Optimal Control
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Lecture - 46
Loop-Transfer Recovery (LTR)

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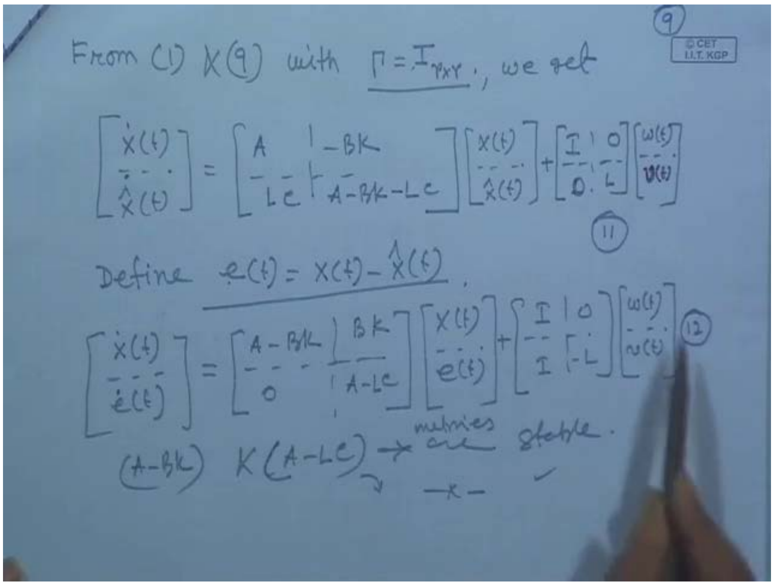

Last class we have discussed what is called LQG problems. Finally, we have to come to the next expression that dynamics equation of this system is processed dynamics and estimated dynamics. We have come across this expressions where V of t this measurement noise, this equation can be retain into error that state equation augmented with error equation which will come into this form.

If you look the equation number twelve, you see this is the matrix, this matrix is diagonal matrix and this we can control what is called the close loop system. The observer or the estimator again will lose or Kalman filter will again use separately, one does not depend on the others. So, this A or B k can be designed independently from a minus L C Kalman filtered gain both are contoured gain and the Kalman filter can be designed independently.

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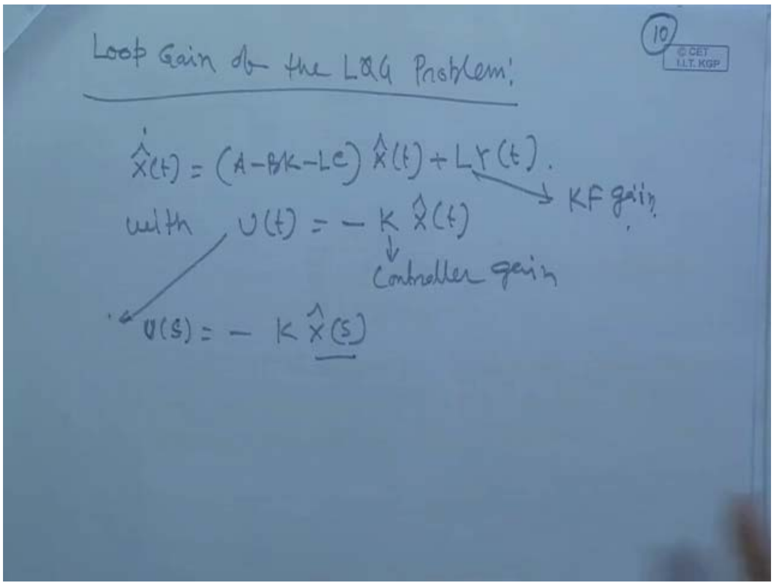

That means separation principle holds, now let us look at this stage that what about it's looped gained. That means if you talk about the reverse nest of this control art as we have seen in the what is called earlier case LQR controlled designed when the states assuming available to us. Then, we have seen that controller S gives what is good stability margin that means gain margin varies from what is called great than equal to half to infinity and phase margin is greater than 6 degree and where this property of the systems holds.

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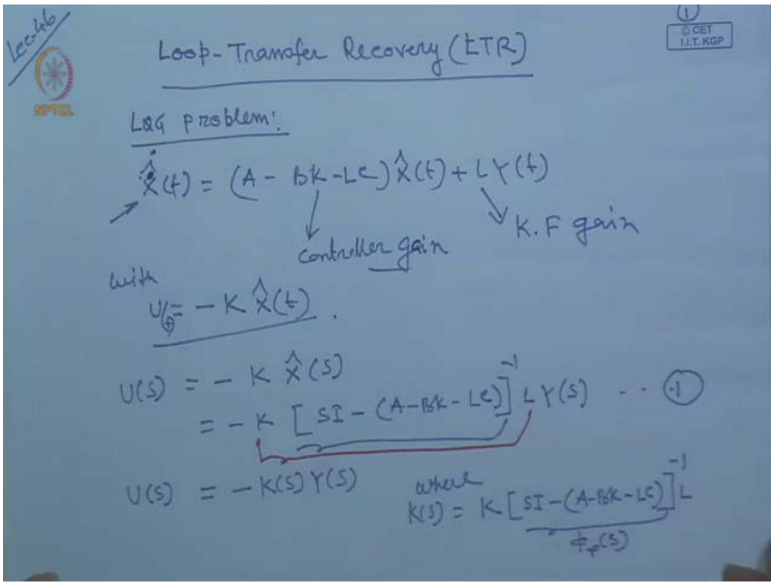

In case of LQG problem or not that we will study, now so called the loop transfer recovery. So, let us start the loop transfer recovery that means LTR and if you see this one and this, let us find out the loop gain of these systems, so we know the estimated LQG problem, we know our LQG problem. The estimated equation LQG problem estimated equation is dynamic equation is x dot hat x hat dot of t this is x hat dot of t is equal to a minus BK minus LC x hat of t plus L of y of t this is L of y of t is Kalman gain, Kalman filter gain.

This k is the controller gain and u is equal to with you can write with u is equal to minus k x hat of t u of t is equal to this. Now, if you take the terms of this quantity that u of S is equal to minus x hat of s, so which you can write it because if even you take this, these equations then we will get x hat of s. We can write it this minus that will be a SI minus a minus BK minus LC, this whole inverse, then this will be a L y of S because you take the transform of this one. You will get it SI x hat of x, then transform this one is x hat of S, bring to this side and take the well of x hat of S.

Then, x I minus as a minus BK minus LC whole inverse this will be multiplied by L into ys taking from this one, we will get it this to this. So, x hat of S is replaced by this one from this equation by this, so let us call this equation is 1 and we have S is equal to minus whole quantity. I am writing is k S into y s, the whole quantity from this to up to this this is denoted by y s. So, now where we can write where y S is where k of S is equal to k SI minus A minus BK minus LC, this whole inverse into L this is our k s. Now, let us this whole thing this we denoted by let us call phi r of S phi r of S this whole thing is nothing but A which is called resentment.

This phi r of S is called phi r regular resolvant matrix, regulated resolvement matrix regulated resolvant matrix phi rho of S because of the controller. If you see on this one observer gain matrix also involving that sense is called regulated resolvement matrix phi of s. Now, let us call this equation is that is what is considered that equation is two this equation is two. Now, if you draw this circuit diagram or block diagram this is our referencing input r is 0, then this is our if you block diagram if you put, this is our plant. This is equal to SI minus A whole inverse B is nothing but A phi of S this I phi of S into b si minus resultant matrix of matrix phi of x. This is our symmetric, the output of this S this is y of S and from this one, we have that our Kalman filter output of the Kalman filter is our x hat of x and this is this.

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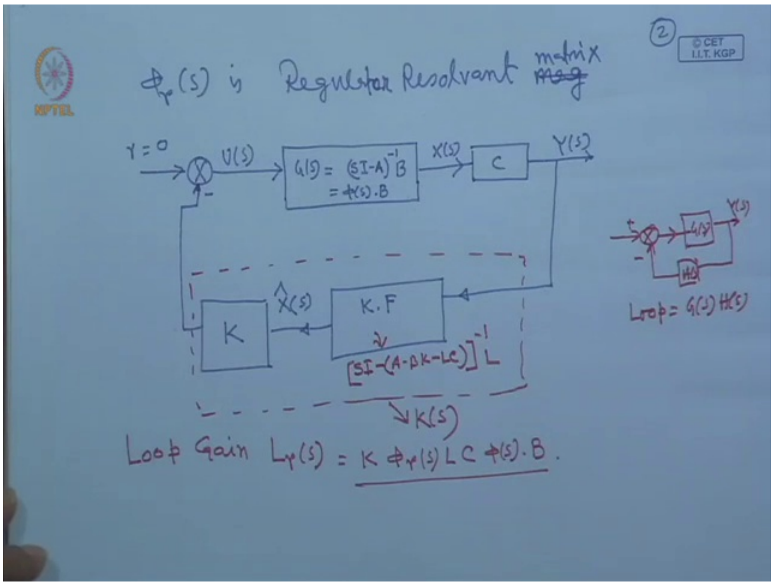

Then, this will go with a controller gain, then this is minus of that one V of S and this inside this one had Kalman filter structure is there input information is also there output information is also there. So, this inbuilt in their basically this is SI minus A minus BK minus LC, these whole inverse L that is our phi L into k that will be our k of s. So, this is nothing but our k of S see our k of XS mission a into phi SI minus a whole inverse L is our kS whole thing. This is our phi of r the whole thing is our k of s, now find out the loop what is called this this is u of S find out what is called loop trans function for this Kalman filter controller loop functions.

So, if you loop gain or loop trans function loop gain that is denoted by LR of S is nothing but from here to here. If you just move it here to here, this loop you have to break it find out the loop trans function just like in our basic block diagram. If you see basic block diagram, if you see this is our g of S and this is our feedback path of h of x. So, this is y of S in order to study this one, then our sensibility are compliments to loop transfer function loop trans function is g x h of x. Similarly, in this case also, we have a loop trans function loop gain of this one, I can write it this equal to k into this quantity I have considered is equal to phi this quantity if you see it is phi r s.

Then, C phi S into B this is a loop gain of this Kalman filter base controller when we have designed Kalman filter controller. Then, the loop gain k phi r, this is phi r, then L then C, then B phi, so this is loop trans function if you look carefully with the what is called LQR, they are our loop trans function is k phi S into B because this filtered is not present in the LQR. We are assumed that our states are measureable, all states are measurable, so next is let us call this equation is three, this equation is three. So, we find out the loop trans function for the LQR problems, loop trans functions or loop gain for LQR problem is equal.

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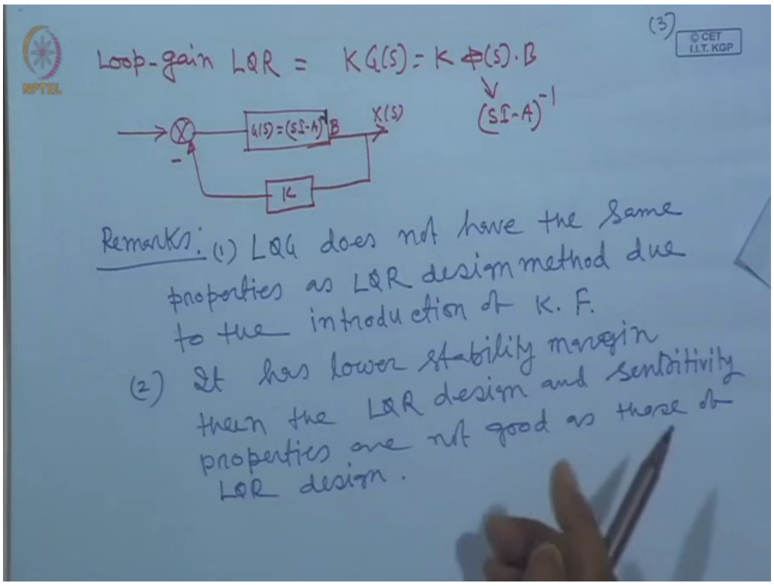

This is our loop gain and clearly you see this one this loop gain of the Kalman filter base controller design and LQG is a loop gain is different and we have seen in what is called LQR design. That controller what we have designed k the controller design the system what is called gain can be very form k is equal to half. You can increase the gain up to infinity present from up gain, whereas phase margin that will give at least 60 degree for the design k from the closed loop systems. So, that we have seen in other words draw the block diagram form of this one, it will look like this GS.

This is nothing but SI minus A whole inverse, whole inverse into b that output will be that output will be x of S and these states are all measurable. So, this and this loop gain if you see the loop gain for this one k into G S is what this is phi S resolvement of the given matrix S I minus A inverse. This is inverse S I minus A, this is inverse, this all two loop gains are different, we may expect that LQR, LQG controller. They may lose the what is called the excellent properties of LQR, what is the excellent property in LQR excellent property in sense of gain margin and phase margin.

We got it, we may lose that one, so remarks LQG does not have the same properties as LQR design method due to the introduction of Kalman filter because the loop gain is different. That part is coming addition to this part, if you see this part here to here, sorry from here to here, this from here to phi L S, L C. This part is coming exact, this is definitely there will be a role of what is called gain role as frequency is changing. So, first remarks is this on second, it has lower stability margin, it has lower stability margin, then the LQR design and sensitivity properties that means sensitivity properties are not good those of LQR sensitive properties are good as that of those of LQR design.

Sensitivity stability margin means gain margin will do not able to maintain as we got LQR problem due to the introduction of this Kalman filter, again not only this the sensitivity disturbance rejections or noise rejection. It will not be good as it is in case of LQR third one is your that third one, it is possible to recover the what is called stability margin or good properties of LQR, what we got it at properties will be able to recover while we are design the LQR. Again, LQG problem when you have designed LQG problems, we have seen that there is the loss of what is called stability what is called an stability properties when a stability margin.

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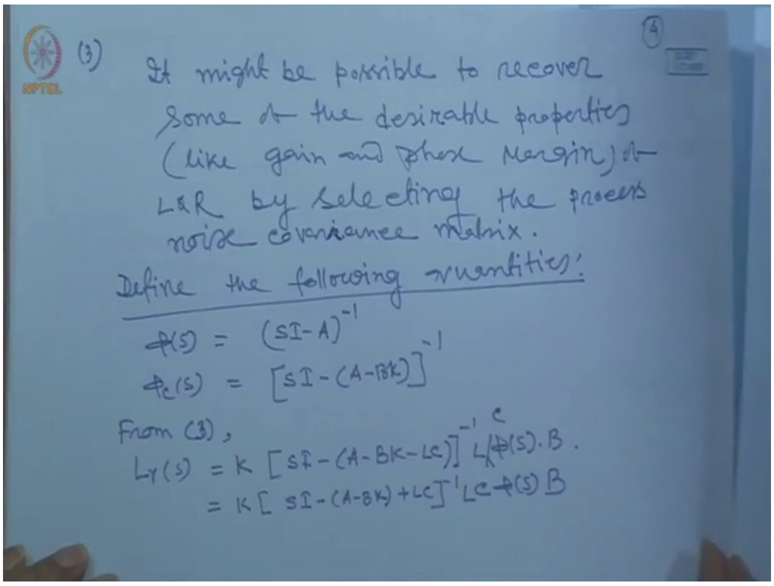

So, that we can regain or recover by introducing some noise in the process signal, so may be possible, it might be possible to recover some of the desirable properties like gain. Phase margin LQR by selecting the noise the process noise process by selecting the process noise covariance vary variance matrix. Let us say how we can regain what is called stability margin by introducing what is called by what is called selecting the process noise covariance matrix. So, let us define there are some mathematical we have to done we have seen how the process noise, if you can select properly, then it is possible to regain the stability properties of LQR.

So, define the following quantities, so it is conclusion is that if you design the LQR problem that stability properties of LQR is not retained. So, we can able to recover this stability margin of LQR by introducing by selecting the noise process, noise covariance matrix. So, let us define phi of S is equal to SI minus A inverse that is resemble matrix of the system, so next we consider phi c is equal to SI minus A minus BK whole inverse with this symbol using any equation number one.

Let us say equation number one, this I will use that one, then will be in this equation I use the phi c, so from one loop gain not one loop gain expression. This is the loop gain expression from equation three from to k, then you have A, if you see this one SI minus A minus BK minus LC whole inverse that whole inverse minus L. So, this L phi of S into B, so this I can write it k into SI minus a minus BK A minus BK, this I am clamp together a minus BK plus LC whole inverse L than C, I missed here phi of S into B.

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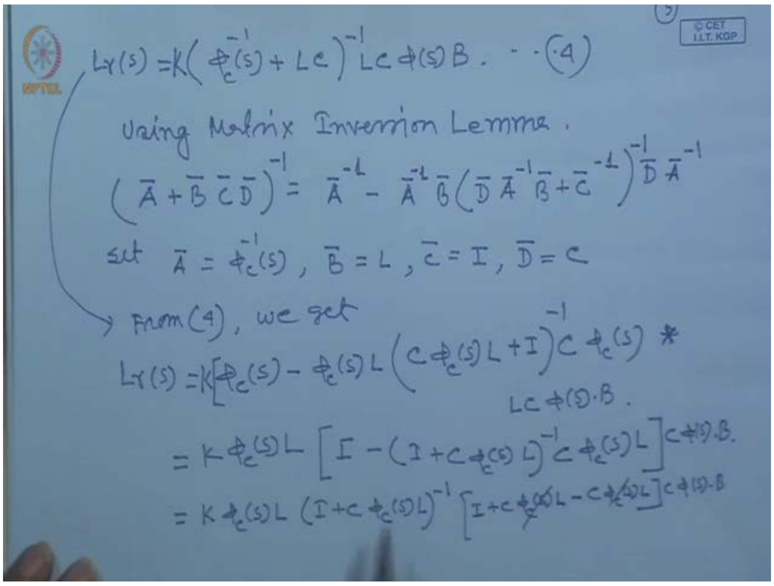

So, this equation I can write further LR of S, then I can write it this that this part SI minus A, this inverse is a your phi C that is what A phi C. So, I can write this is nothing but our phi CS inverse plus what is left LC, then whole inverse LC into phi of S into b it has called four from this equation this is equal to I have written phi c inverse. Then, LC in this one, so this and using and in equation four, we will be using one matrix inversion lemma using matrix inversion lemma, what is this method. Let us call we have matrix A bar, B bar, C bar, D bar whole inverse is equal to a bar inverse minus A bar inverse.

Then, your B bar, then D bar, then A bar inverse then b bar plus c bar inverse, then whole inverse D inverse A bar inverse. This matrix inversion can be retained to this one, so we are using in this expression this matrix that should be lemma for that, we set A bar is equal to phi C inverse of S, then B bar is equal to LC bar is equal to in the matrix and D bar is equal is equal to C. So, I will use this lemma in this inversion, so our using in this equation four, from four using the lemma in 4 we get LR of S is equal to and this is phi inverse of A bar, you say a bar phi inverse.

So, this is the phi c of S phi c of S then minus and you see this one there k is there is missing here k bracket minus a bar inverse then phi c of x b bar is here. Then, bracket d bar is what c then a bar phi c of S d bar is what L then your c bar is I so this whole inverse D bar is what C, then A bar is phi c inverse that inverse is phi C of S. So, this equation is re detained into this form into that into star into L c phi of S into b, so this equation is complete. Now, we are doing some manipulation of this, so I push it see this one, I push L in this expression phi C L and this L will also go here.

So, phi CL, so if you take it phi c L see S L common that I will remain here I minus, so this will be A I plus C phi C S L whole inverse. Then, phi C S into L into then C L I pushed inside, so remaining part is C phi of S into b, so this now, so this I can write it further I can write it this phi C S into L. So, what will come here phi C L this and this I can write it this matrix into inverse of this matrix, so I am taking common of that inverse of this matrix I C phi C X L whole inverse.

Then, what I left here because I am taking this matrix into I plus phi C L, so it is left is I C phi CS into L minus this one, this I have taken common C phi CS into L. So, close this one, then C phi C S into B, these cancel, so ultimately we get it here.

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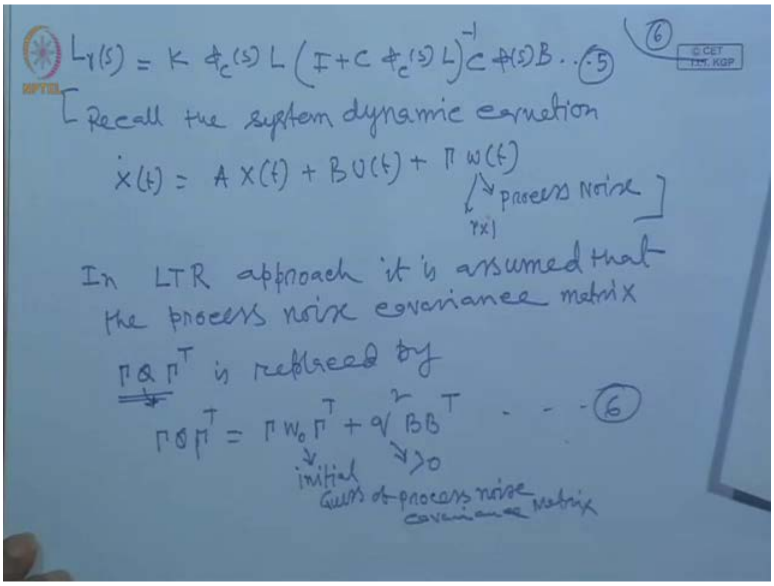

If you see this one is the simplification the LCR, LR of S is equal to K D C of S L I C phi C of S L whole inverse C phi of S B. Again, this is the equation, let us call this equation is equation number five, this is the equation number five. Now, recall what is called basic equation system equation, if you recall our system equation; we call the system dynamic equation from just recall from previous lecture the dynamic equation. So, equation number 1, so x dot is equal to a x plus b u plus gamma w of t. So, in the process in this system this is a process noise w of thing process noise, again we know the dimension is let us call dimension r cross 1.

Other dimension we have already specified in the previous statement of the LQG problem, so this process noise covariance matrix. So, if you recall this one, now in LTR loop transfer recovery approach, it is assumed that the process noise that the process noise covariance matrix. In the process noise covariance matrix, we have considered w in our last lecture matrix is nothing but a process noise covariance matrix gamma w into w transpose is into q that gamma transpose.

This is the noise covariance matrix of w noise considered as small omega that is q, so this is the noise covariance matrix, if we recall our the LQG problem statement, w is the process noise and its covariance matrix is your q. So, since going through a channel w gamma, so the noise covariance process covariance matrix, suppose this is gamma u gamma transpose.

So, this is replaced by now by gamma q gamma transpose is now replaced by gamma w 0 gamma transpose plus small q b into b transpose. Again, this is any positive scalar quantity and gamma 0 is the initial case of process noise initial guess of process noise covariance method noise covariance methods initial guess. So, let us call this is equation number six, now using this thing expression in our dual of what is called algebraic retake equation of contradicting.

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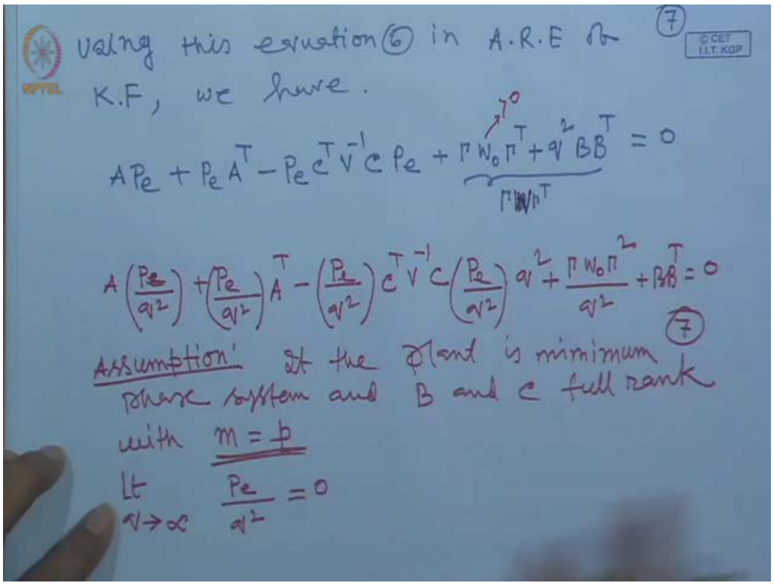

So, there is a transpose p, here a transpose by A, so P e is for algebraic retake equation for the Kalman filter equation this is the error covariance matrix. Then, P e refer to last lecture, so P e than there is P b is transpose by C transpose, so this P b R inverse, here is measurement noise covariance matrix. This one P b R inverse b transpose means C, then P plus gamma 0 gamma transpose plus q square b, b transpose is equal to 0 and that quantity is nothing but A our gamma q gamma transpose.

So, this is the algebraic retake equation for Kalman filter of that one, so let us see what we have considered u or w I am not able to recollect what is considered. Let us see in the algebraic retake equation of the Kalman filter noise covariance matrix, what we have considered the noise covariance matrix. I think I have considered w, so better this is w I have considered that w.

So, this is w this is w already I have considered w please write it same notation here, w noise covariance matrix consider there is a w there. So, this gamma w times replaced by a two parts, one is the initial guess and another is q transpose, this one this is a positive finite matrix, so all these things w is a w is a greater than 0 in a positive matrix. So, this equation we can rewrite by dividing by q square. So, you if you divide by q square that means a P e P suffix e q square P e suffix x square, this is whole k transpose, then P e q square I am dividing by C transpose V inverse C P e, I divided by q square multiplied q square.

It does not change anything, so that will be a gamma w 0 gamma transpose q square plus b, b transpose is equal to 0, let us call this equation is equation number seven what I did I divided by u these things. Now, we made an assumption the plan to that we have assumed that is minimum phase system the plant. If the plant is minimum phase system means all 0 s are stable and b and c full rank with number of inputs is equal to number of outputs m number of outputs number of inputs.

Then, one can write it limit q tends to infinity P e divided by q square is equal to 0, by the way we are increasing the q the p is increasing p is increasing more slowly than covariance matrix and which in term which q is very large. That will tends to 0 that means if we increase in the q again the p is increasing the error covariance matrix p is increasing more slowly. Then, the q is increasing, so this employs this will be q is very large in the infinity this will be 0.

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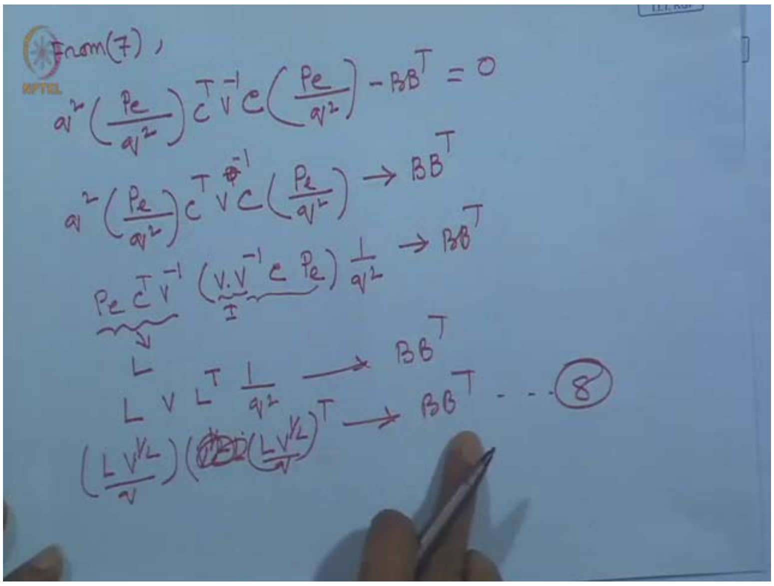

Now, from equation seven, if you use this expression seven what is logic behind this q, the way q is increasing, the increasing in the p is more slowly than increasing in the q, so with this logic, we can rewrite the equation form eight.

This quantity 0, but you see p is also increasing this part I cannot write is a 0, so what is term is left here is q square P e by q square c transpose V inverse C P e q square. We are described into q minus b, b transpose is equal to 0 because q, q cancel, now this is approach in 0 is also large. So, p is also large, so I cannot write this is equal to 0, so further we can write it if you see this one that q square P e by q square C transpose V inverse C P e q square tends to this. I can write, this tends to b, b transpose when q is increasing that this tends to this quantity b, b transpose which I can write it.

I can write it also P e C transpose, this not transpose this is B inverse C transpose V inverse. Then, I can write it into V into V inverse, this is identity matrix, then C P e into 1 by q square tends to b, b transpose.

Then, I can just give cancel this is if you see this is nothing but a Kalman gain L, so this will be a L that will be also V is there if you see this one this L this will be your L transpose and this is V. So, I can write it that L V L transposes this is L, so from here to here, this is L transpose, then V and 1 by q square tends b, b transpose. Then, V is a noise covariance matrix of the measurement noise covariance matrix, so I can write V half V is a symmetric matrix.

This I can write it V half L, this I can write L V half whole transpose tends to b, b transpose because b is a symmetric matrix the square root of b of half transpose, then manipulating this one I am writing this one. So, there is a q square 1 q I keep it here, another q I keep it here, so this expression remains same, so if you look at this expression number eight the this quantity tends to be b transpose if you select the L.

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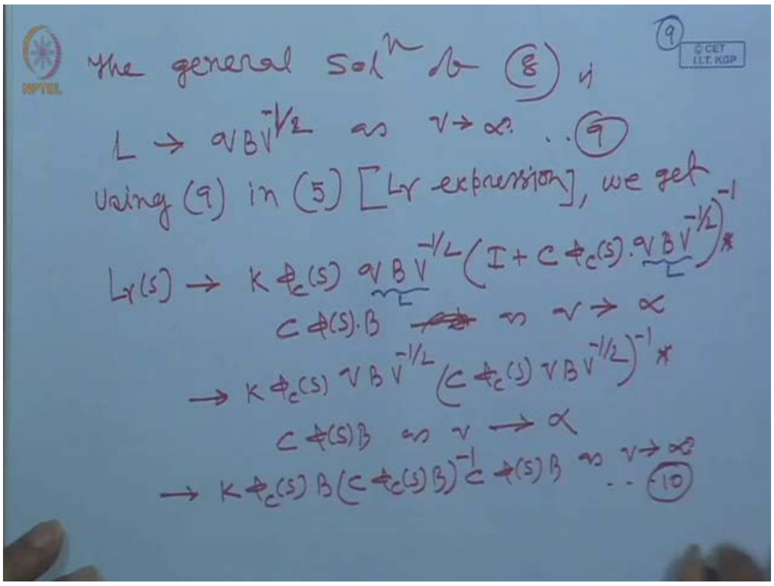

Then, the general solution of equation eight is L tends to when L tends to q b V half L tends to be q b V minus half if the L is q b minis half minus half plus half is cancelled. So, that will be a b q cancel, similarly this becomes for the approach if you selection of this one p half as q tends to infinity, so that is the solution of that one q b V inverse of as V tends to infinity. So, substituting using this, we had known nine using equation nine in five or L R expression L R expression the loop gain expression of LQG expression. So, I am using loop gain expression in this expression, I am using well of what q b and V half that means square root of your noise covariance measurement noise covariance.

If we use this one in this expression, we get L R of S is equal to k phi C S implicit of L I am writing q B v minus half then bracket I c phi c of S q b V minus half whole inverse. Then, star multiplied by this C phi of S into b tends to infinity as q tends to infinity, look this is our L we are substituting L and this is our L in the expression five. So, this further we can write it this, finally we can write it since this is large quantity of this one compare to one in diagonal elements. Then, we can neglect this one to this one, so that tends to phi c of S q phi c of S q, then V of half then c phi c of S q b V half whole inverse star c phi of S b as q tends to infinity.

Now, look at this one this inverse q q cancel then V, V is a identity matrix, so this expression, finally, we will come k phi C of S b than C phi C S b phi, C S b whole inverse C phi of S. This is phi of S as it is from there are C phi b inverse this as q tends to infinity, let us call this equation number ten.

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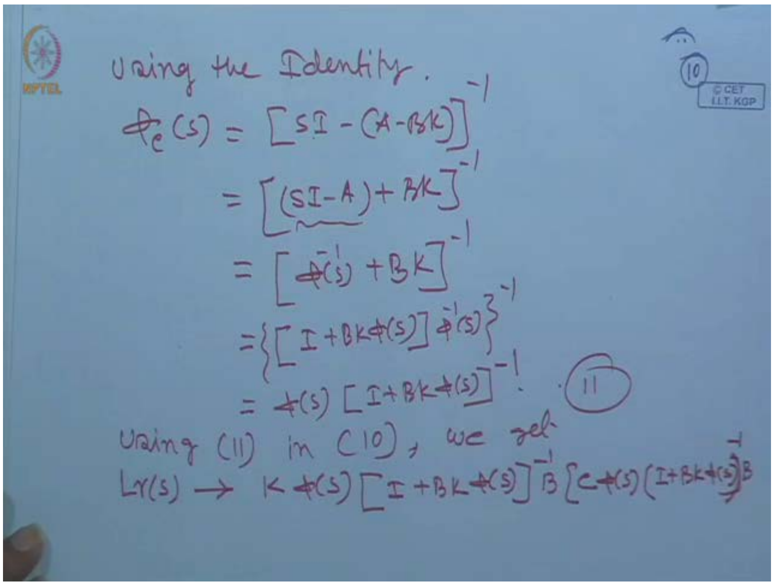

Now using another identity in this expression ten using the identity phi c of S, we know is nothing but S I minus a minus B K whole inverse which I can write it S I minus A. This plus B K whole inverse which I can write it this is nothing but A phi S inverse, this one plus B K whole inverse, so I can write it this as a nothing if you write it this one phi I of phi S take it common. Then, I B K phi of S I B K phi of S whole into phi of S this inverse, so this is nothing but A phi of S because A into B whole inverse B inverse A inverse reverse order, this inverse is phi of S.

Then, it is I plus BK into phi of S whole inverse, let us call this equation 1, this identity I am just writing S I minus a minus BK inverse, so I can write phi S 1 plus BK into phi S whole inverse this expression. I am using equation number twelve, sorry ten, so using eleven in ten and ten is this one; where ever there is a phi C, I will use that one using this one in ten get L S. This whole inverse into B, look at expression this k phi C, I am using that that one phi S, this expression this expression is nothing but A phi C of S.

Then, what is the expression B, then C phi, C phi, C expression is what from here to here is phi C expression. This is phi C, and then B, then this multiplied by this thing, then thing multiplied by C phi S of B as q tends to infinity. Let us call this equation as twelve that means in equation ten, I used this expression whenever there get the phi C is there phi C is there using the in equation and we got these expression.

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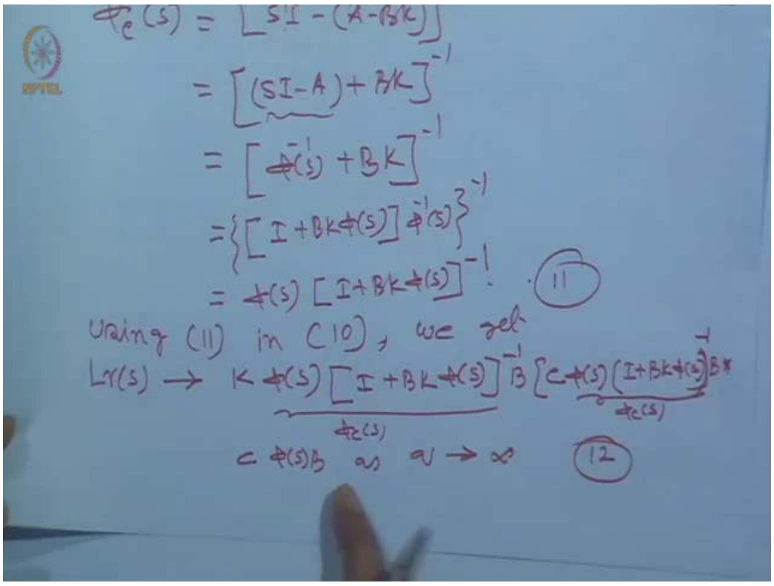

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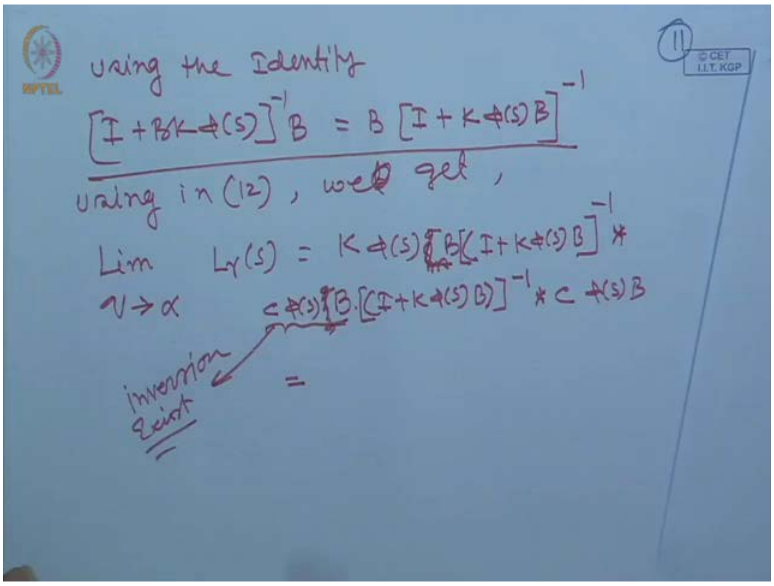

So, finally and once again we are using this identity I plus B K phi of S whole inverse B, you can write it this reverse order B I plus k phi S b inverse. So, I plus B K phi S whole inverse is A B I plus we will get later, I am using this identity using in twelve, we get that limit q tends to infinity L R of S is equal to k say in phi k phi S k phi S. Then instead of that one, I am using the identity that one B first, then I made B, then I plus k phi S into

B whole inverse star. So, this is then B then B C phi S C phi S that is identity, this part identity, so that is B that will be B I plus k phi S into B whole inverse B this I plus this inverse is here.

So, this inverse is here, then whole inverse this inverse into C phi S B, so what is left in here, this will this inverse and again that I miss something. So, this will be finally, it will be because we have considered that number of inputs and outputs is same. So, this inversion is exist that whole inversion is exist this inversion exist, so ultimately we will see will derive this thing in next class some 2, 3 lines is left.

So, we will discuss in the next class that what is our choice of that our loop gain choice through noise process what will be the ultimately loop gain will come. It will come same as the linear quantity regulated loop gain, so one can regain or recover the loop gain of LPG by selecting the properly the process noise covariance, so the rest of the portion we will discuss next class.