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Lecture - 45
The Linear Quadratic Gaussian Problem

So, last class we have discuss that linear quadratic regulator problems, and we have proved that linear quadratic regulator problems has a good robust. I mean properties gain margin is infinity or you can say LQR problem gain margin.

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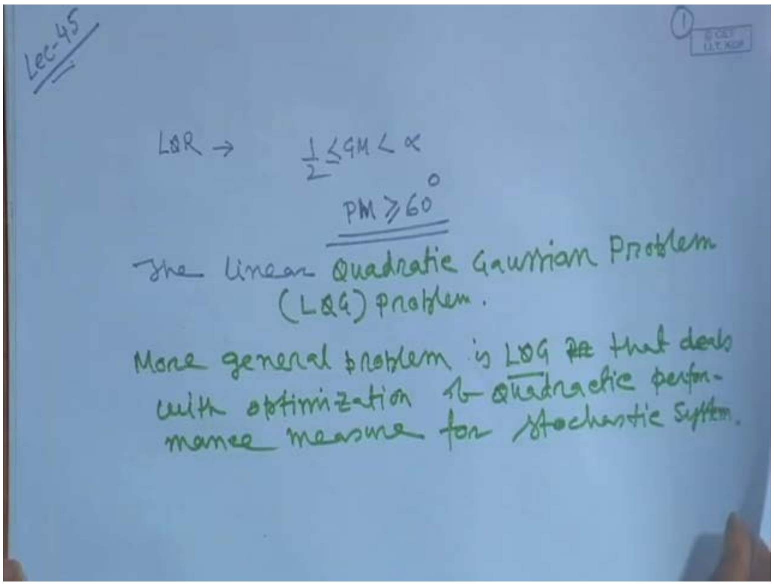

You have seen it is a infinity and greater than equal to half, that means what are the present gain is there of this system. We can increase the gain up to infinity still the system is stable if you designed up controller based on the LQR. And also the gain you can reduce to half whatever the present gain is there that gain you can reduce to half of its below. Still the system is stable at least it may be less than half also it is ((Refer Time: 01.18)), but this range is granted if you designed a controller based on LQR, then system will be stable. This is the gain margin and phase margin, phase margin phase margin will be at least the phase margin is 60 degree.

So, this LQR design if you do the LQR design based on the, what is called performing index elevation, then you will get this gain margin. And phase margin is greater than 60 and gain margin infinity, how greater than half and less than infinity. So, today we will discuss that is what is called linear quadratic. Linear Quadratic Gaussian problem in short it is LQG problem, LQG problem will discuss. So, in LQR design problem there is a restriction while we are designing the controller. First restriction is that the system is deterministic systems again in real practice or real world. The what is called system is under stochastic environment, so we have to design a controller under that environment.

So, more general control problem you can say that more general general problem problem is LQG problem LQG, that deals with optimization of a quadratic performance index. Quadratic performance measure quadratic performance measure for the stochastic systems. In LQR what we did it we deals with the optimization of quadratic performance under deterministic systems. System is determination that is that means the system is not corrupted with noise info, state is not corrupted with noise. And measurement equation is also not corrupted with noise, but more general problem you see that is, what is LQG problem deals with the quadratic minimization quadratic minimization under stochastic environment. So, first we will state the statement of the problem.

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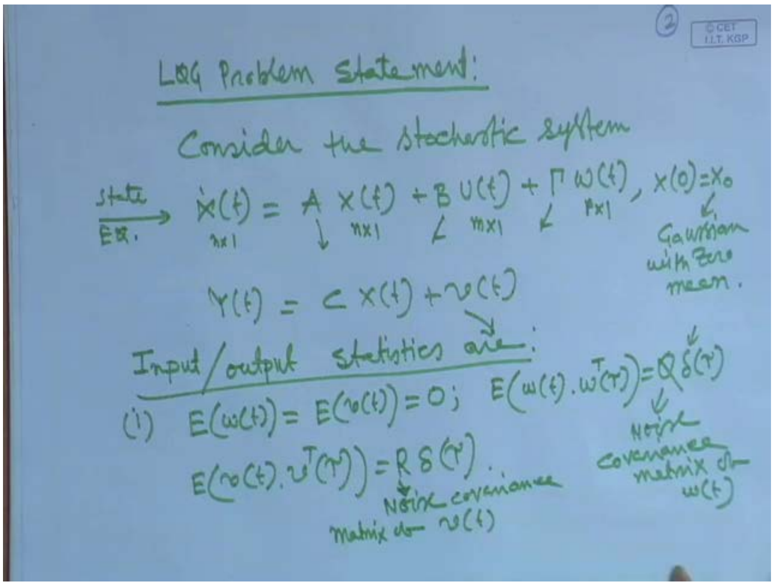

So, LQG problem statement LQG problem statement. So, consider the consider the stochastic systems X dot of t is equal A of X t plus B of U t plus gamma W of t and X. Initial state is depend which is in stochastic in nature that random in nature. So, this is Gaussian which zero on mean X zero in Gaussian with zero mean.

Now, let us call the state of this number of state in the system is n. So, number of inputs is m and number of the noise corrupted to the state equation corrupted with noise through gamma channel is dimension let us call small r into 1. So, once you define the state of this dimension of the X W U t and W of t, W t is the noise. And this noise is zero mean white noise Gaussian process, again the W mean zero mean white noise Gaussian process. And once you this dimension immediately you know the dimension of A B and gamma. So, and this is the state equation you can say this nothing but a state equation state equation and our output equation Y of t is equal to C of X t plus v of t.

And again this W of t sorry v of t is zero mean white noise Gaussian process. Whose characteristics is known means noise covariance matrix is known to us before that, so this is called measurement noise. This is called input noise W is input noise v is called the measurement noise their statistics is known to us. Mean and covariance is known to us W and v as I mention it they are zero mean white noise Gaussian process, and they are uncorrelated. So, there are few assumptions are made before we what is called get the solution of controller solution. In under stochastic and environment, so input so output statistics are like this way. First is E expected value of omega of t input noise is equal to expected value of v of t is 0 mean both input and measurement noise.

Input measurement noise in zero mean and not only this they are white noise that mean E W of t into W transpose of tau that equal to Q delta t. Delta tau is a operator when tau is equal to 0 tau is equal to 0, then this below is one other than 0 this below is 0. So, this indicates the t is equal to t is equal to small t is equal to tau then their covariance matrix is the Q and this is the delta operator again this is the delta operator. Similarly, this indicates that this input noise covariance matrix is Q again and there independent different because at t time, what is the noise and t plus some other time of t 1 time they are uncorrelated.

That means noise and t 1 time is no way related to noise at t 2 time. So, this indicates this is the noise covariance input noise covariance matrix input noise covariance matrix again. So, in place of this one when tau is equal to t when tau is equal to t this value is 0 sorry this value is 1 when tau is not equal to t this value is 0. That means it indicates that they are uncorrelated and this W of v of t into v transpose of tau is equal to R delta of tau. And this is the noise covariance of measurement matrix noise covariance of the measurement noise covariance of noise covariance matrix of v of t. And this is of v W of t this is the this indicates that W is W and v are what is called white noise with zero mean and v and w are also that uncorrelated.

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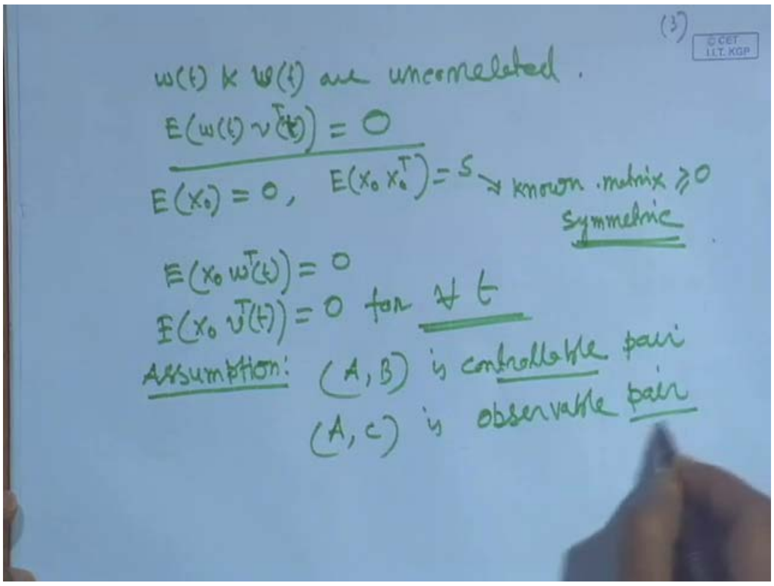

You can say that v and W you can just say then v and W W and v are uncorrelated and uncorrelated how will you depended W of a v tau transpose of t this equal to they are different ((Refer Time: 12.11)) or in A v of t is equal to 0 they are uncorrelated t. And also E assume that initial state has a zero mean and they are covariance is equal to S which is known matrix. That each noise covariance matrix Q and v R this is a this matrix this matrix is positive semi definite matrix. And this is A your that that we use the v because we have use v if we recollect we have use the R when we were discussing the linear quadratic regulator problem that R is the weightage in the control input again. So, it should not use the same symbol here, so we are using the noise covariance matrix of the measurement noise covariance is v.

So, this is this is also we will not use Q, so we will use these values is W capital W here capital W is the noise covariance matrix of W because Q we have use in a LQR problem in state waiting matrix. So, in R we have to avoid a confusion we have use the W and this is v. So, then this is the noise covariance matrix and this matrix is positive definite matrix and this is positive semi definite matrix and the measurement noise covariance matrix is positive definite matrix, we may guess in it and there symmetric again. So, this matrix is positive definite matrix and symmetric, so and they are uncorrelated this initial state is uncorrelated with W of t of this 0.

Similarly, there is initial uncorrelated the input noise as well as it is uncorrelated the measurement noise. So, this for all t, so this are the assumptions is made it before this solve the LQG problem. There are further this correct statistic assumption on characterised input noise covariance matrix output noise covariance matrix. And also initial state what is called covariance matrix is ((Refer Time: 15.08)) variance is known to us.

Then our made in assumptions A and B pair is controllable because then we able to shift the all the poles of the systems open up poles at desired location by LQG problem also just like LQR that must be controllable a then A and C is controllable pair. A and C observable pair and we assume that all the state variables are not measureable. That is why we need an estimated to estimate the state of the systems, so our quadratic the objective function.

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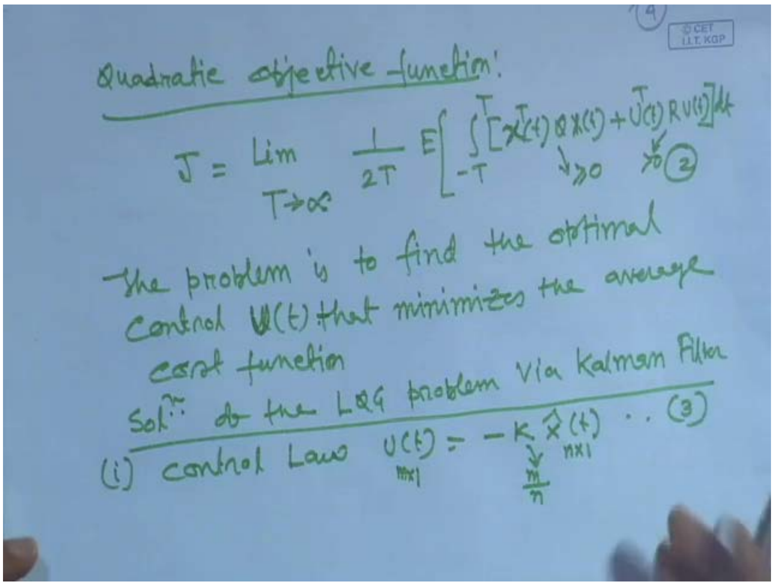

The quadratic objective function first we define J is equal to limit T time T tense to infinity 1 by 2T expected value of that performance index minus T2 plus t and it is nothing but 1 by t I am taking the average of that one X X transpose Q X transpose Q X of t plus U transpose R U of t whole dt whole dt agree. So, this so this we want to minimise this one the, now the problem bulge down to this way the problem is to find is to find the optimal control U of t capital U of t that minimises the above performing index that minimizes the average cost function.

Let us call this is the equation number 2. And now our basic equation what we given describe this two is equation number 1. So, state equation and measurement equation we have given the name equation number 1 and the performance X. What we are assume again in stochastic movement this is equation number 2 we are taking the expected value of this just like a quadratic function, that you are giving weightage on the state an as well as on the input vector U. So, this Q and R is same as before when we have discuss the what is called LQR problem. The Q is positive semi definite matrix R is positive definite matrix and they are symmetric as early as possible.

So, the now next is we just solution of LQG problem. Solution of the LQG problem via Kalman filter you see we have consider the system is corrupted within that. That means state equation is corrupted with noise of measurement equation also corrupted with noise. So, when you are implementing the LQG problem that U is equal to minus control gain into because states are not accessible we have to estimate the X hat of t, we have to use. So, first we have to find out what is call that how to estimate the state of the system on that stochastic environment that one. So, details of this one I will not discuss about the Kalman filter here again.

So, I will just tell you how to design a, what is call Kalman filtered here briefly. So, first step is first step is control law our control law U of t is equal to minus X of t. So, let us call this equation number 3 this is dimension in n cross 1 and this dimension since it is m cross 1 the dimension of k is m cross n. In LQR problem we have assume the all the state variables are excisable to our to us, which will be used for implementing the control law where as in stochastic environment LQG problem. The states are not accessible or measureable, so what we did it we have estimated we have to estimate the states except to this except of t.

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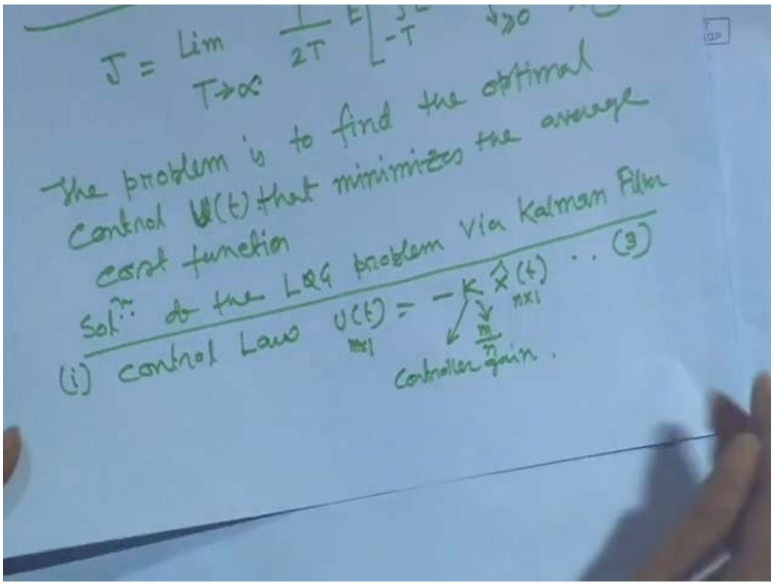

So, how will you estimate the state that will discuss step by step now. So, here if you see the K is the controller gain and the controller gain is design same as what is call the LQG problem and we will prove it here that estimated or the Kalman filter estimated. And the controller design estimated design and controller design they can be done separately or independently. Because they are very separation theorem is valid in you have to design controller as well as the estimated separately, so let us see this one.

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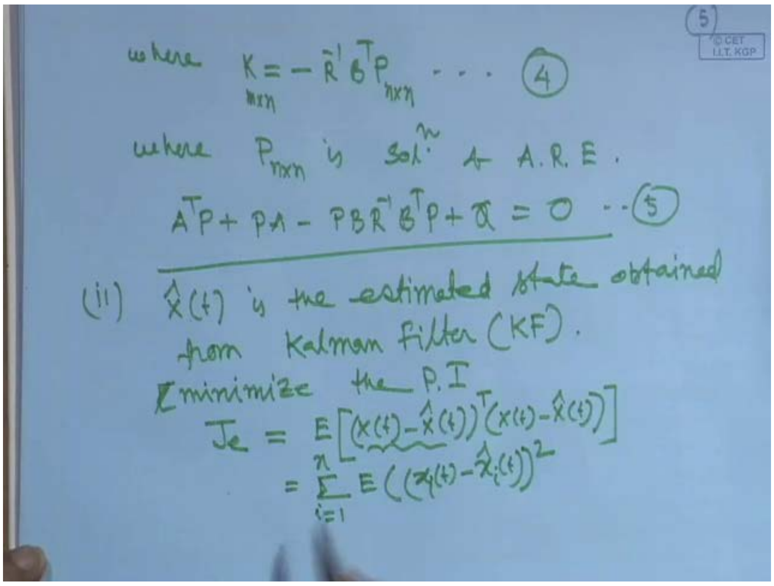

Where K is equal to minus R inverse B transpose P let us call this is equation 4, as I mention earlier that K is design as same as LQR problem. So, I am not we have already discuss how to gain and we got this expression. So, since K dimension is m cross n and n dimension is this immediately other R dimension we know it. So, that is equation number 4 where the P is the P is n cross n is solution of our solution of algebraic Riccati equation. So, what is the algebraic Riccati equation for this controller design A transpose P PA minus PBR inverse B transpose P plus Q is equal 0. So, this the infinite regulator problem we know already how to solve this one different technique.

So, let us call this is equation number 5 next is once we design K because I cannot implement a U of t until unless estimate the states because our system equation are corrupted with miles with a known statistics. That means we have use the conventional that noises are, so this is known means they are zero mean white noise Gaussians process. And some other assumption also we may did, now next is how to estimate X of t.

X is the estimated state as an estimated state obtained from Kalman filter in short K F. So, that so what is the fifth line I am just get link. So, ultimate in Kalman filter there are in many ways in perform index and that performing index is the error covariance matrix trace of error covariance matrix. So, minimize the that means minimize the performance index performance index what is this performance index consider for estimated that J e is equal to expected value of X t of X hat of t. So, this is nothing but a error between actual state and estimated state. So, that into transpose X of t minus X hat of t, so that will be a scalar quantity that means we are doing the some of the errors error square over the period of time again.

And then we are taking expected value of that one this is same as this is same as if you see summation of component voice of state t variety. The error n states are there i is equal to one to n then expected value of x small x of t or small x i of t into small x hat of t. So, this is state error square is same as that expression or minimizing the minimizing the trace of trace of error covariance matrix. So, one can write it this also like this way minimize J.

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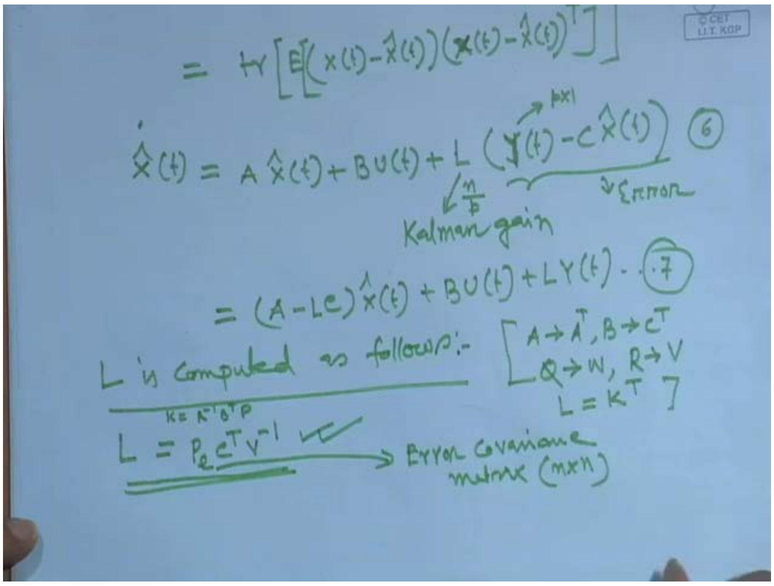

Minimize J that means minimize the trace of E X of t minus hat of t this into X t. This is capital X write capital X capital X minus X hat of t whole transpose then this and this take that trace minimize that trace of that one that means the problem estimated. Minimize the performance index that either you write expected value of the error square or you write it this minimize the trace of this the error covariance matrix. It is the error we find that means all the diagonal elements of this matrix is nothing but a that quantities. So, this we do it the so this way we have to estimate the state of the parameter. So, I am writing directly that what is the estimated equation the Kalman filter of this equation is same as what we did it for the deterministic is structure is same.

But, only the implementation wise we have to do in different way, so this plus L into Y is capital Y of t minus C X hat of t. Look that one this L is the Kalman gain and this is the error. So, this third term is the error correction each duration or each time this will correct the error, so this the error term agree. So, this dimension Y dimension you know this dimension P cross 1. The beginning I just mention it you just see the dimension of P P is not mention it here dimension of P dimension of Y is P cross 1. So, similarly immediately we can find out C this dimension of number of outputs is P. So, if it is this P immediately you know the dimension of this one will be n cross P.

Well Kalman filter gain is an n cross P, now how to obtain the L the next question is how to obtain the L. So, this if you can simplify I can write it A minus LC this into this LC X hat of t plus BU of t plus LY of t. So, let us call this equation last equation number is what we have given it 5 then this is equation number 6, we give it this equation number 7. So, the estimated design is the dual of controller design agree, so how L is computed this. This Kalman gain is computed L is computed as follows, so you write the expression for LQR controller or L P J controller design expression. You write it there you put it in place of A replaced by A transpose.

So, you replace A by A transpose, B by C transpose and Q by W and R by R replace by V and then L replace by K transpose or K replaced by K transpose replace by L, so let us call how it is framed. So, our expression for L and whose dimension is n cross P is equal to what is the if you see side by side I am writing what is the controller expression R inverse B transpose P are inverse B transpose P. So, I replace the, because our K is what K will be replaced by L transpose but I want to find out L transpose. So, let it be L then it will be K transpose L is K transpose. So, it will be a P first will come P let us call for estimator I am denoted by P e.

Then B transpose and then transposed B, B is replaced by C transpose. So, C transpose then it will be coming R inverse R is replace by V, so V inverse again. So, this is the our Kalman gain you have to find out for you know C you now. V is the noise V is the measurement noise covariance matrix, but you do not know P, P is the error covariance matrix. P is the P is the error covariance matrix of dimension m cross n. So, that we do not know look here how I am find competing the Kalman gain Kalman gain I am finding about what is the expression for controller gain. That K is equal to R inverse B transpose take the transpose of K if you take the transpose P transpose. But, I am writing is P symmetric matrix P. So, P is replaced by P e the error covariance matrix.

And then B is replaced by C transpose then R inverse transpose is R. So, it will be a R is replaced by V agree. So, this is the Kalman gain expression once I get a but here is unknown is P e error covariance matrix of dimension m n. How to solve the covariance the way we solve algebratic Ricatti equation for L Q L problems or LQG problem while we are designing the what is called the LQG controller. We have used this expression if you recollect that one this expression this used to calculate the LQG controller again gain. So, in this expression I will replace A by A transpose and P by P e, B by B will be replaced by C transpose and Q will be replaced by W that we what we mentioned. So, if you do this one that L is computed the P e is computed.

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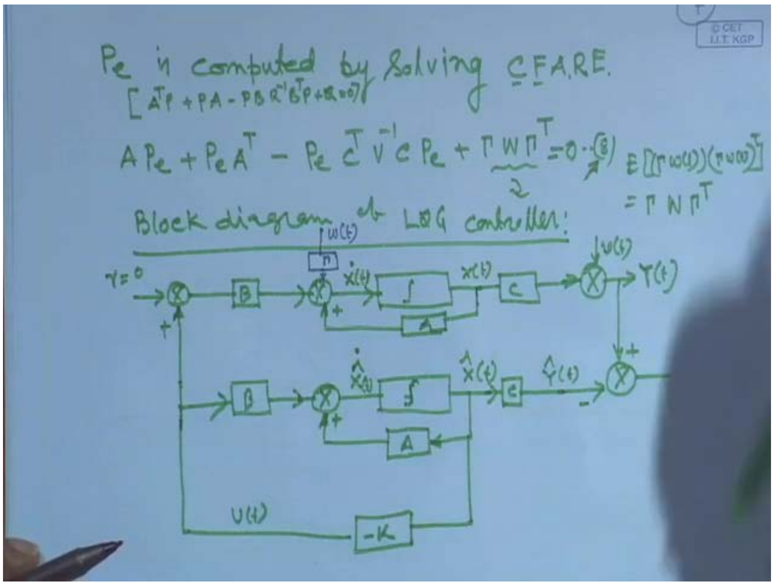

P e is computed by solving continuous filter algebraic Ricatti equation algebraic Ricatti equation what is this continuous filter. So, filter expression for designer filter and designer controller is dual of each other. And how we are doing a dual replacing controller whatever the A is there replaced by A transpose B is there replaced by C transpose and your Q is replaced by noise covariance matrix. And R is replaced by measurement noise covariance matrix this replaced by in their original Ricatti equation. So, if you replace by this one it will be coming AP e plus P eA transpose then P eC transpose V inverse again.

Then CP e then gamma transpose gamma W gamma transpose what is this one because it is expected value of gamma into W of t. So, this is nothing but this come and getting from this one is expected value of gamma W of t into this into transpose of that one transpose of that gamma. This gamma Y of t whole transpose and that will come gamma that is W noise covariance of this one into gamma transpose that one is write this one. So, any way this equal to 0 and that is equation number. How I am writing it you see the controller design how we doing A transpose P plus PA minus PBR inverse B transpose P plus Q is equal to 0, so A is replaced by A transpose.

So, A transpose and its transpose A P is replaced by P e error covariance matrix. And in this way B is replaced by C transpose you see C transpose R is replaced by measurement noise covariance matrix V. And so on and Q this is now as shown you this Q is what noise covariance matrix that one this Q. So, this is equation number 8, now if you see the our LQG controller structure how it is then the detailed diagram that block diagram of block diagram of LQG controller. So, we have a this star then B is then integrated I am just writing the block diagram from actual system, this is X of t then C and this is our output actual output. But, it is corrupted with noise v of K and then we are getting the Y of t measured output.

So, this output from this output and this is our B and it is corrupted to here you see this the W W is coming it is passing through the gamma. This block is gamma and is coming here this is W of t input noise input noise entering through channel W gamma again. So, this now again you can see here that from here this is reference input let us consider 0 then we have a B.

The structure of controller structure estimated is same as the structure of the actual plant structure I am talking about. So, this equal to this one then integrated and here is not completed. So, here from here from here there system matrix A is there system matrix A and it is coming here plus here is X dot of t this signal is X dot of t again. So, this then it will be coming is X hat of t.

So, in this is integration so X dot X hat dot of t then C then it is it is coming here Y hat of t this minus this is plus agree. So, this signal is coming and from here this A this and the if you see this one this is the Y hat and from the estimated state. The controller I am telling the estimated state will go to the controller K minus K. And that is the controller law that is the controller law again U of t then this sign is plus U of t. This is B this now if you see the observers the estimated equation A X plus this is the correction term correction term.

We have not build up here, now this is the this signal is Y minus Y of t minus Y hat of t agree, so this will go to the Kalman gain L. So, that L that L will come ultimately it will come that L or you will write it here instead of showing here this is the Y hat of this one. I will show you in the red one this is the Kalman gain agree, so that will come this Kalman gain will come from here plus that one agree. So, this is this and now show you this block this red with this block this is your plant this is your plant and this is your controller agree. And this part from here from this part from here to here part is your Kalman filter Kalman filter which will estimate the state of that one. So, this is the this block gain indicates the controller along with the Kalman filter where the states are not available.

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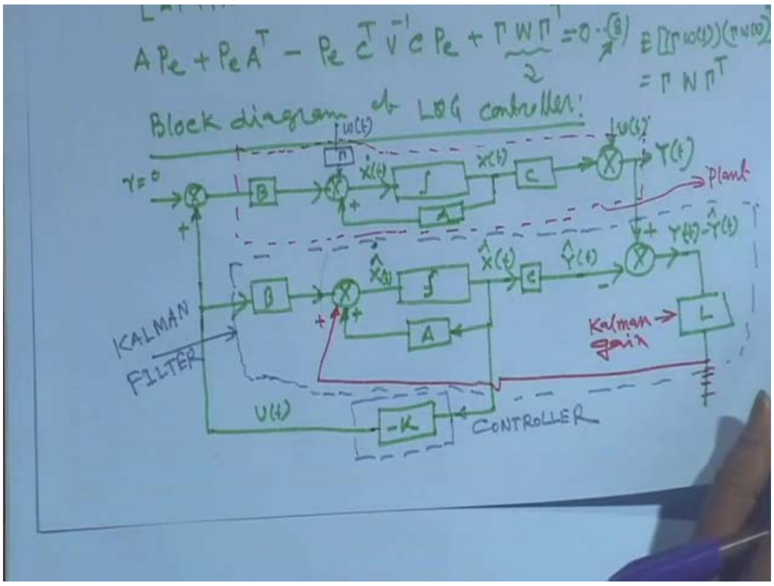

So, you see the Kalman filtere what this dynamics involved here X X hat dot is coming one signal A X hat plus BU plus there is a correction term Y Y hat error multiplied the gain it is coming here. So, this the our this blue dotted line is the Kalman filter and this dotted line small dotted line small box dotted line is a controller and this red dotted box is our plant. So, this is the you a Kalman filter best controller is designed now we have to see.

Will see it what in the LQR designed we have observed that if you design the controller based on LQR that we have a gain margin is infinity. And the present gain you can increase to infinity at the same time present gain you can reduce to a half in this region the system is stable from agree. As well as you can save the phase margin of the system when it is controller designed based on the LQR. The phase margin will be at least 60 degree that is where we have done and when we are using the estimated along with the controller that properties is totally lost. That means robustness is of the LQR LQG is compare to LQR is much less of this one, then we will see this one how to that. So, first we have see we will see this LQR, LQG separation principle, so since we have done already at.

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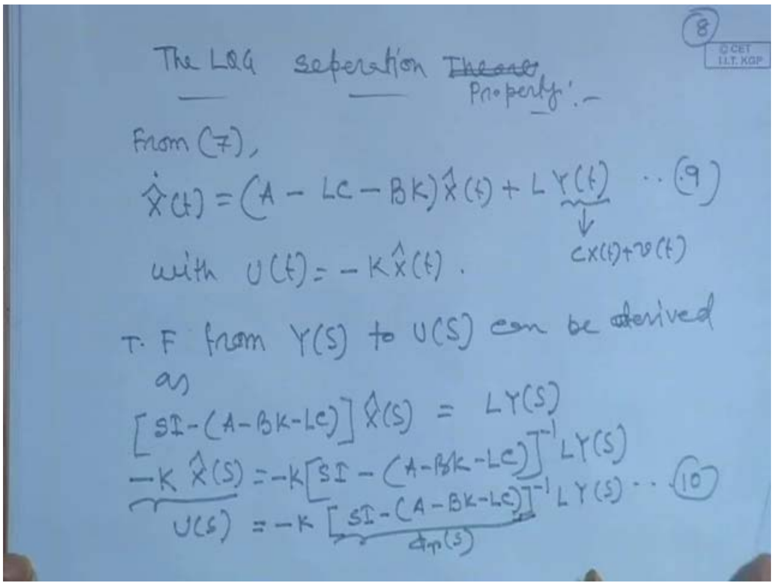

So, I will not go separation property since separation property for LQR we have derived it will not go in details in this. So, recall the equation number 7 this equation number 7 you recall that one this equation putting U is equal to minus K X hat. So, from 7 we can write X dot of t is equal to A minus LC and BK because B U K U K is replaced by minus K minus X hat of t. So, that is why it is X hat of t plus LY of t agree. Expression measurement equation is nothing but a C of X t plus v of t because output is corrupted with noise this one. So, with U of t will replaced by minus K X hat of t again let us call this is equation number 9.

Now, we will find out the transformation from this you just see from Y to that U what is the transformation of that one Y to U. If you see the transformation from Y to U, now find out the transformation from Y S to U S. Why you are finding out the transformer again to show that our loop gain is different from the loop gain of LQG is different from the loop gain of LQR. And there we will show it that we are the properties of LQR noise properties of LQR is lost when we are designing a LQG problem agree. So, can be derived as, so take the Laplace transform both side if you take this one I can write SI minus A minus BK minus LC again.

This into X hat of S is equal to L Y of S which I can write which I can write X hat of S is equal to what is this SI minus A minus BK minus LC whole inverse into LY of S agree. Both side I multiplied by minus KX both side I multiplied by minus K X K minus K. So, both side I multiplied so this is nothing but a our U of S agree. This is U of S this and this is nothing but a minus as it is what you got it SI minus A minus BK minus LC that whole inverse LY of S.

Let us call this is a equation number 10 and we denote this one when there was no observer. SI minus SI minus bracket A minus BK when there is no observe that we have denoted by phi S. If we recollects in the LQR design problems, now this whole thing I am denoted by phi R of S agree. This part SI minus A minus BK whole inverse this one. Now, from system state equation and from 9 one can write it.

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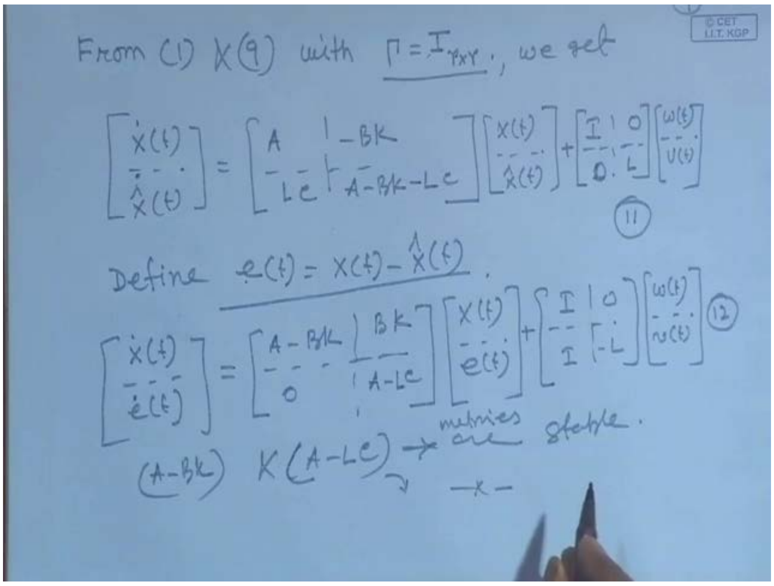

From 1 and 9 with gamma I just consider is an identity matrix. Again whose dimension will be a r cross r. Because noise W of t noise we have consider the dimension is r cross 1, now it is a that one dimension gamma. Let us call it assume or in general you can keep it gamma and also in the expression. So, we get X dot of t and X hat of t X hat dot of t is equal to A minus BK LC A minus BK minus LC. From 1 and 9 we only gamma you have replaced by identity matrix plus. So, this but from this thing we cannot come to conclusion, that whether the estimated design Kalman filter design and the controller design can be done independently or not we cannot say that means separation principle is applicable we cannot say.

So, what we have to do that is call this equation is equation number 11 what we have define that error equation now. So, you define the error e of t is equal to X of t minus X hat of t this you define and then you express X as in terms of error equation. So, I will get it finally X e of t is equal to A minus BK BK 0 A minus LC plus I I 0 minus L W of t that will be v of t. What I did it X hat I express in terms of e agree, so X hat is equal to X minus e. In this if you manipulate this express equation you will get this one now see the structure of that X dot an e and e dot expression. There is upper triangular form the Eigen values of this equation 12.

Eigen value of this 12 note Eigen values of this system is same as Eigen values of A minus BK and Eigen values of L minus A minus LC. This Eigen values is nothing but a controller Eigen values A minus LC Kalman filter Eigen values. So, they are the couple when we are designing this one we do not need any information of Kalman filter gain information and when we designing this and we do not need any information of the controller design information. So, this and both the matrixes that is A minus BK and A minus LK both the matrixes are stable matrixes are stable agree. However you see the how to design the controller gain are observer gain L.

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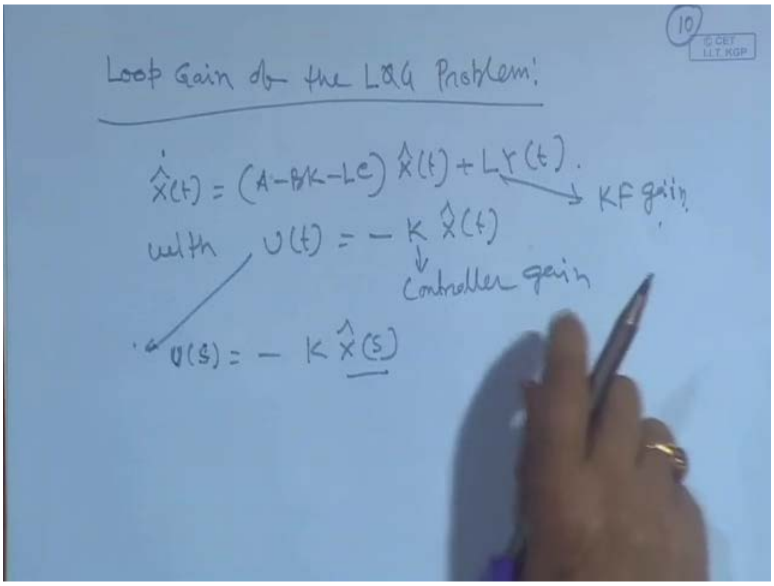

So, you have to design its better than Eigen values of observer Eigen values of estimated A minus LC it should be at least five to ten times of the Eigen values of the controllers agree. In that way we can get better performance or the better close look behaviour of the system. That mean Eigen values of this one should be five times five to ten times more distance from the Eigen values of controller Eigen values and that will give you the what is called good close loop system behaviour this is.

So, now let us investigate our loop gain loop gain of the LQG problem if you look at the LQG problem our basic structure, it is a system that Kalman filter then controller. So, this what is our loop gain of this thing we can make it in simple weight form. This is the estimated equation A of minus BK minus LC X hat of t plus LY of K Y of t agree. Now, take the with U of t is equal to minus K X hat of t this is controller gain and this is the Kalman filter gain agree. So, if you take the Laplace Transform of both side of this equation U of S U of S is equal to minus K X hat of S. And you know the X hat of S expression in details from equation number that is what we have given it X hat of S equation number that is 10, we can get it this one.

So, I will discuss this that expression that how to find out the loop gain of the LQR and compare with the sorry LQG. How to find out the loop gain of LQG and compare with LQR that we will discuss in next class. But, LQG when you will see there are different loop gain you will get it and that will show that we are losing the properties of excellent properties of property LQR, when we are using the Kalman filter along with the controller. So, next class we will discuss in details that one.