Optima Control
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Lecture - 47
Dynamic Programming for Discrete Time Systems

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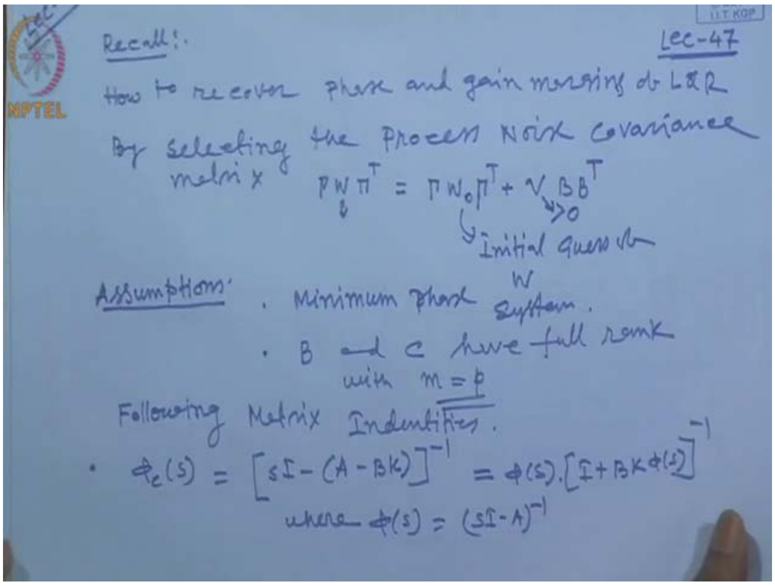

So, last class we have discussing about the loop transfer recovery of a LQG problems. Then just if you recall this, what you have done is that LQR problem provides the excellent robust properties in the sense that, gain margin is infinity and phase margin at least 60 degree. This is the robust properties excellent robust property we obtain in simply LQR problem, but when you are designing the LQG problem assuming that state equation and measurement equation is corrupted with noise.

The noise statistics is known to us a pairing, then in that situation since the states we are estimating using Kalman filter. In that situation loop transformation is different from LQR loop transformation, which in turn that robust properties of LQR is totally destroyed. In the sense that we are not able to obtain the gain margin to infinity and the phase margin is not at least 60 degree. So, in order to recovered that what is call robust properties, we have adopted some technique which is call the process noise covariance matrix. We can split up into two parts.

So, if just we can we are just, what our earlier procedure the how to recover the what is call LTR agree, so how to recover phase and gain margin of LQR and phase margin. We know we know the 6 at least 60 degree in LQR and gain margin is infinity but, in case of LQG this property is totally lost, so how to recover that one. So, that can be recovered by selecting the process noise covariance matrix. That is gamma W gamma transpose into two parts that we have mention it, W gamma Q 0 gamma transpose plus q B into B transpose. Where this is the initial guess of noise covariance matrix, initial guess of W and q is a some positive scalar quantity.

So, this we have split up into two parts and we have shown in the last class how we have what is call regain the loop transformation again as LQR problem. So, in this process if just following assumption are made. First assumption is made that system is minimum phase systems. We also made another assumption that number of inputs that means input matrix and output matrix, then another matrix is input matrix B and output matrix C have the full rank, with number of inputs M is equal to number of outputs.

This assumption we made it to derived to modify the what is call loop transformation of LQG. Using the matrix identity that last class what we have derived to have briefly we are summarising this, that one using the matrix identity, two identities we have used it. Then we have derived that linear quadratic regulator loop transformation is same as the linear quadratic regulator loop transformations. We using this that noise processing noise covariance is spilt up into two parts and we made the assumption that minimum phase systems and VB and C as have a full rank with M. With this assumption and using the following matrix identities, what is this matrix identities we have used it.

That is first matrix identities we have used that phi c of s is equal to loop transformer that is close loop transformation is equal to S I minus A minus B K whole. So, this is a close loop transformation, when there is no noise process and noise in the state equation and measurement equation. Another words when Kalman filter is not used. This is this matrix this equals to we have written like this way phi of S into I plus B H, phi of S inverse this matrix identity we have used it. So, where phi of S is equal to S I minus a whole inverse that we have derived plus. This is one matrix identity we have used it.

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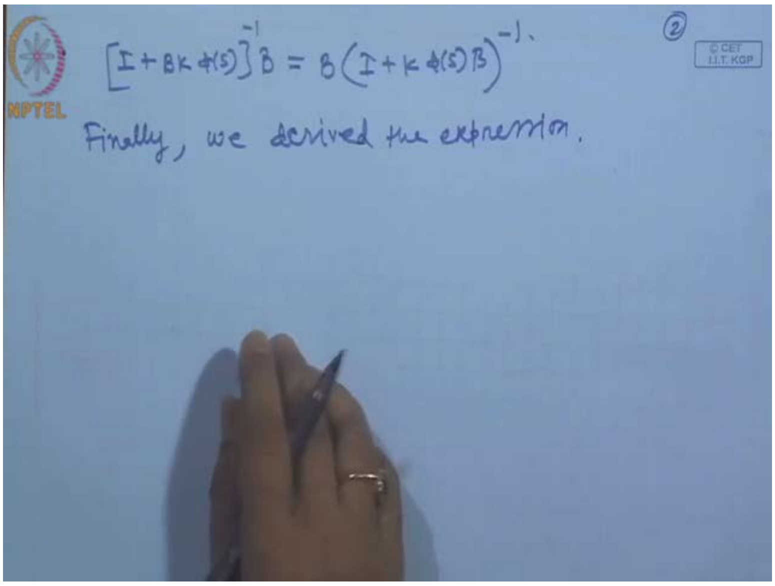

Another identity we have used it that that I plus B K phi of S whole inverse B is equal to B I plus K phi of S into B whole inverse. So, that is another matrix identity we have use it using this two matrix identity. We finally, derived the expression. Finally we derived the expression. Last class if you recollect this one that expression we got it.

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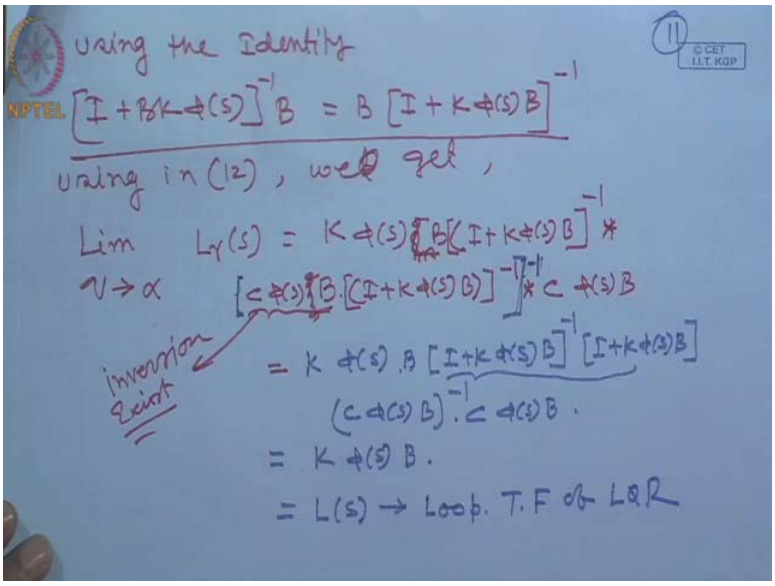

Limit q tensed infinity L r phi of S, this into C phi of S B bracket I plus K phi S B bracket whole inverse. That bracket another bracket is there we missed in class whole inverse. Now, you see this if you can see this and this is identity matrix because A into B into C, whole inverse is equal to C inverse into B inverse into A inverse, provided A B C all inversion exist. So, if you do this process then ultimately you can K phi S into that this B then I plus K phi S B whole inverse.

Then this is I plus k phi S B this then this whole inverse C phi s into b whole inverse agree into then C phi S into B. So, ultimately we are land up on this is this identity matrix this and this identity matrix this inversion is exist with the assumption that B C with a number of inputs and outputs are same. So, this assumption, we are left with this one and which is nothing but the loop transformation of LQR that is nothing but L S which is nothing but a loop transformation of LQR.

So, in short if you are summarise, if you select the process noise into this form and make the assumption that system is minimum phase system B and C have full rank with m is equal to n and following identity is are applied during the derivation, what we did it during the derivations of last class. Then we finally, land up with these expressions and which in term we can write it this one. This is nothing but the loop transformation of LQR. So, which is indicates that we have regained the loop of a robust properties of LQR by selecting the process noise into these mineral. Where with if you select like this way will get the gain margin is infinity and the phase margin at least 60 degree. So, this is call loop transfer recovery.

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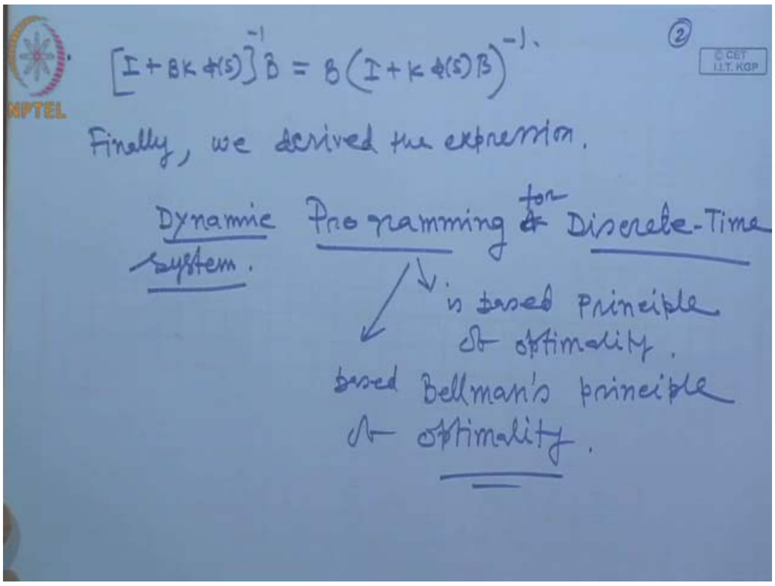

Now, let us go in a new topics, which is call dynamic programming of discrete time systems. So, let us call and this dynamic programming is based on the principle of a e t is based on principle of optimality. What is principle of optimality, we will discuss. In other words it is based on the based on Bellman's principle of optimality.

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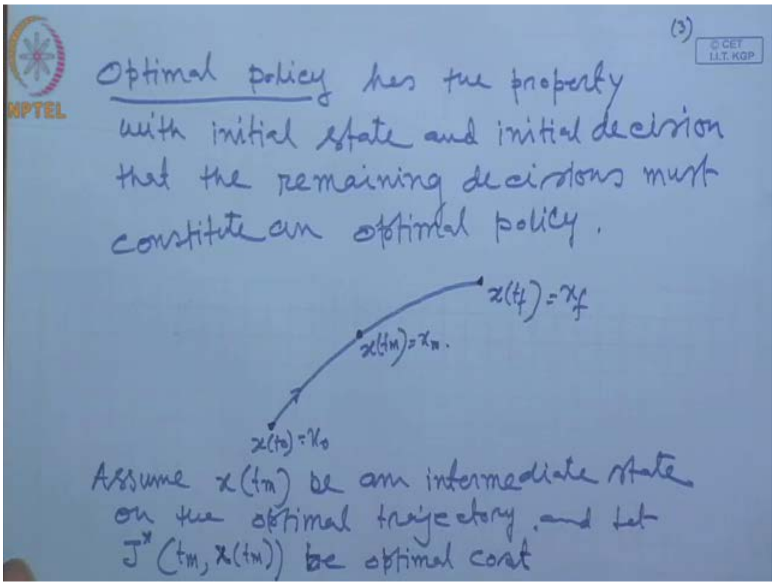

Let us see what is this? So, the optimal policy has the property with initial state and initial decision. That the remaining decision must constitute an optimal cost policy. Let us explain this thing, suppose we have a trajectory whose initial state is x of t 0, at time t is equal to t 0 state position is x of x of t 0. Here the final terminating state is x of t f. This one that means x of t 0 is the if you assume this the trajectory is the optimal trajectory starting from initial state x of t 0 of time index t 0. It is terminating at time t is equal t f at t is equal to t f, is terminating the at t is equal to a x f and this x 0 x time 0 along this optimal trajectory.

So, and there is another point is there here, intermediate point let us call that is x t is equal to t m is t m that trajectory is x m. So, that point at t is equal to t m time index the state value is x m. So, assume x t m be an intermediate, this is a intermediate state or point on the optimal trajectory. Let the cost J star x t m of x t m be the, indicates be the optimal cost for travelling the state form this point to terminal point. The J star x t m from where it is studying x t m and what is the position of time t is equal to t m x. Position is x t m is the this J star indicates the optimal cost transferring the state from this to final state or terminal state. Here the cost is this generated by J star from where is studying at time t is equal to t m x. What is the state position at x t m, this is the call optimal cost of the trajectory moving from thus intermediate point x m to terminal point x f. Similarly, if we just note J star t 0 x t 0.

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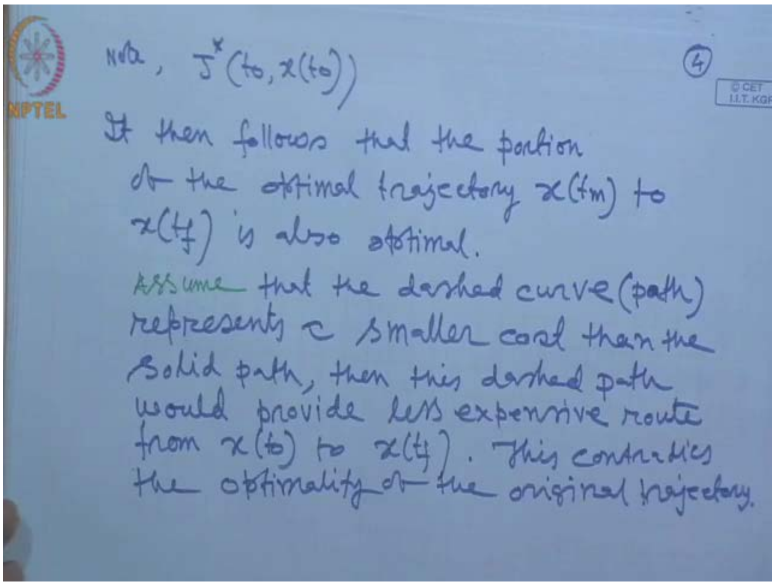

What is mean, this indicates that optimal cost transferring the state or moving the state from initial state x t 0 or x 0 at time t 0 to final terminal terminating state x f at t is equal to t f. What is the optimal cost, that is generated by this symbol agree. So, with the knowledge of this symbol then it you can say this one, this is the intermediate point or state. Then it follows that the portion of, you see from what is the optimal cost from intermediate point to terminating point. Optimal cost is J star t m, x of m. This is the point.

So, that I am writing if then it follows the optimal, that the portion of the optimal trajectory x t m to x t f is also optimal because we have consider this along this curve transferring the state form this along. This stand from initial state to final terminal state this path cost is optimal. This intermediate point belongs to the, what is call the our optimal path being this way. So, from here to here, what is the cost to obtain it that is also should be optimal.

Now, let us assume that this is not optimal and this path is the optimal, the dashed line. From this to this if you proceed like this way I am reaching the terminal point here along this path is optimal. Now, what is the optimal cost is required, optimal cost in during this process that from here to here, what is the optimal cost plus from here to here. What is the optimal cost is there this cost is different from this one. We told that this the optimal and which contradicts our initial assumptions.

Initially we have turning along this line, if you move the state from initial state x of 0 2 terminal state x f the t is equal t f. Then you will get the optimal cost agree. So, this if you move from this to the this and this to this, assuming that this path cost from terminal point intermediate point to the terminal point. This is the minimum or what is the lesser cost compare to these.

Then the whole state the state when we are transferring from x 0 to t f along this path and dotted path this will be the optimal, but that contradicts our initial assumptions agree. So, which cannot be done, which cannot be possible. So, and just writing this things once again. Assume that the dashed curve or path represents a smaller cost. Then the solid path or if you just include this is A B C, this path is D and E path.

Then this dashed path would provide less expensive route or path from x t 0 to x t f. This contradicts the optimality of the original trajectory. So, this is the optimal path on me. Now, this optimal control strategy is, my question is that from here to here it will move to travel from this point to the initial path state. This to this under the control action if you think that from control point of view the stares we have to move from this point to this point agree. If you move along this line that will be give you the optimal cost and what should be the choice of control action, the state to be move along this path.

So, the optimal control trajectory or optimal control strategy or a control law strategy can be obtained by backward pass, from final stage using principle of optimality. This indicates that, if you know the terminal cost of this n per m, then back backward you find you what is the cost is required from here to here. Then from here to here backward pass that means optimal control strategy can be obtained. Optimal control strategy many control law can be obtained backward pass from final stage to the final stage using the optimality principle. The decision this is the thing, decision is nothing but a choice of alternately living path from a given note.

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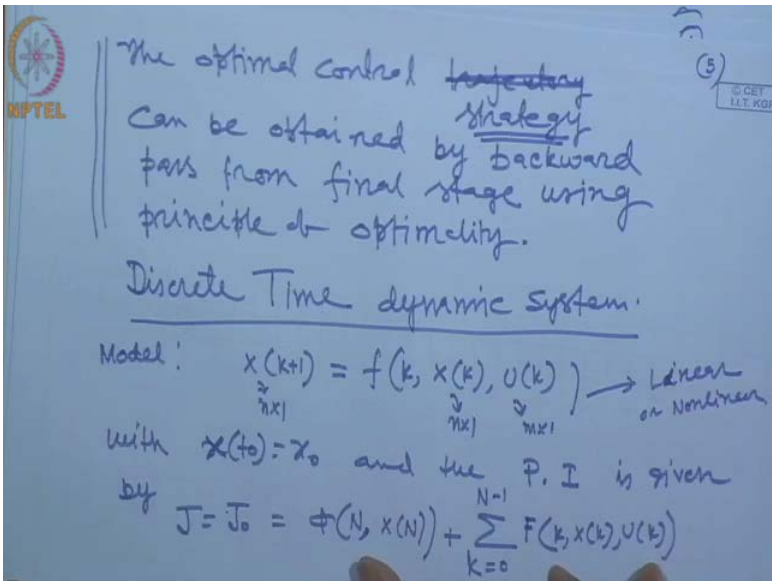

What is the decision which path will move this is nothing but a alternative, living path from a given node. So, let us take an example and see how we can derive the principle of a optimality, based on optimality to find the optimal control or you in case of discrete time systems. So, discrete time dynamic systems, this principle optimality apply in all to implement the control of based on the principle of optimality. So, let us call our discrete model is given by that x k plus 1 is equal to f of k x of k and u of k and this dimension of this, let us call number of state are n cross 1, number of inputs n cross 1 and this number of states are n cross 1.

In general this is the function of x and time index k and u. Any discrete is 10 in this state is from one can write it x k plus 1 k. Indicates the time index which is the function of state and input and the time index. So, this function can be linear or can be non-linear. So, with our initial state x of t 0 is equal to x 0 and the corresponding performance index and the corresponding performance index P I performing index of the given system is given by J is equal J 0. Then phi n capital X of this plus submission of k 0 to n minus 1.

Then it is a F k x k and u k. This capital F is the performance index, which is in generally may be contradict form or a new form. So, this is our performing index is deriving. So, our job is to find a control u in such a way so that j is minimised. Correspondingly optimal trajectory of the state we can obtain. So, we are mention it that we are moving backward. So, terminal cost final terminal cost you know then we let us call x.

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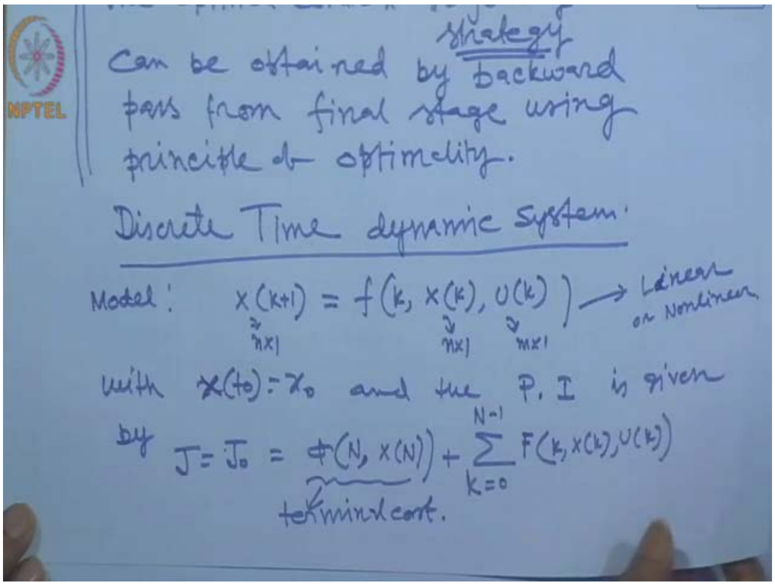

This is the call terminal cost terminal cost and our terminal point t that is k is equal to n.

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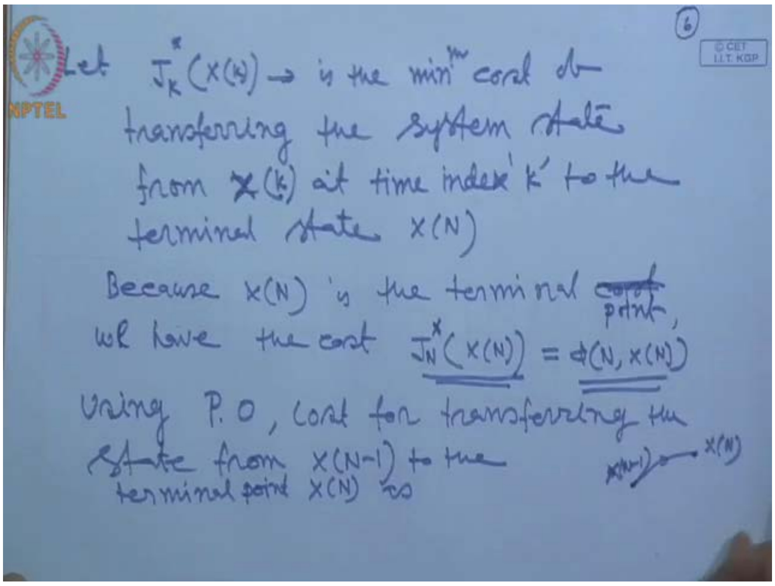

Time index the terminal cost k is equal to n is the terminal time index. So, let J k, J suffix k star X of k or full k what is this indicates. If you are considering this indicates that we are moving the states at k things stand to final time index k is equal n index for that 1 what is the minimum cost of this one. So, this we are writing is J is the minimum cost of transferring the system state from capital X of k at time index k to the terminal state X of n.

So, this indicates from where do I study in k thing stand the state is transferred to a final state that indicates which in stand the state is transferred and corresponding this is x of k. Where transferred the state to a final state. If k is equal 3 and N is equal to 10 that means we are transferring the state x of 2 k is equal 2. To a final state k is equal to k of 10 because capital N is 10. So, that corresponding cost e to denoted by J star of 2, if we k is equal to 2.

So, this is the same definition what we have consider earlier, that what we had talking about this. So, what is a J star of t m a the state at time t is equal to t m. We are transferring the state to a final state agree and what is the final state, that t is equal to t f what is the cost is required. So, because X of N is the terminal cost or terminal, X or this not that cost, this is the terminal point because X of t is the terminal point we have.

The terminal cost or cost terminal cost J n star X of N. That means J n final state to final state, what is the cost that is the cost of this one by defining that way. That will bigger terminal cost J n star and whose values is what in this expression. If you see just now in this expression if you see, that means is equal to n you see what we have to be done, k is equal to you see this k is equal to n. So, J X of N this value will the terminal cost is that much because k is equal to N, means upper lower limit is N, upper N minus this expression does not valid only this term is left.

So, the terminal cost is phi N of X N. So, this is the terminal cost. Now, using principle of optimality, cost for transferring the state from X N minus 1. So, our state u is X N is here, now X N is X capital X N minus 1. So, what is the cost or for transferring from this to this. So, that we are going that using, cost for transferring the state from X N to the terminal point X of N, as what you can write it, so J minus 1 star of X N minus 1. So, this meaning is, we are transferring the state at index time index n minus of k is equal to n minus 1 to k is equal to n in final state terminating state.

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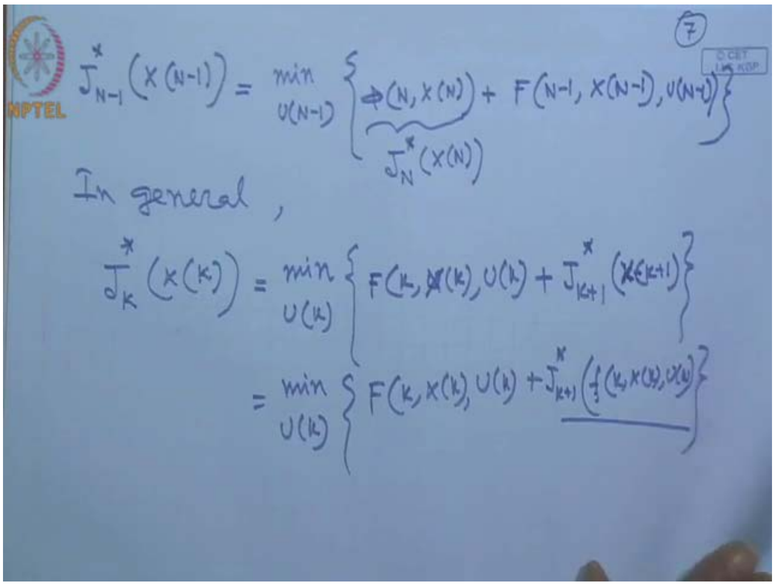

For that to on what is our cost mini away to minimise that you minimise u n minus 1 and what is our performing index ,phi terminal cost is phi n of X N this plus phi N minus 1 X N minus 1 u n minus 1, bracket closed. Now, this is the terminal cost. So, what is our n, k N minus 1 step to final step. So, final step cost is this one that k is equal to N minus 1 to N minus 1. So, we have k is N minus 1, X of N minus 1, u of N minus 1.

That is what we have written now we have to minimise this one and which is a function of u. So, we have to minimise this one. So, if you recall this is nothing but as a according to our notation this is nothing but X of N terminal cost. Now, this is from N minus time index two terminal time index this N. Now, in general we can write in general we can write J k X k of this star, that mean what how you will read this one, we are transferring the state at k is equal to k th time index to a terminating time terminating state at k is equal to capital N. Time index k is equal to N, which we can write it minimise U k, you see when N minus 1 minimise because of information is only n minus 1.

So, minimise this then k 2 that is your k state at k is equal to k th instant to a final terminal condition, then what we can write it for this one. That will be a F k U X k, then U k then what is required for this one because J k plus 1 k plus 1 star k plus 1 stars k plus 1. You followed this one, in this expression I will write k is equal to k N minus 1. So, let us term N minus 1 of this because we are doing back ward first calculation we are doing if this one. So, this will be that which I can write it minimum U of k, F of k X k U k plus J k plus 1, then X k plus 1 what we can write is nothing but, F of k X of k U of k.

Now, you see U of k is a function of that f small x is a function of small f then dynamic equation is function of U k that means I am writing into a x k plus 1 into this form. So, this then you do the minimise that thing with respect to U k. So, this is the basic principal that you calculate the what is called control strategy that word pass. Ultimately you find out the optimal strategy of the dynamic systems again. So, it is clear to you because I am finding out the optimal cost from the k th instant k th instant to final terminal state.

When the state is transformed from k th to k th, I mean index that mistake value is X of k to final terminal time index k is equal to N. Then what is the optimal cost. So, this is nothing but, I am writing from this one, k is equal to k N minus 1. So, from k th instant that last k th instant is, k is equal to I write it and other terms I club to this one, which I have written into this form. So, let us take one example and see how we can use the principal optimality to find the control law in case of dynamic this circumstance. So, example.

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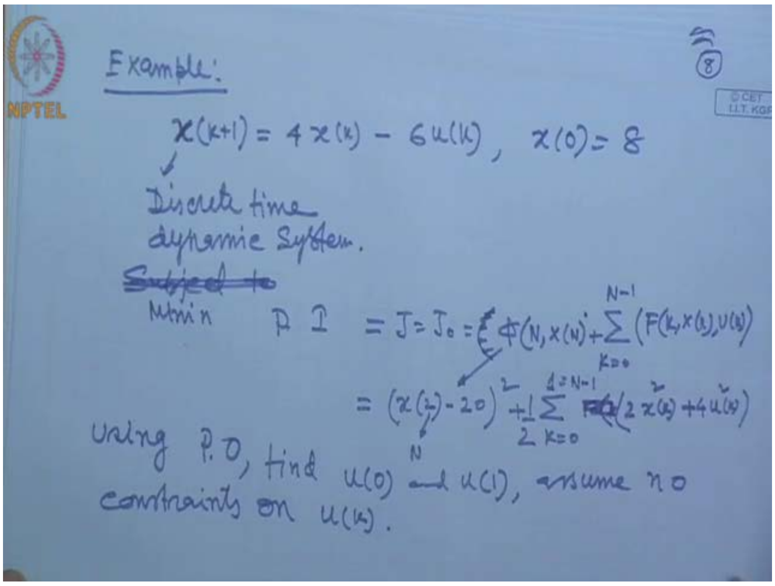

So, our problem is we have given a discrete time system, whose dimension is first to difference equations. So, let us call and write it since it is a small x four x of k is equal to 6 u of k. Your initial condition is given state is x of 0. So, this is discrete time dynamic systems subject to our. So, this is the system and minimise not subject to minimise the perform index and what is the perform index J of J 0 from initial state to final state as given.

So, this is given to phi in general expression I am writing first phi N X of N this plus summation of k is equal to 0 to N minus 1. Then your F k X of k agree then U of k. So, for this particular problem this is given like X of 2 minus 20 whole square. So, what is our what is our phi N X of N is this one your our phi N of X N is that one. What is our capital N that capital N is 2 plus summation of half k is equal to 0 to N minus 1, N is 2 this is our n is 2.

So, it is nothing but a 1 which is nothing but a N minus 1 of F k, that F value is given this expression is given I am just writing here 2 small X k whole square plus then U 4 U k whole square bracket closed. So, the phi capital phi X can is given this one. So, our performed X is this one, our problem is now find out U k that mean I have to find out U k mean 1 inverse U k to get 2 terminal state 2. This 2 terminal state X 2 I have to find out U 0 and U 1 by applying the our optimal control study. This is our perform index. So, using problem is using P O principal optimality, find U of 0 and U of 1 assume no constraints on U of k agree, no constraints of way you minimize that one. Now, what is unknown is here, you see I have to choose the control law U such that this performance index is minimize. Once you find out U 0 immediately you can out X 1, then once you know X 1 find the optimal loop of U 1 you will know X 1

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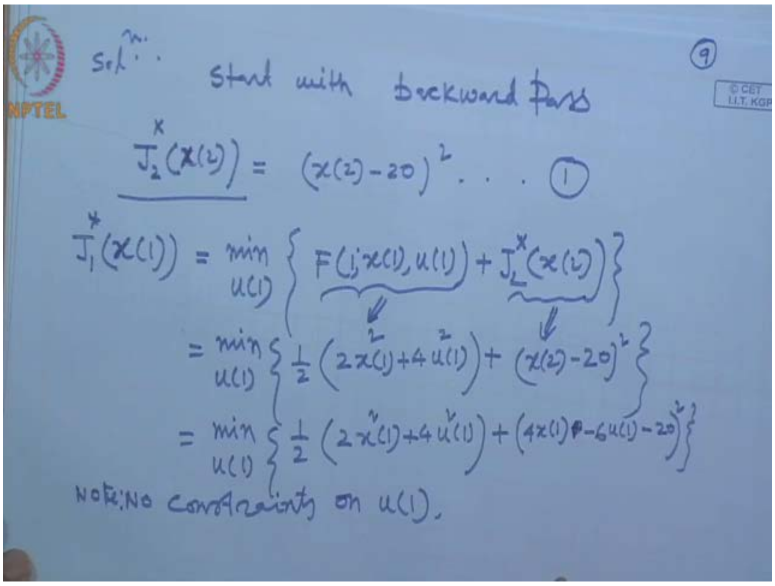

So, if you move along this trajectory the performance index will be minimized this one. So, how to solve that problem solution. So, we start with backward pass. So, we have. if u recollect that J 2 star of X of what does it mean small x 2. This mean what is the terminal cost is equal to know what a terminal cost is that one only because k is equal to 2. If you write k is equal to 2 here in a break is equal to 2 ,1 that this expression does not valid. So, what you are writing for this one, this equal to our terminal X of 2 minus 20 whole square is equal our terminal cost.

So, this next we can find out what is call J 1 of x of small x of 1 star. What is it mean I know the terminal cost x at k is equal to 2. This is the terminal cost, then I will find out backward from x 2 to x 1 in other words I can form a x 1 to x 2 what is the our cost is involved. That is nothing but a minimize u of 1, F of 1, x of 1, u of 1 plus J of star x of 2. Look at this expression so k is equal to 1 to 1. So, it will be F of 1 x of 1 less then this J phi to phi of 2 or x of 2 is nothing but a terminal cost of J of 2 this one.

So, that is minimize, now I will write expression min of u of 1 what is the expression for this one. We know already this expression is given that one. So, we will write it that half twice x of 1 square plus 4 u 1 of 1 square. So, that is our that quantity, plus this quantity is that 1 x of 2 minus 20 whole square that things. So, this is the terminal cost I have written and this is for x is equal to 1 the 2 x of 1 square plus u y of 1 square I have written.

Now, you differentiate this with this with 2 u that will give you if you differentiate with expect to this, u of 1 then half twice x 1 square plus 4 u square of 1. This I can write in terms of x 1 and x 2, then that is nothing but if you see this one, k is equal to 1 that means 2 x of 2 is equal to 4 x 1 minus 6 you want. So, I can write it 4 x 1 plus 6.

The 6 of minus that is minus 6 u of 1 minus 20 whole square that you form differentiate with of into F. Then by principle of what is call how you differentiate this with respect to this. Then you can write it note no constraints on u of 1. So, optimal u 1 is obtain optimal u of 1 is function of x of 1. So, this if we difference equation that cannot difference equation that optimal cost in discrete demand with differentiate with respect to that 1, I mean u of 1 and that u of 1 you will get function of x 1 only this is what we have written it here.

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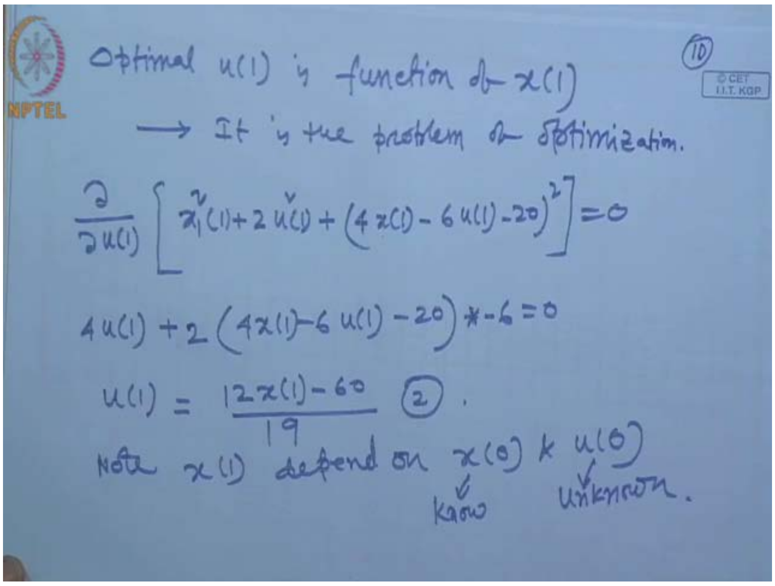

This is nothing but it is the optimisation problems, it is the problem of optimisations, discrete optimisation. So, what will generate dou u of 1 then whole thing what is this whole thing this half cancel this. So, I will write it x 1 square of 1 plus 2, that is 2 plus of u square of 1 plus 4 x of 1 minus 6 u of 1 minus 20 whole square whole bracket. So, if you differentiate with the respect to 1. It will be 4 u 1, then u 1 is coming here.

So, this will be twice 4 x 1 minus 6 u of 1 minus 20 into minus 6 is equal to 0. This equal to 0. So, if you solve this one, you will get the expression of u of 1 is nothing but a 12 x 1 minus 60 divided by 19 this is equation 2. So, this x 1 now look at this one in basic equation of dynamic equation k is equal to 0 mean x 1 x of 0 y of 0. So, x 1 depends on u 0 and x 0. Note x 1 depends on x 0 and u 0, which we do not know only x is known this is unknown. So, let us see how to do. So, you are now where we are now x 1 from x 1 to u 1 to shift from x 0 to x 1 what is the optimal cost required optimal path.

That we have cost is J 1 star of X of 1. It indicates it what the optimal cost transferring the state from X is equal to x 1 to x 2 what is the optimal cost because we know u 1 expression is that one, if you put this value in the expression that half twice x 1 square plus 4 u of 1 square plus J 2 star of x of 2. If you put this one, then J 2 star is what we got it. Just now we have found of J 2 star, say J 2 star of this is that one.

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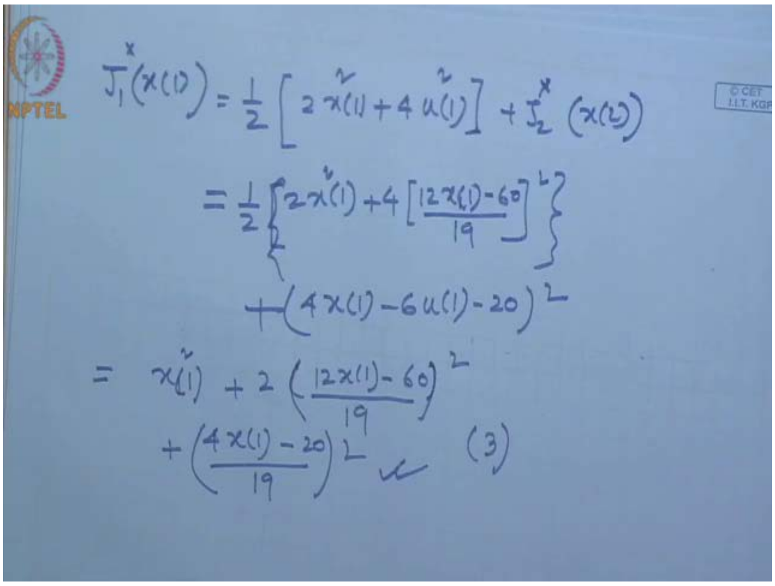

So, we can write it that half of twice x square of 1 plus 4 u 1. We know what is u 1 12 x 1 x of 1 minus 60 divided by 19 that square. So, that is I am writing for this path plus J 2 square that one. That is 4 x 1 minus 6 u of 1 minus 20 whole square. If you simplify that one finally, we will get it if you simplify this one. We will get it finally, that x 1 x of 1 square then twice 12 x of 1 minus 60 divided by 19 whole square plus 4 x of 1 minus 20 by 19 whole square.

The three parts x 1 square plus twice 12 x 1 minus 60 by 19 whole square plus that term, if you use the value of u 1 here then 4 x 1 minus 20 then this one. So, this is the final expression let us call this expression is three and expression number this is one and two. We have consider that u 1 expression, what is u 1 expression that is two. So, what is our next, then we know the optimal cost up to backward, then up to the state x of 1. Say now we are moving the state x of 0 to x of 1 what is the optimal cost of that one.

So, J 1 star x of 1 this cost we have to find out. How will you find out, you see that our basic expression for this one. See now this is k is equal to 0 n is equal 2. So, k is equal to 0 n is equal to 2, n is equal to 1, k is equal to 0 F of 0 x of 0 is 0 then F of 1 x of 1 u 1 that cost we have to calculate backward. So, our expression will be half, minimum of u of 0, then half twice x of 0 square plus 4 u of 0 square that 1 plus JU 1 star of x of 1. What is J 1 star of the x 1 that optimal cost required to transferring the state of x 1 x of 1, means k is equal to 1 time index 2 x of 2. This is the time index and now this is we have to find out minimum value of u, so that that this, what is the trajectory from x 0 to x 1 that we can find out this one. So, we will continue this thing in the next class.

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