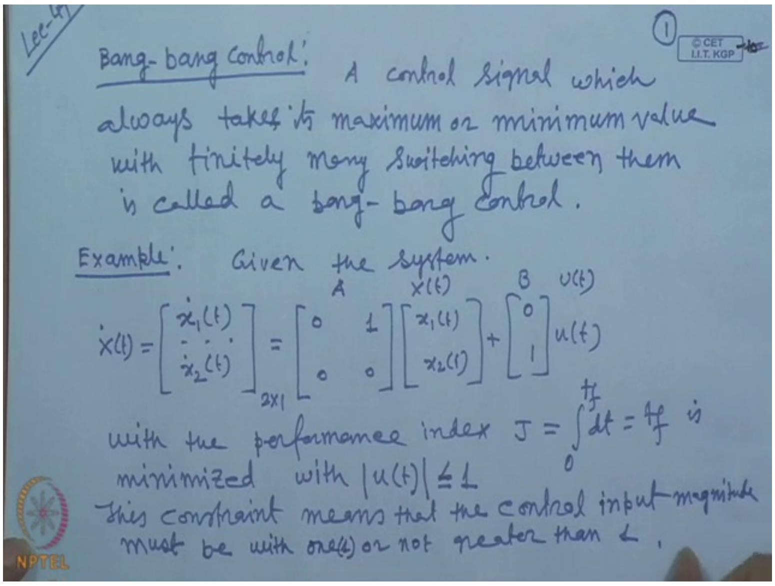

Bang-bang control, so what is bang-bang control, a control signal which always takes maximum always minimum value with finitely many switching between them minimum and maximum value between them is called a bang-bang control. So, the two maximum minimum state is there for a control input. Either you switch on plus 1 or you switch in minus 1 and the state will drive from initial this control action will drive the state from initial state x t is equal to 0 to the final state x t f is equal to 0. When at the at the origin in minimum time t f.

For any initial condition x 0 to a final state x t f is equal to 0 will drive the initial state to 0 by proper sequence of control actions. That means either you switch on to plus 1 or minus 1. In this way and many times you have to do depending upon the order of the systems. So, let us take any one example and illustrate the problem that minimum time optimal control problems. So, given the dynamic system, whose dynamics are expressed in this form, that x 1 dot of t that has two state. Let us call we consider the second order system which is expressed in time demand.

In general we can write as dot of t which has two states up there dimension of this equal to it is given at, 0 1 0 0. There is x 1 of t x 2 of t and that is 0 1 of u of t. So, our equation is this is a matrix, this is state vector, this is the B matrix if it is single input we write generally small b and this is the u of t, if it is multi input capital B and capital U.

So, this is our dynamic equation with the performance index J is equal to t 0 to t f, t 0 is equal to 0 and d t which is nothing but a t f. That means we have to minimize that performance index. With the performance index minimize the performance index, this is the performance index is minimized, not only minimized with the constraint control input constraint mode of this is equal to less than equal to 1. This constraint means that our control input is plus 1 to minus 1. This constraint means that the control input must be control input magnitude must be within 1 or not greater than 1. This one means this, that is our problem. So, how to solve this problem.

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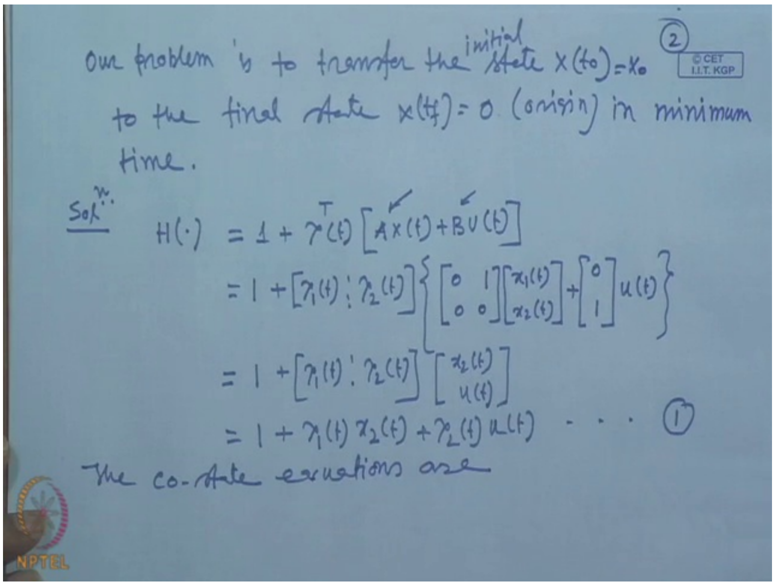

So, our basic problem is to transfer the state x of t 0, initial state transfer you can write it initial state x of 0 to x of 0 is to the final state x of t f is equal to 0, means origin in minimum time mean t f must be minimum. So, if you recollect just state given the system our problem is, to transfer the state initial state to the final state t x is equal to t f to 0 ,means origin in a minimum time by minimizing this performance index. Not only that a subject to constraint control input constraint, that magnitude of u cannot be greater than 1 or within 1.

So, that is solution of the problem, if you follow the what we have discussed last class. First you form a Hamiltonian matrix that our performance index. If you see that integral part is 1 coefficient of this is 1. So, 1 plus lambda transpose of t into A of x t plus B u of t that is our this. Let us write since we have a number of states is two. So, the dimension of lambda is also two row one column. So, it is a transpose. So, if I write the each element of lambda 1 of t partition lambda 2 of t write corresponding matrix A B in this expression.

So, our A matrix is if you see the our X limit is 0 1 0 0 multiplies by x, x means x 1 of t and x 2 of t. So, multiplied by B, B is 0 1 then u of t. Now, you simplify this one, if you simplify this one you will get it 1 plus lambda 1 of t lambda 2 of t then this into this, this into this. So, it will be A x 2 of this and this into this into this 0 plus u of t. So, that which is nothing but a further expanding this 1 lambda 1 of t x 2 of t plus lambda 2 of t u of t, let us call this equation is 1.

Now, see their lie Hamiltonian matrix, that is a linear function in u and not only that, you can if you use the what is call pontigion minimum principle. This function will be minimum when this value of this one is minimum, so that only possible by selecting the value of u of t with the knowledge of lambda of t.

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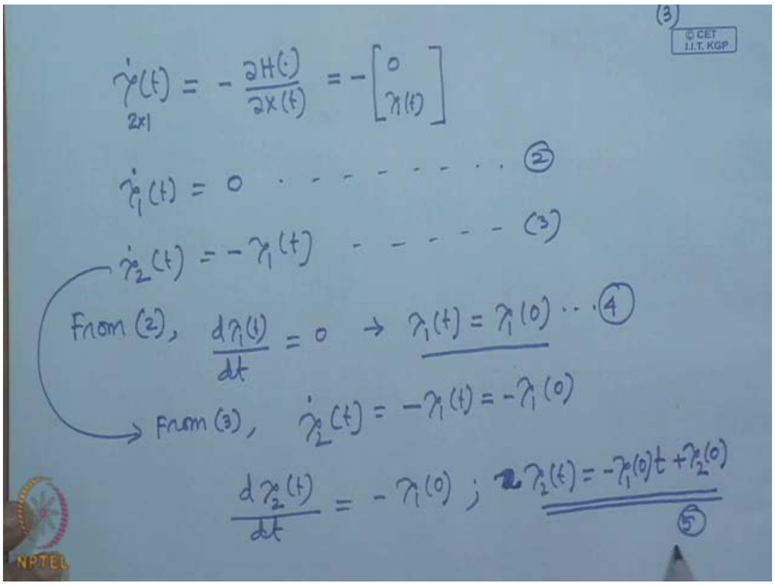

So, the co-state vector, co-state equation are, what is the lambda dot is equal to lambda dot of t is equal to minus del h del x of t. That is our basic equation that we have seen from calculus of variation how to optimize the dynamic systems this is the things. So, this x is the dimension of 2 so lambda also dimension of 2. So, this we can write it further we can write it minus, see each function I have to differentiate with respect to x. So, there is no x term involved in the expression 1. So, that will be a 0, then differentiate this ((Refer Time: 15:14)) this with respect to x 2. So, this is the only term is there, so it is only lambda 1, so lambda 1 of t.

So, the co-state vector expression is this 1 and we can write it because it has a two components and that 2 components is lambda 1 dot is equal to 0 another is lambda 2 dot is equal to minus lambda 1 of t. So, let us call this equation is 2 and this equation is 3. So, we can easily solve the equation 2 and 3. In general even if you get the what is call the matrix is not a special structure in all elements are there, we know how to solve the what is call state equation. Here also will come this type of equation and we know how to this is x dot is equal to something like a x form you will get it you can solve.

So, looking from this 1 I can write it first, see from equation 2, what you can write it integrate that 1 with respect to that is delta 1 is nothing but a delta 1 dot is lambda 1 is nothing but this thing. So, this is equal to 0. So, now this implies that lambda 1 of t is equal to lambda 1 of 0. Let us call this equation is equal to equation number 4. So, you take this, that side then integrate from 0 to t. So, say that will come lambda 1 of t minus lambda 1 of 0 and right hand side is 0.

So, this is the expression and use this expression in equation 3, from equation 3 lambda 2 dot of t is equal to minus lambda 1 of t which is equal to minus lambda 1 of 0 lambda 1 of 0 is replacing this is that one. So, what is the solution of that 1 the solution of that 1 I can get it lambda 2 of t or you write it this lambda t of t divided d of t is equal to minus lambda 1 of 0 this. So, now take d t is this is and then integrate from 0 to t. So, this shows that lambda 2 of t is equal to minus lambda 1 of 0 into t and lower limit t is 0. So, this term will not be there then another term is there lambda 2 of 0. So, this is the our solution of that lambda 2 expression. So, this expression let us call this is equation number 5.

So, from 4 and 5, I can now say from equation from equation 1 we means Hamiltonian equation h of from equation 1 this equal to 1 plus lambda 1 of t into x of t plus lambda 2 of t into u of t. So, using pontigion minimum principle I can say that when lambda 2 is positive, u 2 is negative we will consider. Then this function value will be minimum, then when lambda 2 is negative u 2 value is positive.

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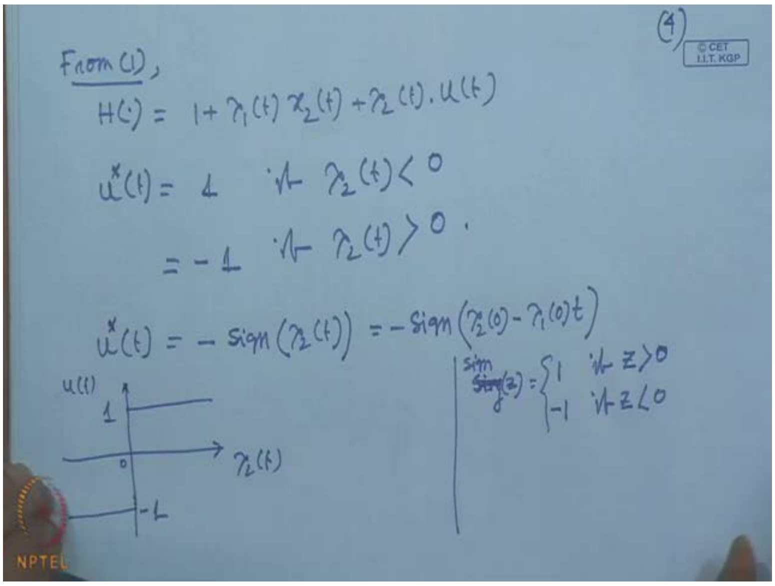

So, what we write in the optimal value of this u star of t is equal to 1, if lambda 2 of t is less than 0. That equal to minus 1 if lambda 2 of t is greater than 1. So, it is just switching depending upon the value of lambda 2 of t and lambda 2 of t expression is this the unfortunate part of this 1. We do not know what is lambda 1 of 0 and lambda 2 of 0.

So, this two equation we can write in mathematical form by using the sign function, u star of t is equal to minus sign of lambda 2 of t. Sign lambda of 2, you know the expression for lambda 2 or sign that lambda 2 expression is that one, lambda 1 lambda 2 of 0 minus lambda 1 of 0 into t. So, that I am writing lambda 2 0 minus lambda 1 0 into t. So, this what is the sign function, I just explain this the sign of z, mean sign of z is equal to it is 1, if z is greater than 0, if it is minus 1 if z is less than 0.

So, the sign function sign of z value will be 1 if z is greater than 1 value will be minus 1 if it is greater than 0 means negative. So, it is switching only, u is switching to plus 1 or minus 1. So, this quantity when it is positive then u I am switching to a minus 1 when this quantity is negativec1 plus I am switching to plus 1. So, the our control action is switching control law either u I am switching to plus 1 volt or minus 1 volt or 1 unit minus 1 unit. So, it is something like sign function is that 1 if you see. So, let us call this is lambda 2 of 0 this is 1, u of t this is minus 1 this is 0.

So, now look at this expression when this is positive, u is negative then we are applying the input to the system is minus 1 and correspondingly system dynamics will change.

When this is negative sign this negative sign of this negative quantity is the negative, negative, negative positive. Then we are applying positive control signal voltage or signal. So, it is again the system dynamics will change it. Now, see one by one what we are doing.

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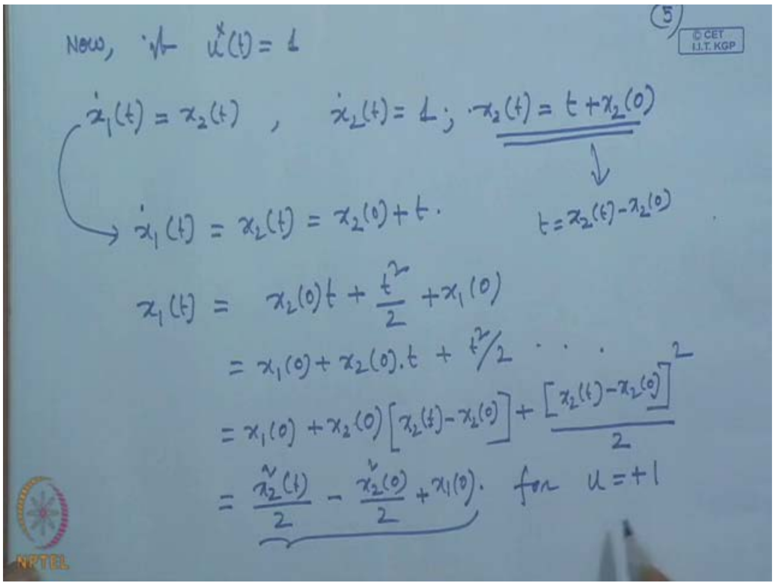

Now, if u star of t is equal to 1, when this quantity is negative. Then sign of negative quantity is minus sign that by definition of. So, if it is 1 let us see our corresponding dynamic equation of the system x 1 dot is equal to x 2 of t. Then x 2 dot of t is equal to 1 because x 2 dot is equal to u, u is equal to 1. Now, what is the solution of this one the solution of x 2 of t is equal to minus this is d t, t plus x 2 of 0. So, this is the our solution of x 2.

So, if you put in this expression, that x 1 of dot of t is equal to x 2 which you can write it x 2 0, x 2 0 plus t. Now, what is the solution of that one solution of this one, x 1 of t is equal to that x 2 of 0 into t agree because this lower limit is 0 and upper limit is t. So, it will be coming t. So, this one is t square by 2 upper limit is t and lower limit that lower limit term will not come from this integration part, so plus x 1 of 0.

So, this is our solution of that one, that x 1 you can write it x 1 of 0 plus x 2 of 0 into t plus t square by 2. So, that now this I can write it into, see what is x 1 of 0 I am writing this plus x 2 of 0. The initial condition of the state and t what you can write it from this expression you see t, t is what x t of t minus x 2 of 0 is equal to t is equal to this. So, put the value of t here that x t of x t of t minus x 2 of 0 this is that one put the value of this that is x 2 of t minus x 2 of 0 whole square divided by 2. Now, if you simplify that one because just you simplify this one, then this equation ultimately you see this is x 2 square x 2 and minus b whole square x 2 square by 2.

So, it will come x 2 square by 2 and plus x 2 square by 2 and here will x 2 square only. So, it will be minus x 2 square by 2 then twice x 2 x 0. So, that is 1 x 2 x 0 here. So, that will be plus x 2 x 0. So, let us write it that next is this term is there x 1 of 0 I can write it now see this one. So, minus this into this then minus x 2 square, but here is half. So, minus is coming then x 2 0 into x 2 t, here is twice x 2 0 into x t divided by 2. So, that will be cancelled out.

So, ultimately we will get it this expression after simplifying this 1. So, x 2 square by 2 minus x 2 square by 0 and x 1 of 0 for u is equal to plus 1 positive this let us call this equation what is the equation we have come across up to 5 last equation is up to 5. Let us call this equation is 6. Similarly, we will see when we will apply you see the dynamics of the ((Refer Time: 28:54)) the solution or response of the system is this. When you put the value of input is minus because control action is applied different minus 1.

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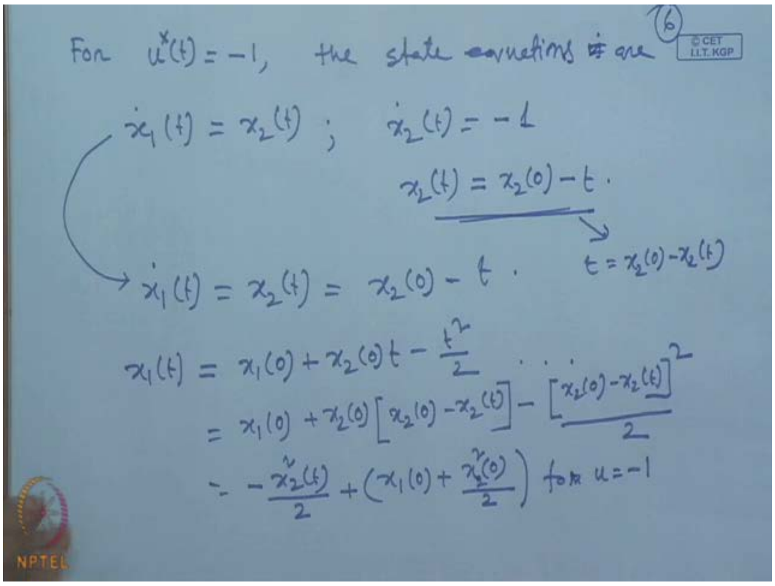

Then what is that solution or the trajectory of x 1 and x 2. We will see for u star of t is equal to negative, the state equation is x 1 of t is equal to x 2 of t. State equation are this 1 another is x 2 dot of t is equal to u and u is minus 1. Similarly, I can find out what is the value f x 2 of t, by certain simple different integrating both side and taking the limit t is equal to 0 to t is equal to t. So, this equal to here will be x 2 0 agree minus x 2 0 minus t. So, this is the solution of this one.

Then what is this x 1 t dot is equal to x 2 t and what is x 2 t is nothing but x of 0 minus t. Now, integrate this both side and with a lower and upper limit t is equal to 0 to t is equal to t. Then you will get x 1 of t is equal to that x 1 of 0, then it will come plus x 2 of 0 into t then t square by 2. Now, put the value of t what is t expression, t is equal to you can write it x 2 0 minus x 2 of t. That is our expression for t, x 2 0 minus x 2 of t. So, put this value here x 1 of 0 plus x 2 of 0 and the value of t is just now here, x 2 of 0 minus x 2 of t, so minus t square and x 2 of 0 minus x 2 of t whole square by 2.

So, after simplification this one as we have discussed earlier that this value will get it, that x 1 of t that is x 1 of t. You will get minus x 2 square of t by 2 plus x 1 of 0 plus x 2 square of 2 by 2. See the difference for u is equal to minus 1, see the difference this if I write it in the same manner of that 1 x 1 x 2 square this expression. You can say is nothing but a mm this one if I write re-write this 1 z 2 square of x 2 square of this by 2. Then this is a plus and I can re-write it is minus plus x 1 of 0 minus x 2 square of 0 by 2. If you see this one so this let us call this is equation number 6, this is equation number 6.

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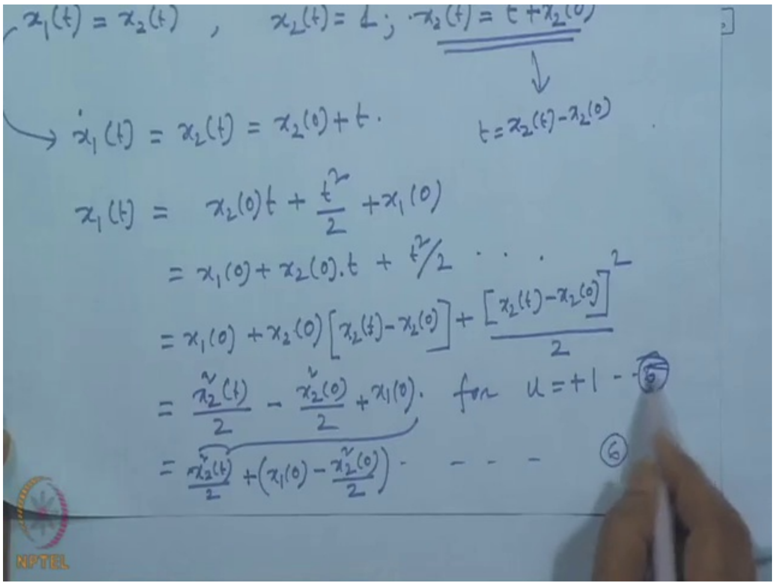

Now, look this expression and this expression this is equation number 7. So, when the control input u is applied the solution of or trajectory of x 1 and x 2 is how they differ.

This x 2 square by 2, so here is minus x 2 square by 2. Then this constant term which depends on the initial condition of the state is x 1 0 plus x 2 square of 2. So, this is are the two equation easily you can plot the trajectory of what is call equation 6 and 7. You can plot it in phase plane there are two states are there and we can plot it this.

Basically this equation 6 and 7, if you look the equation 6 and 7 is a parabola equation. Both represent parabola equation depending upon the initial condition the parabola will be different, but nature of this in phase plane is a parabola. So, let us see, if you plot it this, what is the nature of this curve we will get it.

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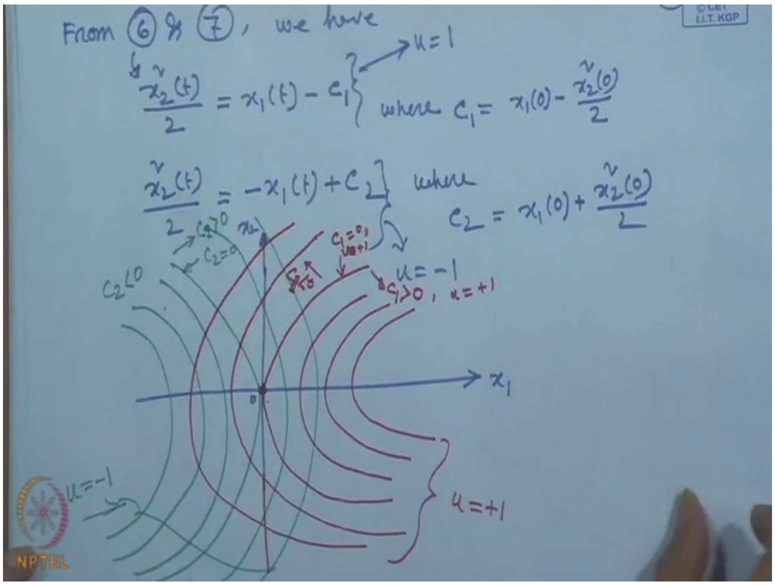

So, from equation 6 and 7 we have 1 you see x 2 square of t divided by 2, 6 let us call this one. I am writing this x 2 square of this is equal to x 1 of t minus c 1. Where c 1 is equal to, see here this I am keeping it is from equation this it is this I am taking it this whole thing I am considering as c. So, c 1 is equal to x 1 0 minus x 2 square 0 by 2. So, that is from equation 6 I am writing. Similarly, from equation 7 I can write x 2 square of t by 2 is equal to minus x 1 plus c 2. Where c 2 is equal to, see equation 7 this equation 7 if I take this is that side x square by t x square t of 2 is equal to minus x 1. So, and plus this is you write it c 2 and this is you write it c 1, so this where c 2 is equal to x 1 of 0 plus x 2 square of 0 by 2. So, this is the 2 parabola equation, the y square is equal to 4 a x plus some constant term. When constant term is there that what is this shifted to the right or left of this one of the parabola.

So, let us see if you plot this one and this is for this expression for u is equal to plus 1 and this is u is equal to minus 1. This corresponding u is equal to minus 1 and this equation corresponds to is equal to plus 1. So, let us plot it in phase plane. So, there are 2 states are there. So, x 1 and x 2, I am showing here x 2 and this is 0. Let us call c 1 is equal to 0 that c 1 is 0, then what is this one, this is the equation of parabola when x 1 is 0 x 2 is also 0. So, it is, when x 1 is some positive quantity then x 2 is plus minus. So, this nature of the curve is like this way agree and this is correspondence to c 1 is equal to 0 now tell me when c 1 is greater than 0 when c is greater than 0 what will be the its nature of this curve.

When c 2 is c 1 is greater than 0 what will be this value. So, this value now c 1 is greater than 0 if it is there this curve will be like this way. So, I am just writing in this side c 1 is greater than 0, but all this cases u is c 1 greater u is plus 1. All this cases are u is I can write it u is plus 1 here also u is plus 1, u is equal to plus 1. See now c 1 is less than 0, less than 0 means if you put it that side less than 0 is sum below of x that this will be looks like this way. So, in this side c 1 is equal to less than 0 and this is c 1 is 0 is this 1. So, this is corresponding to this all this thing corresponding to u is equal to plus 1.

Now, let us see for this one when u is control input is applied, when u is equal to minus 1 what is this we will get it. So, again consider c 2 is 0 if c 2 is 0, then this is you see y is equal to minus 4 a x. When x is equal to x 1 is equal to c 2 is 0, when x 1 is equal to let us call 0 x 2 is 0. So, it will start from this one, when x 1 is minus agree minus - minus plus. So, x 2 is plus minus. So, this will be a curve is like this way. So, this is c 2 is equal to 0.

When c 2 is positive, tell me which side i have to draw it that 1 c 2 is positive, but x 1 is negative c 2 is positive x 1 is negative. So, this will be a because at same value of x the x 2 value will be more. So, c 2 is positive, then our curve will be like this way. So, this side c 2 is positive and now c 2 is negative and x 2 also negative. So, that value will be below the c 2 curve. So, this is c 2 less than 0 and all these curve this curve u is equal to minus 1. This curve is u is equal to minus 1. Now, see our initial condition now question is coming look at this expression our initial condition can be anywhere in the four quadrant. Let us call our initial condition is here I am just showing our initial condition is here, x 0 is here then what is the dynamic equation is there.

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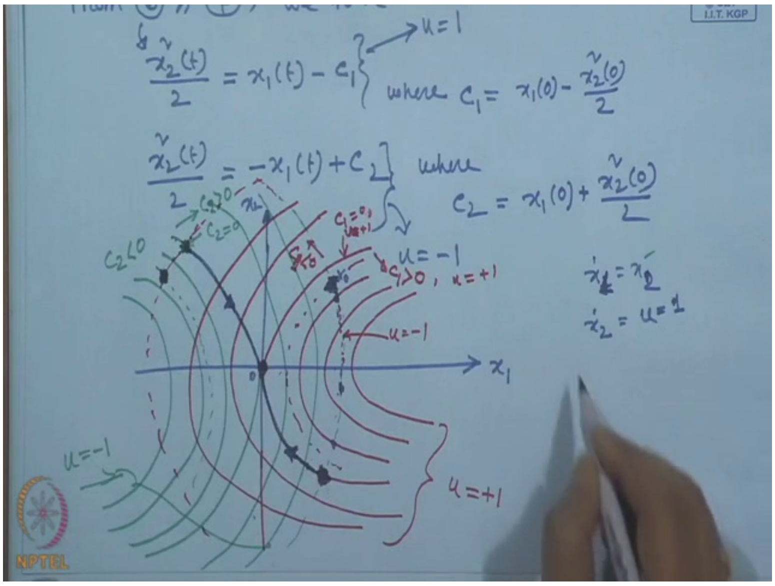

X 2 dot is equal to x 1 and x 1 dot is equal to x 2 and x 2 dot is equal to u and u value is your maybe plus 1 or minus 1. So, u 1 is equal to either plus 1 or minus 1. Let us call this is that one. So, this point can lie either this green curves parabola or red parabolas, anywhere you can write it. So, let us call it is here. So, what is the value of x 2 positive x 2 value is see here x 2 value is positive. If x 2 value is positive, then x 1 dot is positive then which direction the velocity is positive.

Then which direction x 1 will move, x 1 will increase because this (Refer Time: 44:08) is positive, x 1 is increased that means it will go like this way. As I told you it can be in the red colour parabola or green coloured parabola. So, if it is red colour parabola is lies then it should increase. So, it is going away from the origin, but if it is in the what is call that green parabola. So, x dot x 1 dot is positive. So, x should increase.

So, the curve x is increasing along this curve see this one. So, x 1 increasing it is going like this way. So, it is approaching to because x 1 is increasing agree, but if it is in this curve let us call it is a red curve. So, x 1 is increases. So, it will move like this way it will move like this way. So, it is going away from the origin, suppose it is on the green curve green parabola. So, x 1 should increase. So, it is going in this way agreed and it is approaching to the origin. So, if it is approaching if it is like this way, let us call it is coming here. So, still x 2 is positive the x 1 should increase.

When it comes here when it comes here x 2 is negative means what x 1 is negative. That means x 1 should decrease and see it is decreasing it is decreasing and decreasing in this way it decreasing. So, when it hits here, then what is this one, this is the trajectory for when u is equal to 1 and this is your it is going along the green trajectory dotted green trajectory and u is equal to 1.

When it hits here that u is still 1, but it hits that switching surface this 1 c is equal to 0. When it is 0 still this u value is 1 this u value is 1. This u value is what minus this when you are coming this are the u is equal to minus 1 that when it hits here u is equal to plus 1. So, it switch here agree and you see x 2 is negative that x 1 is decreasing and it is ultimately it will come to origin here. So, if you see this one our switching surface is at this point, that x 2 once again I am telling you.

Let us call our initial condition is this 1 x of 0 and x of 0 means x 2 is positive when x 2 is positive see x 2 is free way x 1 should increase. So, it may fall in this parabola or it may this point may lie on this parabola. I am telling you suppose it is lying on this parabola. So, x 1 should increase. So, it will go this way it is going away from the origin because we have to drive the state to the towards the origin. Now if it is on the green parabola that means equation number 7, u is minus 1 and this trajectory is u is equal to minus 1. So, it will drive this one in this direction because x is x 1 dot is positive since x 2 is positive.

So, it is going and x 1 is increasing in this way. When it is at this at this 1 x 2 is x 2 value is negative, negative means x 2 x 1 value should decrease you see it is decreasing this one value from here to here x 1 is decreasing. When it is here u value will change previously u was minus 1. Now, you see this it hits here u is equal to your 1, it hits here also you see no doubt, but if it hits it cannot go it cannot reach to the origin it is going like this way.

So, it if it hits here, then along this path u is equal to minus 1 along this path x 2 is always negative means x 1 is decreasing and approaching to 0 this. So, our trajectory of this switching trajectory is that one when c 1 is equal to 0. This is the our trajectory switching trajectory. Now, what about this one, in this case suppose our point is our point is here. So, our initial point is here. So, what is the value of our state x 2 is positive x 2 is positive. So, what is this x 1 should increase. So, once again it may fall what is call in the, this is the dotted line at it may be a fall with this dotted line. That means this is corresponding to the equations 6 and this dotted line equation 7, suppose it falls on the equation of that red line.

So, x 2 now what is x 2, x 2 is positive x 1 should increase, means it should go towards this because minus 1, suppose minus x 1 is minus, then it should go towards direction x 1. So, t is going away from the origin, suppose it is coming here. Now, suppose this agreed. So, our at this point x 2 is positive, now whether it will move along the red parabola or along the green parabola. That is our question now suppose it is falling like this way in the red parabola that means u is equal to our plus 1. The x value is increasing.

So, when it hits here when it hits here then u value is your negative, but at this moment u value was positive, when it hits here the u value that mean switching occurs here. When c 2 is this is this c 2 is 0. So, our trajectory switching curve will be that one. Now, one make why not it follow with this green. Suppose, it is in the green curve, that mean x 2 is positive, x 1 must increase this one and it is increasing. So, it will go like this way and it is going away from the origin. So, it has to go in this direction. So, u is equal to plus 1 and then is switch to u is equal to minus 1.

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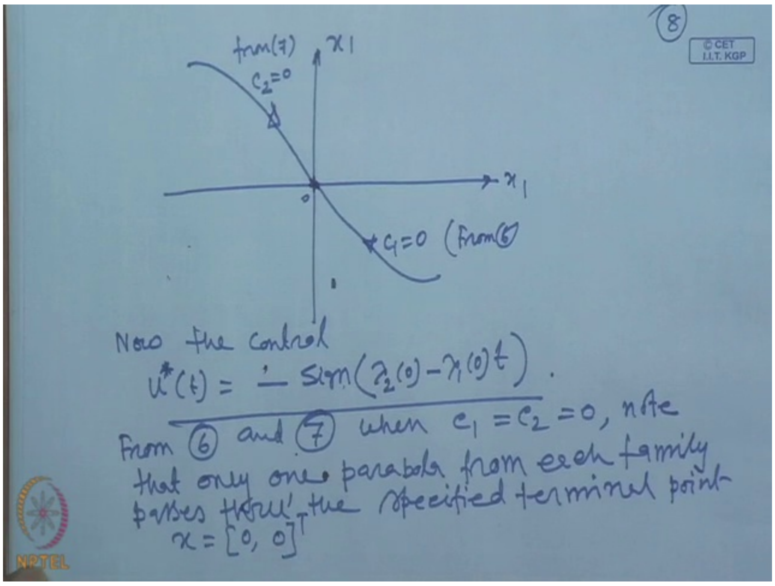

So, our switching curve in this ultimately it is the that is our switching curve. So, this curve is c 1 is equal to 0 from equation 6 and this is c 2 is equal to 0 from equation 7. So, once it is hit here it will follow along the switching curve. Follow this switching curve and ultimately it will reach to the origin. So, we can write our control law, you see now the control law u star of t is equal to minus sign of lambda 2 and lambda 2 is what lambda 2 of 0 minus lambda 1 of 0 into t.

So, this is our control law. So, from 6 and 7 when c 1 is equal to c 2 is equal to 0. Note that only 1 parabola from each family passes through the specified terminal from the specified terminal point x is equal to 0 and 0 transpose. That means if you see this one from 6 and 7, if you this initial condition is there anywhere in first quadrant. Then it will move along the u is equal to first u is equal to minus 1 switch is there, you have to make the switch is equal to minus 1. Then it will go continue along this path as soon as this hit the switching curve agreed switching curve. That means u is equal to plus 1 is said it will follow along this curve and at in reach to the origin.

If it is here again it will switch to a minus 1 plus first and then it will switch to a plus 1 control input. Anywhere here, it will first switch to u is equal to plus 1 and then switch to u is equal to minus 1. Anywhere here suppose here we are calling here, here suppose here it is just above the switching line above the switching line. If it is initial point above the switching curve then it will switch to u is equal to minus 1, then it will switch to u is equal to plus 1. If it is below the switching line below the switching line, first it will switch to u is equal to 1, then it will switch to u is equal to minus 1. So, this is the optimal time it will require along this trajectory of this one.

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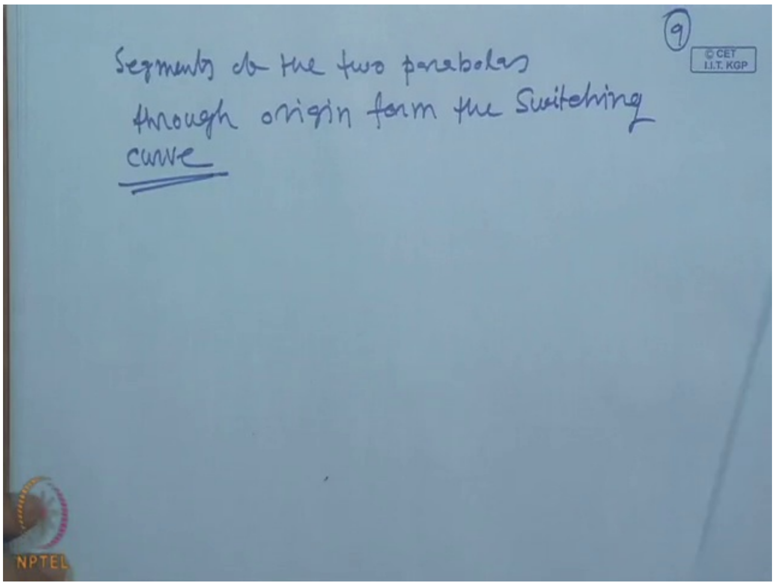

So, segments of the two parabola through origin form the switching curve. So, this one segment of two parabolas, two familiar parabolas that segment of this one is and another parabola. This is a set of parabolas family of parabolas, which is passing through the origin and that intersect and that will form the switching curve. So, equation of the switching curve we will discuss later. How to take the decision when that u plus 1 to minus 1, how to take we will make some logic behind this, we will explain this next class.