Optimal Control

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Lecture - 48
Minimum-Time Control of a Linear Time Invariant System

So, last class we have discussed the, what is called the dynamic programming of a discrete time system using principle of optimality. We have described the basic principle behind this and also we have taken one example, which we could not complete during that time, during last class. So, let us recap the example and complete this example now.

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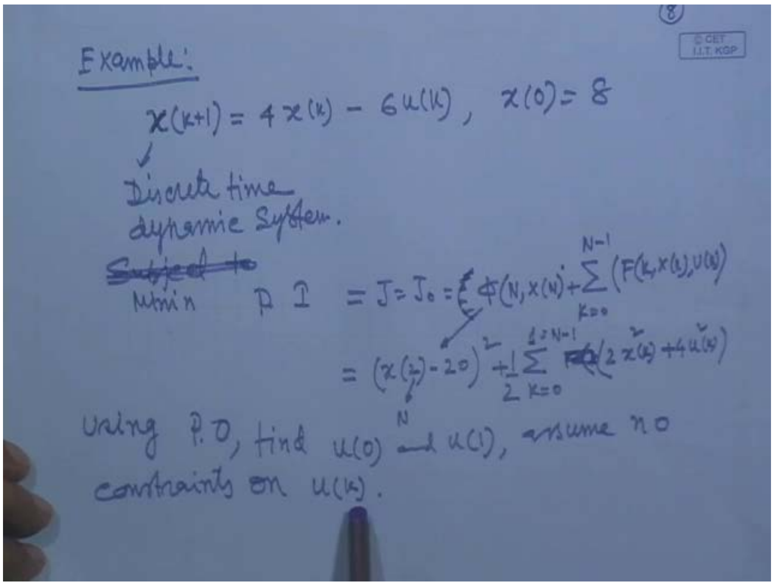

So, our problem is given the discrete time system it is a first order difference equation, initial state x of 0, is equal to 8 is given. Our problem is minimize this performance index, this is the standard performance index and it is in quadratic form and this is the terminal cost, is given. So, our problem is using the principle of optimality, find control sequences u 0, u 1 assuming there is no constraint on the control signal u of k. So, we will start with, what is called principle, optimality principle we will apply. That means from we will start from the backward pass, we start with the backward pass.

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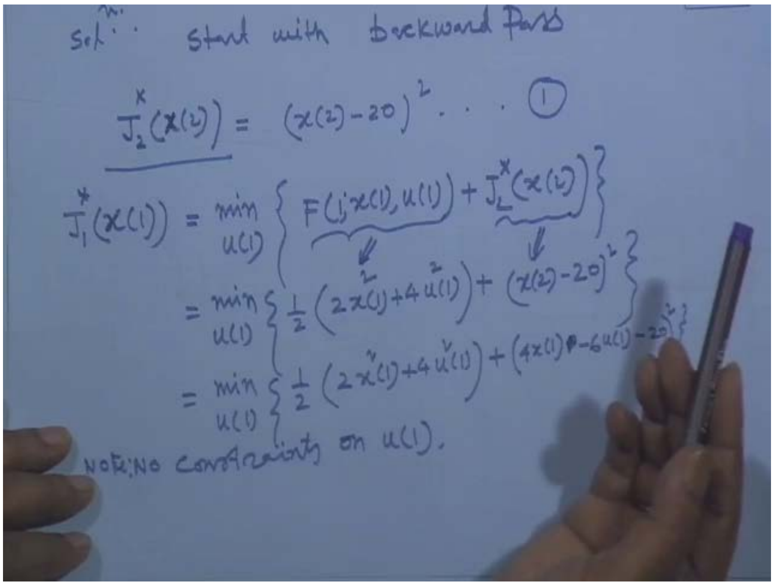

So, if you recollect this one that our terminal cost is that one so, from x 2 to we will move to x of 1 state and for that one, what should be our control sequence u of 1, you have to find out. So, j of 1 performance cost when you move from j 2 to j 1, what is the performance cost, that minimize this performance index. And that is we are writing from this expression, if you see N is equal to j is equal to 1, that x of phi of 1, phi of N, x of N that is the terminal cost to know this one. Now N is equal to that is 1, k is 0 to 1. So, k is equal to 1 then this means 2 x of 1 whole square 4, u 1 of whole square, that means we are moving from the terminal cost j 2 to j 1.

So, that is why we have written it into this form so, what is that from j 2, this is the terminal cost at j 2. Now we are moving with the j 1, j 2 plus j 1, total cost. So, this is the expression now, this x 2 we can write in terms of x 1, from the dynamic equation, dynamic descriptive equation given to this. So, that x 2 is now replaced by this expression and this remaining is same now, you see here that minimization of this performance index depends on the x 1 again. Because, they have to minimize this one selecting the u of 1 and we have also seen there is no constant impose on control input of u of 1.

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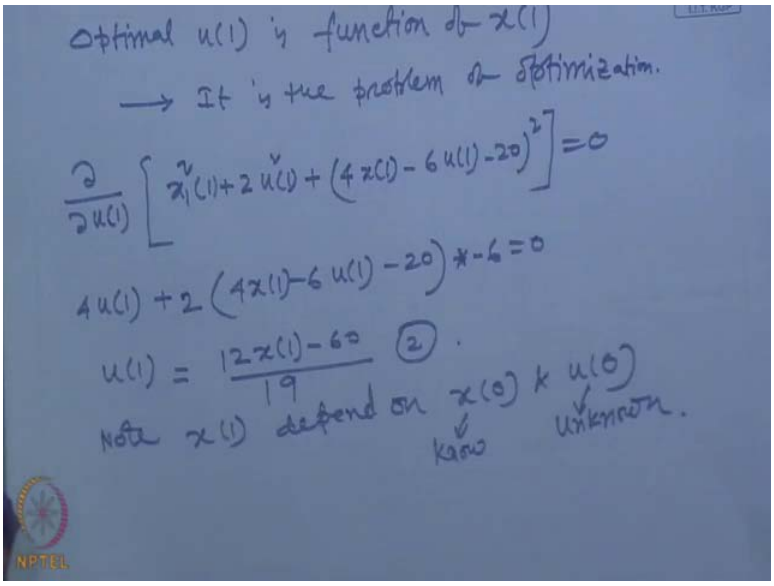

So, that means so, if we see this one then, our optimum u is the function of x 1, from the previous slide you see is a function of x 1 only. So, it is not a, nothing but a problem of optimization problems. So, partial differentiation of u of 1 with respect to this function, cost function with respective to u of 1. So, if you do this one, the u is involved in u of 1 involved here and in this expression. So, this will come with this expression ultimately you will got, u 1 in terms of x 1 is this. So, x 1 again you see I cannot find out u of 1, until and less I know x 1, but I know x 0. So, x of 1 can be expressed in terms of x 0 and u 0, where u 0 is unknown, but x 0 is known to us, known to us.

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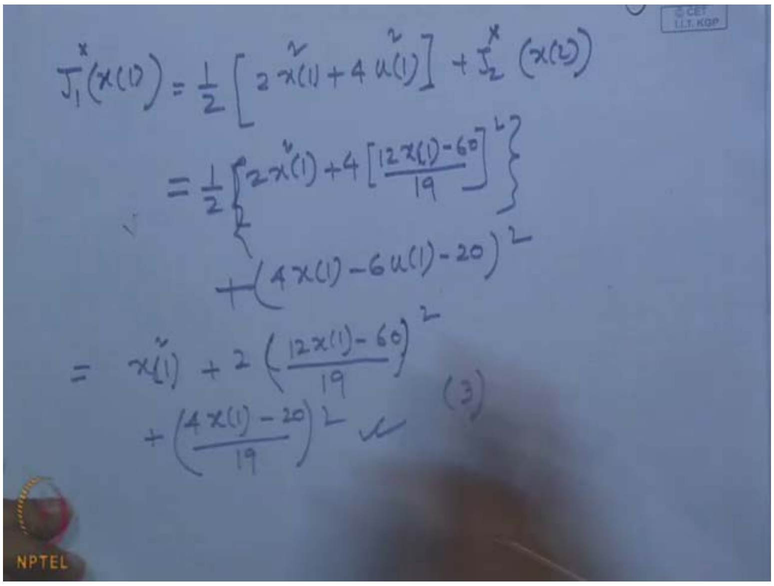

So, if you use this expression into that j 1 expression that is, this and you know x 1, this and u 2 in terms of x 1, I have written this one. And j 2 expression is nothing but a x 2 minus 20 whole square an x 2 is replaced by that one. So, now it is that terminal cost required from the, from j 2 to j 1, what is the cost is involved with this one. Now this, if you can simply write ultimately it will bulge down to this expressions, this after simplification.

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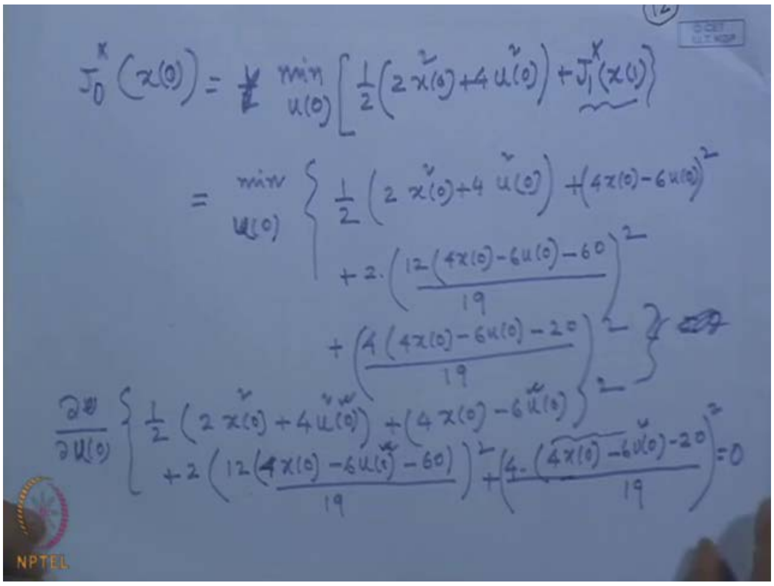

So, next we will go from j 1 to j 0, that means what is the cost for moving the backward pass from j 2 to j 1, total cost. So it is not a j 1, this is j and it is a function of x 0 so, if you see this one, so, this, now you see I will write this expression of j 1 here. So, u of 0 then half twice x square 0 plus 4 u 0 square plus j 1 and j 1, just now we got it this expression. And here we will express x 1 in terms of x 0 and u 0, wherever x 1 is z, we replace by x 0 and in terms of x 0 and u 0, from the dynamic equation of discrete terms systems. So, this expression if you see that first is, first expression is x 1 square so, x 1 square is, nothing but a 4 x 0 minus 6 u 0, whole square.

Because, if you see this our basic expression for dynamic equation, when that k is equal to 0 this x of 1 is equal to 4 x 0 minus 6 u 0, what I have written it here in place of x 1 square, I have written that one. Then, 2 plus see this one that expression 2 into 12 into x 1 so, I will replace x 1 by x 0, 2 into 12 into x 1 is 4 x 0 minus 6 u 0, this is 4 x 1 minus 60, divided by, divided by 19, that whole square. Then, next term is like this way 4, if you see this one, 4 x 1 minus 20, divided by 19 whole square 4 x 1 is there 4 x 0 minus 6 u 0, divided by this 4 minus 20, minus 20 divided by 19 whole square.

So, this and that equal to 0, this one so, we have to write here, this is the, this is 0 we will write it there. So, del u, del j divided by del u of 0, that one, this whole expression that half twice x 0 square plus 4 u square 0 bracket close, plus 4 x of 0 minus 6 u of 0, whole square. Because, this is the performance index, this performance index you have to use, you have to minimize using the proper choice of u 0. So, you have to differentiate gradient of this with affect to u 0, I am finding out this is capital u 0, you can write if it is small type, we have considered small.

Because, this is scalar input u of 0 small so, this plus twice into 12, 4, 12 into 4 x of 0 minus 6 u of 0 minus 60 divided by 19 bracket whole thing square plus 4 into 4 x 0, this same expression. I am writing minus 6 u 0 minus 20 divided by 19, whole square this 4 into x 1, 4 into x 1 minus 20. So, 4 into x 1, 4 into x 1 this multiplied by this whole thing is you 4 x, 4 into this whole thing is x 1 minus 20 divided by 19, whole square is equal to 0. So, if you simplify this one, differentiate this with respect to u term is involved here, here, here in all term, except the first term of this one.

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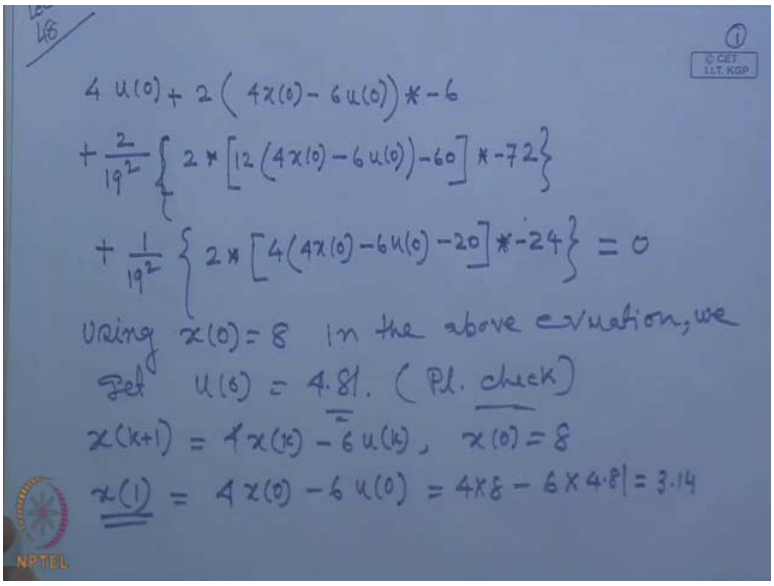

So, if you do this one then we will get it that 4 u 0 just to differentiate that, this is u square 2, 2 2 cancel 4 u 0. So, just check it this one, 2 into 4 x 0 minus 6 u 0 whole star minus 6 from the first term of differentiation, this one will get twice this and differentiation of that one, minus 6. So, twice into this, this then plus twice 19 square then bracket 2 star bracket 12, 4 into x 0 minus 6 u 0 bracket close minus 60. Then bracket close into minus 72 so, that you use this one so, that plus this term differentiation, this we have done it, this we have done it. Now, this one that one 19 square then, 2 star bracket 4 into 4 x 0 minus 6 u of 0 minus 20 bracket close inch star minus 24.

This is simple what is called differentiation of that j 0 with respect to u of 0 is equal to 0. If you solve this one because, I can write it now u 0 in terms of x 0 and then, since I know the initial state of x 0, I can easily calculate u of 0. So, using x of 0 is equal to 8 in the above equation, we get u of 0 is equal to 4.81, please check it this calculation. So, we got it this one, we know u of 0 and we know x of 0, then immediately you can find out x of 1, by using what is call the dynamic x, dynamic state equation of this system, that means ((Refer Time: 14:13)) system.

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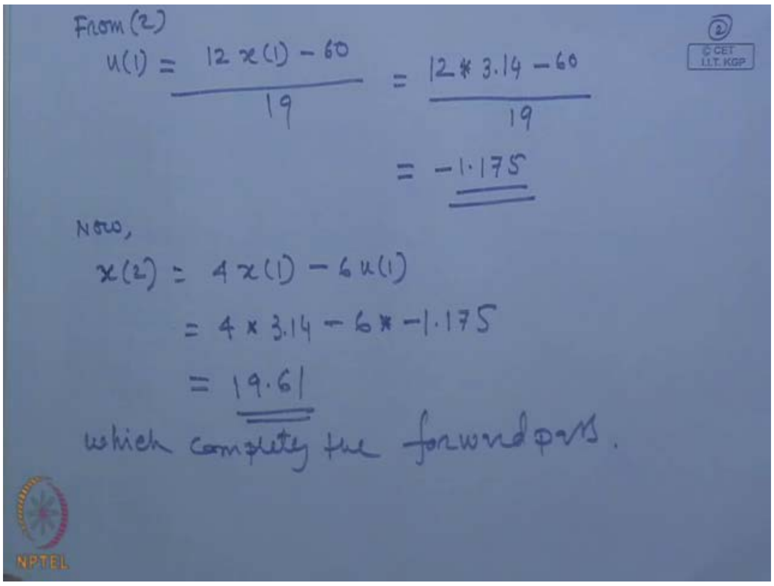

So, basic equation of this one if you recollect this is x is equal to j is equal to 4 of x k minus 6 u of k and x of 0 is equal to 8. So, immediately I can find out x of 1, k is equal to 0 if you put x of 1 is equal to 4 of x 0 minus 6 of u 0 and you know the value of x 0 u 0. So, 4 into 8 minus 6 into 4.81, whose value will come 3.14 so, in its state value you know it, then you can find out, next you can find out x of. So, u of 1 you can find out, look this expression u of 1we calculated the last class if you see the u of 1, we have calculated in the last class.

That u of 1 expression is nothing but a 12 of x of 1 minus 60 divided by 19. I will show you this one, that last class from equation 2, you can write it that from 2, you can write it from 2, we can write it we know x of 1. So, this is nothing but a 12 x of 1, you got it just now, you got it 3.14 minus 60 divided by 19. So, that value will come your minus 1.175, check this value so, once we know u 1, you know x 1 you may need to find out x of 2, x of 2 is equal to 4 of x 1 minus 6 of u of 1 is equal of 4 of x 1, I know that x 1 value I got it 3.14 minus 6 into u 1 value is minus 1.175.

That will come near about 19.61 so, and from starting with back pass we have found out the trajectory of x 0, x 1, x 2 and simultaneously we are getting the control, optimal control sequence of u 0, u 1. So, which completes the forward pass so, first we have done backward pass, then once we complete the backward pass, that means j 2 because terminal cost is equal to j 2, you got it. You know the terminal cost from j 2 to j 1 you found out, found out by selecting the proper, by selecting the optimal choice of u of 1.

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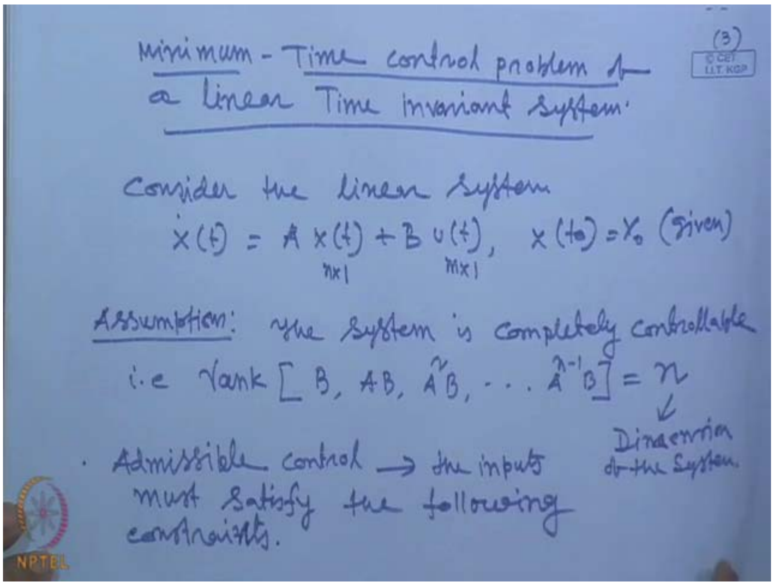

Then, from j 1 to j 0 we found out by proper selection of, means in other word it is an optimum selection of u of 0. Once you completed the backwards pass we have to do the forward pass which completes like this way, finding out x 1, x 2 then u 1 then x 2 in this way we found out. So, this is the complete solution of the problem, which we have discussed last class.

So, now we will consider new topics, which is called time optimal minimum, minimum time control problem of a linear time invariant system, linear time invariant system. By the name itself that, this statement of the problem is like that given the, system dynamic equation our job is to find out the control sequence, in such a way so that, we can reach from our initial state to final state within a minimum time. So, let us take this one consider, consider the linear time invariant systems, linear system, time invariant system described by x dot of t is equal to A of x t plus B u of t and our initial state is given x t 0 is given, you can given.

So, let us assume the input, number of inputs is m and number of states is n and immediately you can find out the dimension of A matrix, system matrix and input matrix dimension, immediately you can find out. So, our job is to, that to transfer the state of x 0 to final state x at t is equal to t f is x f transfer this, with the sequence of control with the, with the optimal control in a minimum time, that is our. Before that we may do an assumption, first assumption the system must be, the system must be, the system is completely controllable as at, the rank of B, AB, A square B dot dot A n minus 1 B is equal to n. Where n is the dimension of the system, dimension of the system or state vector dimension.

So, this is the first assumption, this is the system must be completely controllable, the second assumption is made admissible control, means the input must be satisfy the following constraints, in the sense input is constant. So far, we have discussed we have not made any constant on the input, when you did the optimal problem solution by using calculus of variation, we have not made any constraint on the inputs.

As well as on the states, here you just consider there is a constraint on the input, if you see the physical point of view that control effort or control signal we cannot apply, whatever is coming from the control output, that we cannot directly apply to the systems. Because, all physical system has a limitation of control signal input so, you have to restrict it so, you have to constrain the control input signal, which is coming from the control, we have to restrict it in general. So, in general it is saturation element is used it here, here we are telling admissible control, the control inputs, the inputs must satisfy the following constraints.

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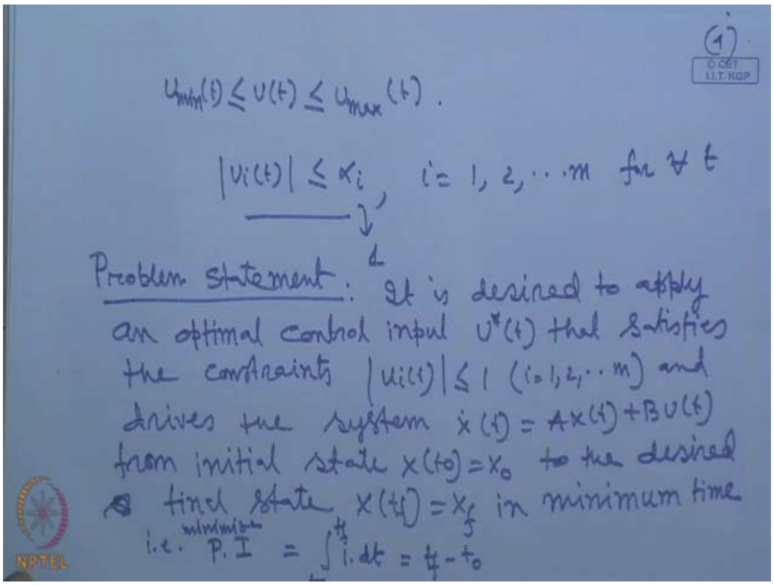

So, what is the constraints are imposed on the input controller, that u of t, what is the control input is coming from the controller or from the design of the controller, that must be restricted with a u min of k or less than equal to u max of t. Or in other words you can say, if the u min value lower limit and upper limit are same, suppose this u max is 5 volt and u minimum is also 5 volt. Then in that sense I can write it u of t, then each component of u mod, must be less that equal to alpha i magnitude, where i is equal to 1, 2 dot dot m for all t. Suppose we are considering input magnitude in lower, upper bound is 10 volt and lower bound is minus 10 volt.

Then, I can write it absolute value of this is equal to less than alpha I, alpha i is 10 volt so, different control inputs that upper bound and lower upper bound agree and lower bound, if are same that alpha i may be different. Suppose control one into upper and lower bound, if both are same in negative sign with the 10, then it will be a 10 volt u to maybe 5 volt, all these things.

So, this is the thing say, alpha i you have restricted to 1, that upper value of the control input is positive 1 volt and lower bound of the control input for i is equal to u i, i is equal to 1 is equal to let us call minus 1 volt. For all cases it is plus minus 1 volt so, problem statement though we have explained problem statement, now we are writing this statement of the problem, problem statement. So, it is desired to apply an optimal control input u star of t that satisfies the constraints u of i of t let less than equal to 1. You can say, i is equal 1, 2 dot dot m, since is the number of, for all cases the limited restriction we made it same.

And drives the system x dot is equal to A x t plus B u of t, that the system state system from initial condition or initial state from initial state x t 0 is equal to x of 0 to the desired state, desired final state x t f is equal to x f, in minimum time. In other words the, our optimization, performing index optimization problem is taken optimization problem is described with performing index P i. In otherwise minimize the, per minimize the performing index this t 0 to t f, 1 into d t is equal to t f minus t 0 if t 0 is 0, minimize that one t f, that is our problem. Now you see this one, that this problem our integrant of the performing index is 1, is not a function of u.

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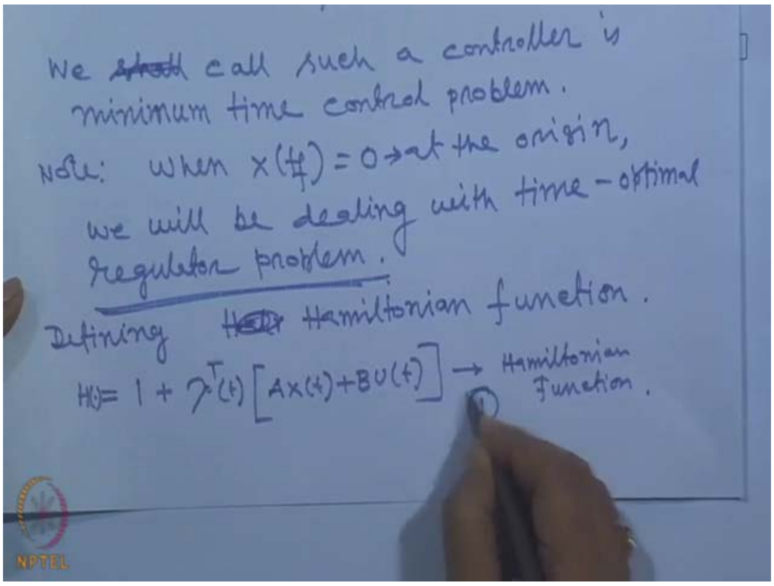

So, in other words you can say our problem is like this way, our initial state x of t 0 x of t 0 is equal to x 0. And we have to find out the optimal path, such that the performance index is minimized in the sense the state x of t 0 x 0, will transfer with the optimal control signal, through a desired state x t f is equal to x f in a minimum time. Let us call this is the optimal path in a minimum time, that is our problem so, we shall call, we call such a controller is the, is minimum time control, minimum time control problem. If the initial, if the final state is terminating is at the, at origin then this problem is called the optimal time regular regulated problem.

If note, when x t f is equal to 0 means at the, this indicate, at the origin of the state, origin comma we will be dealing with time optimal regulator problem or regulator systems. Now how to solve this one, if we repeat our earlier problem, that in opposition problems we have given the description of the systems, then we have given a performance index of the corresponding performance index. Our problem was, to solve the u to find out the optimal control law u of t such that, this performance index is minimized. So, we use the technique calculus of variation to solve this problem, please recall all the steps what we have considered earlier.

That means first we have to consider, that find a, what is called a Hamiltonian function but, in that problem we have not considered any constraint that u was unconstraint, state also is unconstraint. But here now, our problem is coming constraint on u that how to solve this one. Basically, the procedure is similar to that one only, just one step we have to see very carefully, which is different from other earlier procedure. So, first we will define, defining the Hamiltonian function, Hamiltonian function. So, what is the emerging function, if you recollect that our performing index is, our performing index is there.

So, we found the Hamiltonian function by taking the integrant part of the performing index, here if you see the integrant part of the performing index is 1. So, our Hamiltonian function is H is equal to H function of 1, that integrant part of the performing index 1 plus the costate vector lambda, transpose this and that is, that is, next is correspondent of problem A of x t plus B of u t. And in general it is the dynamic equation, right hand side of the dynamic x dot is equal to A x plus B u, costate vector transpose multiplied by that right hand side of the dynamic equation of the systems this.

This is our Hamiltonian function, H function, Hamiltonian function. And this part, first part in our case is the integrant part of the performing index, in this case since it is time optimum problem, the integrant part of this performance index is 1. So, and other process, we will follow the procedure that once we follow the Hamiltonian function so, let us call this is the equation number 1.

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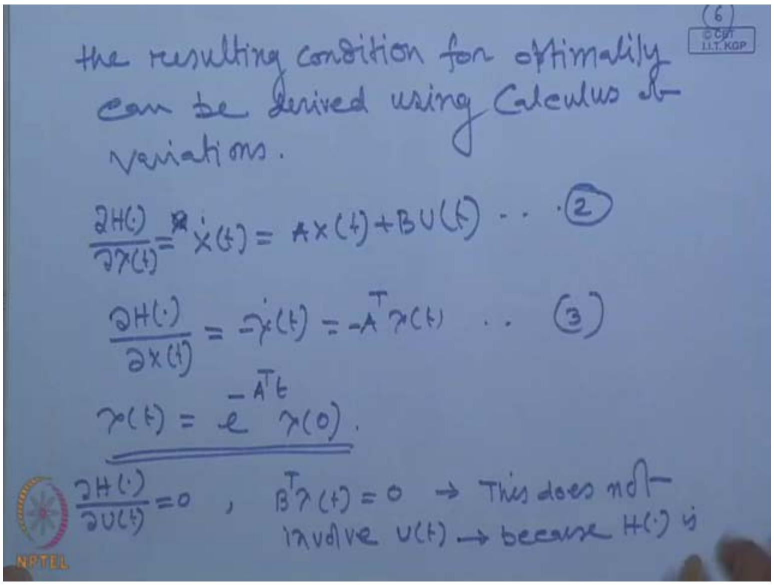

So, the resulting now I can write, the resulting condition for optimality can be derived, similar to earlier case can be derived, using calculus of variations. So, what is our calculation here, if you see x dot is equal to del H, del H del lambda of t is equal to our x dot state vector, which is nothing but A x of t plus B u of t. That you have derived earlier, if you see this one so, let us call this is the equation number 2. Then, costate vector you can write, this is a straight equation costate vector, we can write it del H del x of t is equal to lambda, minus lambda dot of t is equal to lambda dot of t is equal to.

Now if you did, you see that del H through differentiation this, this is nothing but a, A transpose lambda of t minus. So, this is the costate equation so, let us call this is equation 3, now if you solve this one, if you solve with this, this, this one then you can find out this. So, lambda of t is equal to from this one equal to, e to the power of minus A t of t lambda of 0. So, now this is the costate vector equation solution in that, that one now we can write it that another is delta H. Now look at this one, this is more important del H, del u of t is equal to 0, then what is the write it from this you see del H, del u.

So, it is nothing but a B transpose your lambda so, that equal to B transpose of lambda is equal to 0, B transpose lambda is equal to 0. So, now you see this, does not contain u so, you cannot find out that one, but why it is does not contain u because, our in the, what is called in the Hamiltonian function, Hamiltonian function is a linear function in u. That is no contrary function of u so, this we cannot handle like this way so, this does not you can write, this does not involve u of t.

Because, H is a linear function of u because, H is, H is the, this, the, you see this one when you have differentiated this one, that will be a plus not this one, please change this one, that lambda. This is the costate vector equation, now I am differentiating with a respect to x so, that will be transpose of this. So, the solution of this one is, that lambda is the lambda t solution is this one. This one is simple state equation solution is this one.

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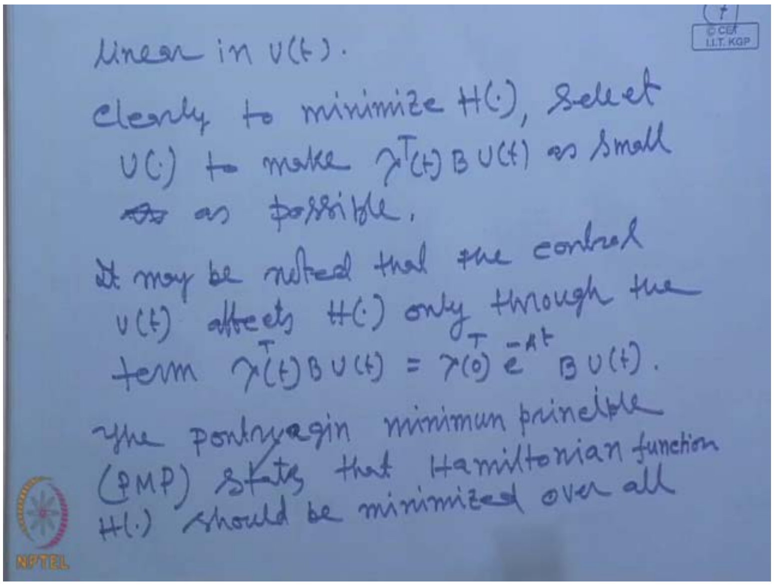

So, because H of t is linear in u of t so, it clearly indicates, clearly to minimize, to minimize u of t, H of t, clearly minimize H of t. Now you see, to clearly minimize u of t you have to select u of t, clearly minimize H of t you have to select u of t, such that the function is minimum. So, clearly to minimize H of t, H of that function, Hamiltonian function, select u of t, u of t to make lambda transpose t B u of t, that part. Lambda transpose B u of t, lambda transpose B u of t, select you have to make this as small as possible so, that is our requirement.

So, this is call Pontryagin minimum principle. So, you have to choose, we have to minimize H because we cannot minimize H, this one by selecting u because, there is no function of this, there is, there is no u terms involved. Because of, that layer what is called Hamiltonian function is a linear function, not a quadratic function or a non-linear function. So, that why you will not be able to do so, what you need to do, we have to see the minimum, Pontryagin minimum principle here.

So, in order to minimize this select u so, that this should be as small as possible so, naturally so, this, it may be noted that, it may be noted that the control signal u of t affects H only through, only through, tell me this expression. If you see this one, that the control u affects H, control u affects H only through this term, lambda transpose B u t.

So, only through the term lambda, lambda of t transpose B u of t which equal to lambda, just now we have found out the solution of lambda of this one, you see lambda of this transpose of this, lambda 0 transpose if you take transpose both side lambda 0 transpose. So, this is equal to lambda 0 transpose, e to the power of minus A t B, B u of t so, the important is the Pontryagin minimum principle. That PMP states that, that Hamiltonian function, Hamiltonian function, H of dot should be minimized, should be minimized by, should be minimized over all, over all possible control input.

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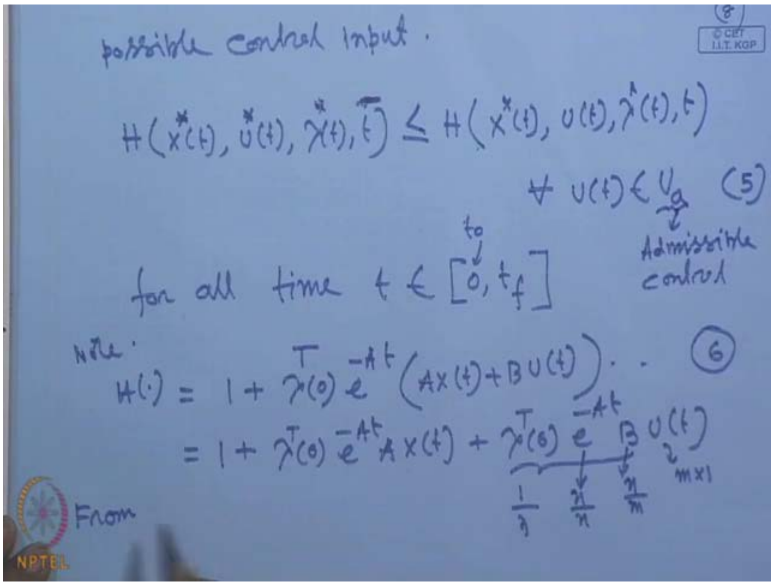

Overall all possible control input so, what does it mean, this permits including the constant you see, H which is the function of, if you see x of t, then u of t, then lambda of t and t. So, you find out the optimal control of u star, the star indicates the optimal what is called quantities, optimal quantities, this is not star. So, less than equal to H x star of t, then u of t, that lambda star of t, then t for all permissible input, that means u of t belongs to u of a, a stand for all permissible, admissible, permissible, no admissible control, a for admissible control, u a this indicates, u a admissible control.

That means that the control input is restricted to this, constraint to this one so, this for all time belongs to that 0 to t f, if t 0 this is our t 0, if t 0 is 0,then 0 to t f and if not 0 then t 0 to t f. So, what is our equation we have given is 3, let us call lambda is equal to this equal to our equation number 4, the costate equation, the costate vector equation is 4. Then, we have written that what is call the Hamiltonian function, we have defined Hamiltonian function is here.

So, this let us call this equation is equation number 5 so, now note what is the corresponding Hamiltonian function, H of this is equal to 1 plus lambda transpose, lambda transpose, we have just written in terms of initial value lambda 0 transpose. Then e to the power of A t, then A of x t plus B u of t, let u call this is equation number 6. Now see this one, lambda what is our Hamiltonian matrix is considered, that is, first is Hamiltonian matrix is considered, that in place of lambda t, I am writing it that quantity. And then, remaining is as it is, remaining is as it is so, this now from this equation you see, when this function will be minimum, that Hamiltonian function.

So, this and now writing into two parts, this into lambda transpose not 0, e to the power A t into x of t plus lambda transpose of 0, e to the power of minus A t B u of t. So, we have a choice with u t now, suppose this quantity is positive so, this quantity is what, this, this vector is n by m, the dimension of B, this dimension is n cross n and this dimension is 1 by 1 row and column. So, if you see its nothing but a, that is 1 row m columns, 1 row m columns and this dimensions is m rows 1 column. So, this is a vector of this one, now I can write this will be minimum that from 5 and 6, from 5 and 6, I can write it.

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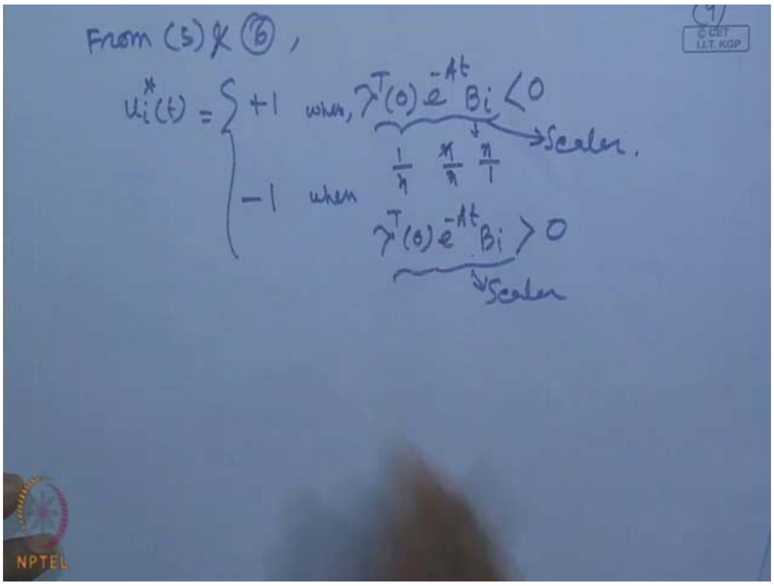

From 5 and 6, we can write each component of, each component of u, you can write u i of t star is equal to plus 1, when lambda transpose of 0, e to the power of A t and B I, what is i, B is a, if you see this one B is a, how many rows are there n rows, m columns. So, first column of B is B 1, second column of B is B 2 and ith column of B is B i so, I am writing this and this will be what, this is a n row 1 column, n row 1 column and this is n by n and this is one 1 n. So, this will be a scalar quantity this, so if this absolute value of this one or this, that, this value is less than 0, each element of B, I am considering, each element this is you will, each element of u, I am considering.

If this quantity because, this is what whole thing is a, your this whole thing is a row vector so, first each element of this row vector, if it is a less than 0, then this should be a positive. Then product of this one, this means which one, the corresponding element of u must be positive, if this is negative corresponding ith element of u must be positive. So, that is why I am writing this positive.

now if lambda star of j, e to the power of A t B i is greater than 0, this is scalar, if you see this is scalar, this is also scalar ,then it is this. So, ultimately the, this function value will be reduced, minimized in this way. So, this so, u you have switch either plus 1 or minus 1, when this, when you followed my, this point this quantity if you see from here to here, just from here to here, if you see up till here, you have a 1 row ((Refer Time: 52:47)).

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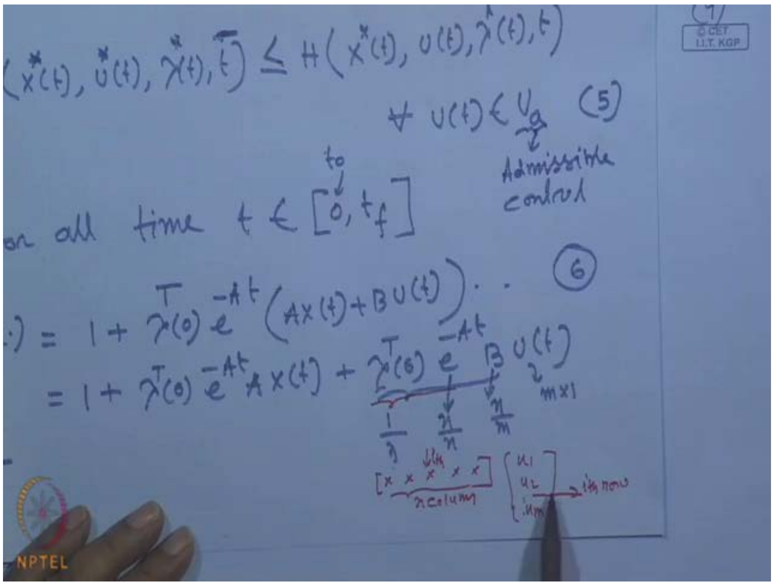

Over here n columns, this is the n columns, this is the n columns and what is this, this one I am writing it u 1, u 2 in this way m rows. So, ith row let us call this is the ith columns of this multiplied by ith row, this is the ith columns, this is the ith row of, it should be multiplied. So, ith column of e to the power 0 it, lambda transpose 0, e to the power of A t, ith columns of that one, if it is negative, then the ith row of u must be positive.

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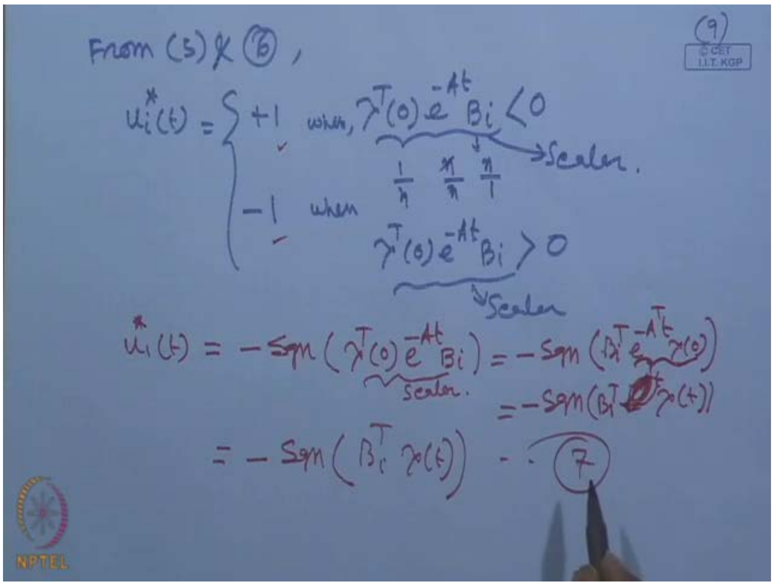

If it a positive, that must be negative and the resultant will be always negative so, that will be, a function value will be minimum as possible. So, this so, we can make it what is call up to infinity, but we have a restriction that input must be a plus minus 1, that 1. That is why we have given this. So, in short we can write both the equation, we can write, combinely we can write it using the signum function, sgn is equal to lambda transpose 0, e to the power of A t B i. So, which because this is scalar, we can write it this equal to minus sgn, B transpose B i transpose, e to the power of minus A transpose of t, lambda of 0. Which is nothing but a sgn B i transpose, u to the power of minus A transpose t, this whole thing, this whole thing from here to here, this is nothing but a lambda of t. See the solution of lambda of t, we have just found out that from the costate vector lambda t, e to the power A t lambda 0.

So, e to the power A t lambda 0 is lambda i, so signum function, if this quantity is greater than 0, then that value is positive, switching to the positive. Since it is preceded with a minus sign so, there is a u i is negative, if this quantity is negative then sign signum function of this one indicates, that it is negative, negative positive. So, it is a positive, u is positive so, that is the our, we can write it this ultimately minus signum function of this one, B i transpose lambda of t is, let us call equation number 7.

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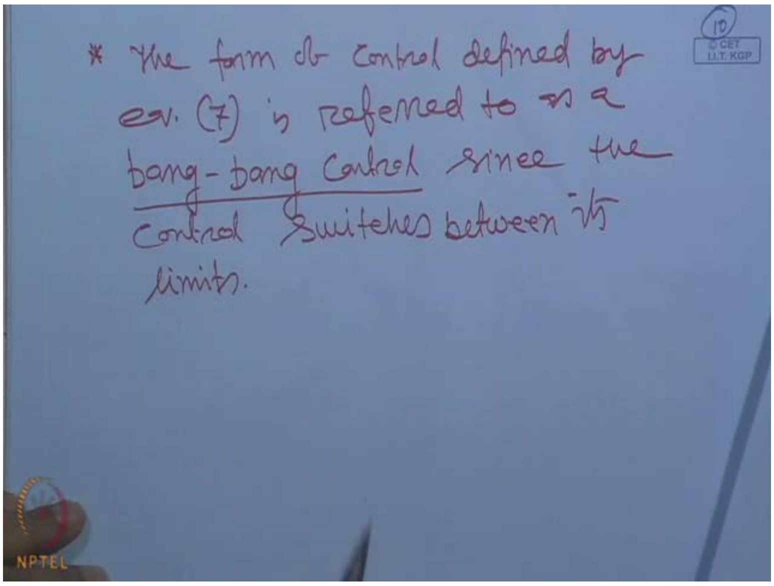

So, the form of control so, what you can write it, the form of control defined by equation 7, is referred to as a bang-bang control. Since, the control is switching between the two limits, since the control switches between its limits, so this is called the bang-bang control. So, next class we will take in one simple example and explain you, how this control action can be taken into consider, for solving the, what is called the, time optimum control problems. So, we will stop it here now.